

# REFLECTION AND TRANSMISSION AT THE INTERFACE OF AN ELASTIC AND TWO-TEMPERATURE GENERALIZED THERMOELASTIC HALF-SPACE WITH FRACTIONAL ORDER DERIVATIVE

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**Abstract.** The present investigation is concerned with the reflection and transmission at elastic half-space and a two-temperature generalized thermoelastic half-space with fractional order derivative. The governing equations in the context of the theory of two-temperature generalized thermoelasticity using the methodology of fractional calculus are used to investigate the problem. The incident wave is assumed to be striking at the plane interface after propagating through the elastic solid half-space. It is found that the amplitude ratios of various reflected and refracted waves are functions of the angle of incidence and frequency of the incident wave. These amplitude ratios are influenced by the fractional-order thermoelastic properties of media. The expressions of amplitude ratios and energy ratios have been computed numerically for a particular model. The variations of energy ratios with the angle of incidence are shown graphically. The conservation of energy at the interface is verified.

**Keywords:** fractional, temperature, elastic, reflection, transmission

## 1. Introduction

In recent years, several interesting models have been developed by using fractional calculus to study the physical processes particularly in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity, etc. It has been realized that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There exist many materials and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading, etc., where the classical thermoelasticity is based on Fourier type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time-fractional (non-integer order) derivatives.

The first application of fractional derivatives was given by Abel [1] who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. Caputo [2] gave the definition of fractional derivatives of order of absolutely continuous function. Caputo and Mainardi [3], Caputo and Mainardi [4], and Caputo [5] found good agreement with experimental results when using fractional derivatives

for the description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity.

Oldham and Spanier [6] studied the fractional calculus and proved the generalization of the concept of derivative and integral to a non-integer order. A theoretical basis for the application of fractional calculus to viscoelasticity was given by Bagley and Torvik [7]. Applications of fractional calculus to the theory of viscoelasticity were given by Koeller [8]. Kochubei [9] studied the problem of fractional order diffusion.

Rossikhin and Shitikova [10] presented applications of fractional calculus to various problems of mechanics of solids. Gorenflo and Mainardi [11] discussed the integral differential equations of fractional orders, fractals, and fractional calculus in continuum mechanics. Mainardi and Gorenflo [12] investigated the problem of Mittag-Leffler-type function in the fractional evolution process. Povstenko [13] proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative of order  $\alpha$ . Because the heat conduction equation in the case interpolates the parabolic equation ( ) and the wave equation ( ), this theory interpolates a classical thermoelasticity and a thermoelasticity without energy dissipation. He also obtained the stresses corresponding to the fundamental solutions of a Cauchy problem for the fractional heat conduction equation for one-dimensional and two-dimensional cases.

Povstenko [14] investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. Jiang and Xu [15] obtained a fractional heat conduction equation with a time-fractional derivative in the general orthogonal curvilinear coordinate and also in another orthogonal coordinate system. Povstenko [16] investigated the fractional radial heat conduction in an infinite medium with a cylindrical cavity and associated thermal stresses.

Ezzat [17] constructed a new model of the magneto-thermoelasticity theory in the context of a new consideration of heat conduction with fractional derivative. Ezzat [18] studied the problem of the state space approach to thermoelectric fluid with fractional order heat transfer. The Laplace transform and state-space techniques were used to solve a one-dimensional application for a conducting half-space of thermoelectric elastic material. Povstenko [19] investigated the generalized Cattaneo-type equations with time-fractional derivatives and formulated the theory of thermal stresses. Biswas and Sen [20] proposed a scheme for optimal control and a pseudo state space representation for a particular type of fractional dynamical equation.

Borejko [21] discussed the reflection and transmission coefficients for three-dimensional plane waves in elastic media. Wu and Lundberg [22] investigated the problem of reflection and transmission of the energy of harmonic elastic waves in a bent bar. Sinha and Elsibai [23] discussed the reflection and refraction of thermoelastic waves at an interface of two semi-infinite media with two relaxation times. Sharma and Gogna [24] discussed the problem of reflection and transmission of plane harmonic waves at an interface between elastic solid and porous solid saturated by viscous liquid. Tomar and Arora [25] studied the reflection and transmission of elastic waves at an elastic/porous solid saturated by immiscible fluids. Kumar and Sarthi [26] discussed the reflection and transmission of thermoelastic plane waves at an interface of thermoelastic media without energy dissipation.

The two-temperature theory (2TT) of thermoelasticity proposes that heat conduction in deformable media depends upon two distinct temperatures, the conductive temperature and the thermodynamic temperature (Chen and Gurtin [27]; Chen et al. [28]; Warren and Chen [29]). While under certain conditions these two-temperatures can be equal. In time-dependent problems, however, in particular, those involving wave propagation, and are generally different (Warren and Chen [29]). The key element that sets the 2TT apart from the classical

theory of thermoelasticity (CTE) is the (theory specific) material parameter . Specifically, if, then and the field equations of the 2TT reduce to those of CTE.

Warren and Chen [29] investigated wave propagation in the two-temperature theory of thermoelasticity. Quintanilla [30] proved some theorems in thermoelasticity with two-temperatures. Recently, Puri and Jordan [31] studied the propagation of plane waves under two-temperatures.

Youssef [32] studied the induced temperature and stress field in an elastic half-space under the purview of the two-temperature generalized thermoelasticity theory. The half-space continuum is considered to be made of an isotropic homogeneous thermoelastic material, the bounding plane surface being subjected to ramp-type heating. Uniqueness and growth of solutions in two-temperature generalized thermoelastic theories were given by Magana and Quintanilla [33].

The convolutional variational principle, reciprocal and uniqueness theorems in linear fractional two-temperature thermoelasticity were given by El-Karamany and Ezzat [34]. They proposed two models where the fractional derivatives and integrals are used to modify the Cattaneo heat conduction law in the context of the two-temperature thermoelasticity theory. They also proved uniqueness and reciprocal theorems and the convolutional variational principle to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors except symmetry conditions. Fractional order coupled thermoelasticity results follow from this model. Indeitsev, Vakulenko, Mochalova and Abramian [35] studied the transport and deformation wave processes in solids. A two-temperature model of optical excitation of acoustic waves in conductors was discussed by Indeitsev and Osipova [36].

In the present paper, the reflection and transmission phenomenon at a plane interface between an elastic solid medium and a fractional-order thermoelastic half-space with two-temperature generalized thermoelasticity theory has been analyzed. In fractional order generalized thermoelastic solid medium with two temperatures, potential functions are introduced to represent two longitudinal waves and one transverse wave. The amplitude ratios of various reflected and refracted waves to that of incident waves are derived. The amplitude ratios are further used to find the expressions of energy ratios of various reflected and refracted waves to that of the incident wave. The graphical representation is given for these energy ratios for a different direction of propagation and different fractional orders. The law of conservation of energy at the interface is verified.

## 2. Governing Equations

Following Ezzat and El-Karamany [37], the basic equations of fractional order theory of thermoelasticity with two temperatures for an isotropic and homogeneous elastic medium in the absence of body forces and heat sources are:

the constitutive equation (Stress-Strain and temperature relations) is given by

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma T)\delta_{ij}, \quad (1)$$

the heat conduction equations

$$K\phi_{,ii} = \frac{\partial}{\partial t} \left( I + \frac{(\tau_0)^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) (\rho C_E T + \gamma\phi_0 e_1), \quad (2)$$

where  $I^\alpha$  is the fractional integral of the function  $f(t)$  of order  $\alpha$  defined by Miller and Ross [38].

$$I^\alpha f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) d\tau, 0 < \alpha \leq 2,$$

the equations of motion

$$(\lambda + \mu)\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} - \gamma\nabla T = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (3)$$

the relation between heat conduction and dynamic heat is given by

$$T = (1 - a\nabla^2)\phi, \quad (4)$$

the strain displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (5)$$

where  $\gamma = (3\lambda + 2\mu)\alpha_i$ ;  $\lambda, \mu$  are the Lamé's constants,  $\alpha_i$  is the coefficient of thermal linear expansion,  $u_i$  are the components of the displacement vector  $\mathbf{u}$ ,  $T = \theta - T_0$  is small temperature increment,  $a$  is the two-temperature parameter,  $T_0$  is the reference temperature of the body chosen such that  $\left|\frac{T}{T_0}\right| \ll 1$ ,  $\theta$  is the absolute temperature of the medium,  $\rho$  is the density assumed to be independent of time,  $\sigma_{ij}, e_{ij}$  are the components of the stress and strain respectively,  $e_{kk}$  is the dilatation,  $C_E$  is the specific heat at constant strain,  $K$  is the coefficient of thermal conductivity.

### 3. Formulation of the problem

We consider an isotropic elastic solid half-space (Medium I) lying over a homogeneous isotropic, fractional order generalized thermoelastic half-space with two temperature (Medium II). The origin of the Cartesian coordinate system  $(x_1, x_2, x_3)$  is taken at any point on the plane surface(interface) and  $x_3$ -axis points vertically downwards into Medium II which is thus represented by  $x_3 \geq 0$ . We choose  $x_1$ -axis in the direction of wave propagation so that all particles on a line parallel to  $x_2$ -axis are equally displaced. Therefore all the field quantities are independent of  $x_2$ . For two dimensional problem, we take

$$\mathbf{u} = (u_1, 0, u_3). \quad (6)$$

We define the following dimensionless quantities

$$\begin{aligned} x'_i &= C_0 \eta x_i, u'_i = C_0 \eta u_i, t' = C_0^2 \eta t, u_i^e = C_0 \eta u_i^e, \\ \tau'_0 &= C_0^2 \eta \tau_0, \sigma'_{ij} = \frac{\sigma_{ij}}{\rho C_0^2}, \phi' = \frac{\gamma \phi}{\rho C_0^2}, \quad i, j = 1, 2, 3. \end{aligned} \quad (7)$$

$$T' = \frac{T}{T_0}, h' = \frac{h}{C_0 \eta}, \sigma_{ij}^e = \frac{\sigma_{ij}^e}{\rho C_0^2}, P_{ij}^{*e} = \rho C_0 P_{ij}^*, P_{ij}^{*e} = \rho C_0 P_{ij}^{*e},$$

$\beta_0 = a C_0^2 \eta^2 = a(C_0 / K)^2$  is the temperature discrepancy,

where

$$C_0^2 = \frac{\lambda + 2\mu}{\rho}, \eta = \frac{\rho C_E}{K}. \quad (8)$$

Upon introducing the quantities (7) in equations (2)-(3) with the aid of (4) and (6), after suppressing the primes take the form

$$(\beta^2 - 1) \frac{\partial e_1}{\partial x} + \nabla^2 u_1 - (1 - \beta_0 \nabla^2) \beta \frac{\partial \phi}{\partial x} = \beta^2 \frac{\partial^2 u_1}{\partial t^2}, \quad (9)$$

$$(\beta^2 - 1) \frac{\partial e_1}{\partial z} + \nabla^2 u_1 - (1 - \beta_0 \nabla^2) \beta \frac{\partial \phi}{\partial z} = \beta^2 \frac{\partial^2 u_3}{\partial t^2}, \quad (10)$$

$$\nabla^2 \mathbf{f} = \left( 1 + \frac{(\tau_0)^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left( \frac{\partial}{\partial t} (1 - \beta_0 \nabla^2) \mathbf{f} + \epsilon \frac{\partial \nabla \cdot \mathbf{u}}{\partial t} \right), \quad (11)$$

where

$$\kappa = \frac{\gamma}{\rho C_E}, \beta^2 = \frac{\lambda + 2\mu}{\mu}, b = \frac{\gamma T_0}{\mu}, \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, e_i = \frac{\partial u_i}{\partial x_1} + \frac{\partial u_3}{\partial x_3}.$$

We introduce the potential functions  $\phi_i$  and  $\psi_i$  through the relations

$$u_i = \frac{\partial \phi_i}{\partial x_1} - \frac{\partial \psi_i}{\partial x_3}, u_3 = \frac{\partial \phi_i}{\partial x_3} + \frac{\partial \psi_i}{\partial x_1}. \quad (12)$$

Substituting equation (12) in the equations (9)-(11), we obtain

$$\beta^2 \left( \nabla^2 - \frac{\partial^2}{\partial t^2} \right) \phi_i - (I - \beta_0 \nabla^2) \beta \phi = 0, \quad (13)$$

$$\nabla^2 \psi_i - \beta^2 \frac{\partial^2 \psi_i}{\partial t^2} = 0, \quad (14)$$

$$\nabla^2 \mathbf{f} = \left( I + \frac{(\tau_0)^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \right) \left( \frac{\partial}{\partial t} (I - \beta_0 \nabla^2) \mathbf{f} + \kappa \frac{\partial}{\partial t} \nabla^2 \mathbf{f}_i \right). \quad (15)$$

For the propagation of harmonic waves in  $x_1 - x_3$  plane, we assume

$$\{\phi_i, \psi_i, \phi\}(x_1, x_3, t) = \{\bar{\phi}_i, \bar{\psi}_i, \bar{\phi}\} e^{-i\omega t}, \quad (16)$$

where  $\omega$  is the angular frequency of vibrations of material particles.

Substituting the value of  $\phi_i, \psi_i$  and  $\phi$  from equation (16) into the equations (13)-(15) after simplification, we obtain

$$[A\nabla^4 + B\nabla^2 + C]\bar{\phi} = 0, \quad (17)$$

where

$$A = \beta - R\beta_0 - \beta_0 \kappa R,$$

$$B = \beta \omega^2 + R\beta - R\beta_0 \omega^2 + \kappa R,$$

$$C = R\beta \omega^2,$$

$$R = \left( i\omega - \frac{(\tau_0)^\alpha}{\alpha!} (-i\omega)^{\alpha+1} \right).$$

The general solution of equation (16) can be written as

$$\bar{\phi}_i = \bar{\phi}_{i1} + \bar{\phi}_{i2}, \quad (18)$$

where the potentials  $\bar{\phi}_{i1}, i=1,2$  are solutions of wave equations, given by

$$\left[ \nabla^2 + \frac{\omega^2}{V_i^2} \right] \bar{\phi}_{i1} = 0, \quad i=1,2. \quad (19)$$

Here  $V_1, V_2$  are the velocities of two longitudinal waves, that is, P and T(Thermal) waves and are derived from the roots of quadratic equations in  $V^2$ , given by

$$CV^4 - B\omega^2 V^2 + A\omega^4 = 0. \quad (20)$$

From equation (14) with the aid of (16), we obtain

$$\left[ \nabla^2 + \frac{\omega^2}{V_3^2} \right] \bar{\psi}_i = 0, \quad (21)$$

where  $V_3 = \frac{I}{\beta}$  is the velocity of a transverse wave.

Using equation (15), (19) with the aid of (16) and (18), we obtain

$$\{\phi_i, \phi\} = \sum_{i=1}^2 \{I, n_i\} \phi_{i1}, \quad (22)$$

where

$$n_i = \frac{\left( i\omega - \frac{(\tau_0)^\alpha}{\alpha!} (-i\omega)^{\alpha+1} \right) \kappa \omega^2}{\left( \nabla^2 + \left( i\omega + \frac{(\tau_0)^\alpha}{\alpha!} (-i\omega)^{\alpha+1} \right) (I - \beta_0 \nabla^2) \right) V_i^2}, \quad i=1,2. \quad (23)$$

The basic equations of homogeneous isotropic elastic solid are written as

$$(\lambda^e + \mu^e) \nabla \nabla \cdot \mathbf{u}^e + \mu^e \nabla^2 \mathbf{u}^e = \rho^e \frac{\partial^2 \mathbf{u}^e}{\partial t^2}, \quad (24)$$

where  $\lambda^e, \mu^e$  are the Lamé's constants,  $\mathbf{u}^e$  is the displacement vector,  $\rho^e$  is the density corresponding to medium I.

For two dimensional problem  $\mathbf{u}^e = (u_1^e, 0, u_3^e)$ , the displacement components  $u_1^e$  and  $u_3^e$  are related by potentials  $\phi_1^e$  and  $\psi_1^e$  as

$$u_1^e = \frac{\partial \phi_1^e}{\partial x_1} - \frac{\partial \psi_1^e}{\partial x_3}, u_3^e = \frac{\partial \phi_1^e}{\partial x_3} + \frac{\partial \psi_1^e}{\partial x_1}, \quad (25)$$

where  $\phi_1^e$  and  $\psi_1^e$  satisfy the wave equations

$$\nabla^2 \phi_1^e = \frac{1}{\alpha^e} \ddot{\phi}_1^e, \quad (26)$$

$$\nabla^2 \psi_1^e = \frac{1}{\beta^e} \ddot{\psi}_1^e, \quad (27)$$

$$\text{where } \alpha = \frac{\alpha^e}{C_0}, \beta = \frac{\beta^e}{C_0}, \alpha^e = \sqrt{\frac{\lambda^e + 2\mu^e}{\rho^e}}, \beta^e = \frac{\mu^e}{\rho^e}. \quad (28)$$

The stress-strain relation in an isotropic elastic medium is given by

$$\sigma_{ij}^e = 2\mu^e e_{ij}^e + \lambda^e e_{kk}^e \delta_{ij}^e, \quad (29)$$

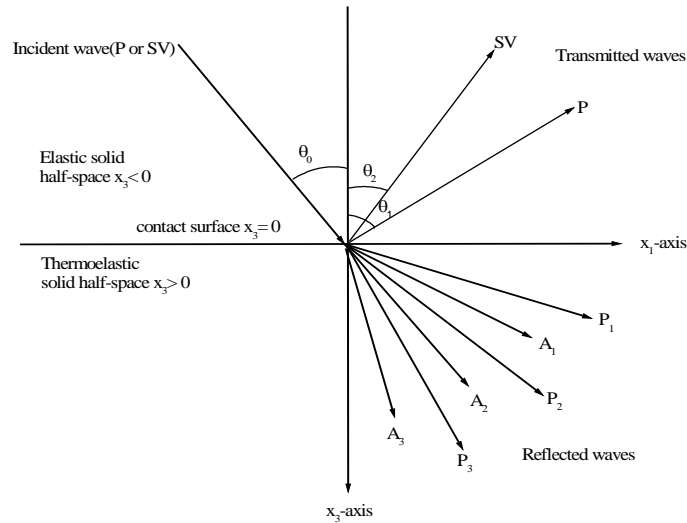
where

$$e_{ij}^e = \frac{1}{2}(u_{i,j}^e + u_{j,i}^e),$$

$e_{ii}^e$  is the dilatation.

#### 4. Reflection and Transmission

We consider a harmonic wave (P or SV) propagating through the isotropic elastic solid half-space and is incident at the interface  $x_3 = 0$ . Corresponding to this incident wave, two homogeneous waves (P and SV) are reflected in isotropic elastic solid half-space and three inhomogeneous waves (P, T, and SV) are refracted in isotropic fractional-order thermoelastic solid half-space as shown in Fig. 1.



**Fig. 1.** Geometry of the problem

In elastic solid half-space, the potential functions satisfying equations (26), (27) can be written as

$$\phi_j^e = A_0^e e^{[i\omega(x_1 \sin \theta_0 + x_3 \cos \theta_0)/\alpha - t]} + A_1^e e^{[i\omega(x_1 \sin \theta_1 + x_3 \cos \theta_1)/\alpha - t]}, \quad (30)$$

$$\psi_1^e = B_0^e e^{[i\omega(x_1 \sin \theta_0 + x_3 \cos \theta_0)/\beta - t]} + B_1^e e^{[i\omega(x_1 \sin \theta_2 + x_3 \cos \theta_2)/\beta - t]}. \quad (31)$$

The coefficients  $A_0^e$  ( $B_0^e$ ),  $A_1^e$  and  $B_1^e$  are amplitudes of the incident P (or SV), reflected P, and reflected SV waves respectively.

Following Borchardt [39], in isotropic fractional-order thermoelastic half-space with two temperature, the potential functions satisfying equations (19) and (21) can be written as

$$\{f_1, f\} = \sum_{i=1}^2 \{I, n_i\} B_i e^{(\bar{A}_i \bar{r})} e^{i(\bar{P}_i \bar{r} - \omega t)}, \quad (32)$$

$$\psi_1 = B_3 e^{(\bar{A}_3 \bar{r})} e^{i(\bar{P}_3 \bar{r} - \omega t)}. \quad (33)$$

The coefficients  $B_i$ ,  $i = 1, 2, 3$  are the amplitudes of refracted P, T, and SV waves respectively. The propagation vector  $\bar{P}_i$ ,  $i = 1, 2, 3$  and attenuation factor  $\bar{A}_i$ ,  $i = 1, 2, 3$  are given by

$$\bar{P}_i = \zeta_R \hat{x}_1 + dV_{iR} \hat{x}_3, \bar{A}_i = -\zeta_I \hat{x}_1 - dV_{iI} \hat{x}_3, i=1,2,3, \quad (34)$$

where

$$dV_i = dV_{iR} + idV_{iI} = p.v. \left( \frac{\omega^2}{V_i^2} - \zeta^2 \right), i=1,2,3. \quad (35)$$

and  $\zeta = \zeta_R + i\zeta_I$  is the complex wave number. The subscripts R and I denote the real and imaginary parts of the corresponding complex number and *p.v.* stands for the principal value of the complex quantity derived from the square root.  $\zeta_R \geq 0$  ensures propagation in positive  $x_1$ -direction. The complex wave number  $\zeta$  in the isotropic fractional-order thermoelastic medium is given by

$$\zeta = |\bar{P}_i| \sin \theta'_i - i|\bar{A}_i| \sin(\theta'_i - \gamma_i), i = 1, 2, 3, \quad (36)$$

where  $\gamma_i$ ,  $i=1, 2, 3$  is the angle between the propagation and attenuation vector and  $\theta'_i$ ,  $i=1, 2, 3$  is the angle of transmission in medium II.

## 5. Boundary conditions

The boundary conditions to be satisfied at the interface  $x_3=0$  are as follows

a) Stress conditions

$$\sigma_{33}^e = \sigma_{33}, \quad (37)$$

$$\sigma_{31}^e = \sigma_{31}, \quad (38)$$

b) Displacement conditions

$$u_1^e = u_1, \quad (39)$$

$$u_3^e = u_3, \quad (40)$$

c) Thermal condition

$$\frac{\partial T}{\partial x_3} + hT = 0, \quad (41)$$

where  $h$  is the heat transfer coefficient.

$h \rightarrow 0$  corresponds to the insulated boundary and  $h \rightarrow \infty$  corresponds to the isothermal boundary. Making use of potentials given by equations (30)-(33), we find that the boundary conditions are satisfied if and only if

$$\zeta_R = \frac{\omega \sin \theta_0}{V_0} = \frac{\omega \sin \theta_1}{\alpha} = \frac{\omega \sin \theta_2}{\beta}, \quad (42)$$

where

$$V_0 = \begin{cases} \alpha, & \text{for incident P-wave} \\ \beta, & \text{for incident SV-wave} \end{cases} \quad (43)$$

and  $\zeta_i = 0$ .

It means that waves are attenuating only in  $x_3$ -direction. From equation (36), it implies that if  $|\bar{A}_i| \neq 0$ , then  $\gamma_i = \theta'_i$ ,  $i=1, 2, 3$ , that is, attenuated vectors for the four refracted waves are directed along the  $x_3$ -axis.

Using equations (30)-(33) in the boundary conditions (37)-(41) with the aid of equations (12), (25), (42), (43), we get a system of five non-homogeneous equations

$$\sum_{j=1}^5 d_{ij} Z_j = g_i, \quad (44)$$

where  $Z_j$ ,  $j=1, 2, 3, 4, 5$  are the ratios of amplitudes of reflected P-, reflected SV-, refracted P-, refracted T- and refracted SV-waves to that of the amplitude of the incident wave.

$$d_{11} = 2\mu^e \left( \frac{\zeta_R}{\omega} \right)^2 - \rho^e c_0^2, d_{12} = 2\mu^e \frac{\zeta_R}{\omega} \frac{dV_\beta}{\omega}, d_{15} = 2\mu \frac{\zeta_R}{\omega} \frac{dV_3}{\omega}, d_{21} = 2\mu^e \frac{\zeta_R}{\omega} \frac{dV_\alpha}{\omega},$$

$$d_{22} = \mu^e \left[ \left( \frac{dV_\beta}{\omega} \right)^2 - \left( \frac{\zeta_R}{\omega} \right)^2 \right], d_{25} = \mu \left[ \left( \frac{\zeta_R}{\omega} \right)^2 - \left( \frac{dV_3}{\omega} \right)^2 \right], d_{31} = \frac{\zeta_R}{\omega}, d_{32} = \frac{dV_\beta}{\omega},$$

$$d_{35} = \frac{dV_3}{\omega}, d_{41} = -\frac{dV_\alpha}{\omega}, d_{42} = \frac{\zeta_R}{\omega}, d_{45} = -\frac{\zeta_R}{\omega}, d_{51} = d_{52} = d_{55} = 0,$$

$$d_{1j} = \lambda \left( \frac{\zeta_R}{\omega} \right)^2 + \rho c_0^2 \left( \frac{dV_i}{\omega} \right)^2 + \rho c_0^2 \frac{(1 + \beta_0 \zeta^2 + \beta_0 (dV_j)^2)}{\gamma \omega^2} n_j, d_{2j} = 2\mu \frac{\zeta_R}{\omega} \frac{dV_j}{\omega},$$

$$d_{3j} = -\frac{\zeta_R}{\omega}, d_{4j} = -\frac{dV_j}{\omega},$$

$$d_{5j} = n_j \left( \frac{idV_j}{\omega} + \frac{h}{\omega} \right) \left( \frac{1}{\omega} + \omega \beta_0^2 \left( \frac{\zeta}{\omega} \right)^2 \right), j = 3, 4.$$

$$\frac{dV_\alpha}{\omega} = \left( \frac{1}{\alpha^2} - \left( \frac{\zeta}{\omega} \right)^2 \right)^{\frac{1}{2}} = \left( \frac{1}{\alpha^2} - \frac{\sin^2 \theta_0}{V_0^2} \right)^{\frac{1}{2}}, \frac{dV_\beta}{\omega} = \left( \frac{1}{\beta^2} - \frac{\sin^2 \theta_0}{V_0^2} \right)^{\frac{1}{2}},$$

and

$$\frac{dV_j}{\omega} = p.v. \left( \frac{1}{V_j^2} - \frac{\sin^2 \theta_0}{V_0^2} \right)^{\frac{1}{2}}, \quad j = 1, 2, 3.$$

Here *p.v.* is calculated with restriction  $dV_{jt} \geq 0$  to satisfy decay condition in fractional order thermoelastic medium. The coefficients  $g_i, i=1, 2, 3$  on the right side of the equation (44) are given by

$$(a) \text{ For incident P-wave } g_i = (-1)^i d_{ij} \quad (i = 1, 2, 3, 4, j = 1), \quad g_5 = 0. \quad (45)$$

$$(b) \text{ For incident SV-wave } g_i = (-1)^{i+1} d_{ij} \quad (i = 1, 2, 3, 4, j = 2), \quad g_5 = 0. \quad (46)$$

Now we consider a surface element of the unit area at the interface between two media. The reason for this consideration is to calculate the partition of the energy of the incident wave among the reflected and refracted waves on both sides of the surface. Following Achenbach [40], the energy flux across the surface element, that is, the rate at which the energy is communicated per unit area of the surface is represented as

$$P^* = \sigma_{im} l_m \dot{u}_i, \quad (47)$$

where  $\sigma_{im}$  is the stress tensor,  $l_m$  are the direction cosines of the unit normal  $\hat{l}$  outward to the surface element, and  $\dot{u}_i$  are the components of the particle velocity.



The time average of  $P^*$  over a period, denoted by  $\langle P^* \rangle$ , represents the average energy transmission per unit surface area per unit time. Thus, on the surface with normal along  $x_3$ -direction, the average energy intensities of the waves in the elastic solid are given by

$$\langle P^{*e} \rangle = Re \langle \sigma \rangle_{13}^e \cdot Re \langle \dot{u}_1^e \rangle + Re \langle \sigma \rangle_{33}^e \cdot Re \langle \dot{u}_3^e \rangle. \quad (48)$$

Following Achenbach [40], for any two complex functions  $f$  and  $g$ , we have

$$\langle Re(f) \cdot Re(g) \rangle = \frac{1}{2} Re(f \cdot \bar{g}). \quad (49)$$

The expressions for energy ratios  $E_i$ ,  $i = 1, 2, 3$  for the reflected P- and reflected SV- are given by

$$E_i = -\frac{\langle P_i^{*e} \rangle}{\langle P_0^{*e} \rangle}, \quad i = 1, 2, \quad (50)$$

where

$$\langle P_1^{*e} \rangle = \frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\alpha} |A_1^e|^2 Re(\cos\theta_1), \langle P_2^{*e} \rangle = \frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\beta} |B_1^e|^2 Re(\cos\theta_2), \quad (51)$$

and

(a) For incident P-wave

$$\langle P_0^{*e} \rangle = -\frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\alpha} |A_0^e| \cos\theta_0, \quad (52)$$

(b) For incident SV-wave

$$\langle P_0^{*e} \rangle = -\frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\beta} |B_0^e| \cos\theta_0, \quad (53)$$

are the average energy intensities of the reflected P-, reflected SV-, incident P-, and incident SV-waves respectively. In equation (50), the negative sign is taken because the direction of reflected waves is opposite to that of the incident wave.

For fractional-order thermoelastic solid with two temperature, the average intensities of the waves on the surface with normal along  $x_3$ -direction, are given by

$$\langle P_{ij}^* \rangle = Re \langle \sigma \rangle_{13}^{(i)} \cdot Re \langle \dot{u}_1^{(j)} \rangle + Re \langle \sigma \rangle_{33}^{(i)} \cdot Re \langle \dot{u}_3^{(j)} \rangle. \quad (54)$$

The expressions for energy ratios for the reflected P-,reflected T- and reflected SV-waves are given by

$$E_{ij} = \frac{\langle P_{ij}^* \rangle}{\langle P_0^{*e} \rangle}, \quad i, j = 1, 2, 3, \quad (55)$$

where

$$\begin{aligned} \langle P_{ij}^* \rangle &= -\frac{\omega^4}{2} Re \left[ \left\{ \begin{aligned} &2\mu \frac{dV_i}{\omega} \frac{\xi_R}{\omega} \frac{\bar{\xi}_R}{\omega} + \lambda \left( \frac{\xi_R}{\omega} \right)^2 \left( \frac{d\bar{V}_j}{\omega} \right) + \rho c_0^2 \left( \frac{dV_i}{\omega} \right)^2 \left( \frac{d\bar{V}_j}{\omega} \right) \\ &+ \frac{\rho C_0^2}{\gamma \omega^2} \left( 1 + \beta_0 \left( \frac{\xi_R}{\omega} \right)^2 \omega^2 + \beta_0 \omega^2 \left( \frac{dV_i}{\omega} \right)^2 \right) n_j \left( \frac{d\bar{V}_j}{\omega} \right) \end{aligned} \right\} B_i \bar{B}_j \right], \\ \langle P_{i3}^* \rangle &= -\frac{\omega^4}{2} Re \left[ \left\{ \begin{aligned} &-2\mu \frac{dV_i}{\omega} \frac{d\bar{V}_3}{\omega} \frac{\xi_R}{\omega} + \lambda \left( \frac{\xi_R}{\omega} \right)^2 \left( \frac{\bar{\xi}_R}{\omega} \right) + \rho c_0^2 \left( \frac{dV_i}{\omega} \right)^2 \left( \frac{\bar{\xi}_R}{\omega} \right) \\ &+ \frac{\rho C_0^2}{\gamma \omega^2} \left( 1 + \beta_0 \left( \frac{\bar{\xi}_R}{\omega} \right)^2 + \beta_0 \left( \frac{dV_i}{\omega} \right)^2 \right) \left( \frac{\bar{\xi}_R}{\omega} \right) n_i \end{aligned} \right\} B_i \bar{B}_3 \right], \\ \langle P_{3j}^* \rangle &= -\frac{\omega^4}{2} Re \left[ \left\{ \begin{aligned} &\mu \left( \left( \frac{\xi_R}{\omega} \right)^2 \frac{\bar{\xi}_R}{\omega} - \frac{\bar{\xi}_R}{\omega} \left( \frac{dV_3}{\omega} \right)^2 \right) - \lambda \frac{\xi_R}{\omega} \frac{dV_3}{\omega} \frac{d\bar{V}_j}{\omega} + \rho c_0^2 \frac{\xi_R}{\omega} \frac{dV_3}{\omega} \frac{d\bar{V}_j}{\omega} \end{aligned} \right\} \bar{B}_j B_3 \right], \\ \langle P_{33}^* \rangle &= -\frac{\omega^4}{2} Re \left[ \left\{ \begin{aligned} &\mu \left( \left( \frac{dV_3}{\omega} \right)^2 - \left( \frac{\xi_R}{\omega} \right)^2 \right) \frac{d\bar{V}_3}{\omega} - 2\mu \frac{\xi_R}{\omega} \frac{\bar{\xi}_R}{\omega} \frac{dV_3}{\omega} \end{aligned} \right\} B_3 \bar{B}_3 \right], \quad i, j = 1, 2. \end{aligned}$$

The diagonal entries of the energy matrix  $E_{ij}$  in equation (61) represents the energy ratios of P, T, SV waves respectively, whereas sum of the non-diagonal entries of  $E_{ij}$  gives the share of interaction energy among all refracted waves in the medium and is given by

$$E_{RR} = \sum_{i=1}^3 \left( \sum_{j=1}^3 E_{ij} - E_{ii} \right). \quad (56)$$

The energy ratios  $E_i$ ,  $i = 1, 2$ , diagonal entries and non-diagonal entries of the energy matrix  $E_{ij}$ , that is,  $E_{11}, E_{22}, E_{33}$  and  $E_{RR}$  yield the conservation of incident energy across the interface, through the relation

$$E_1 + E_2 + E_{11} + E_{22} + E_{33} + E_{RR} = 1. \quad (57)$$

## 6. Numerical results and discussion

We now represent some numerical results following Sherief and Saleh [41], the physical data for which is given below:

$$\lambda = 7.76 \times 10^{10} \text{ Kgm}^{-1}\text{s}^{-2}, \mu = 3.86 \times 10^{10} \text{ Kgm}^{-1}\text{s}^{-2}, T_0 = 0.293 \times 10^3 \text{ K},$$

$$C_E = .3831 \times 10^3 \text{ JKg}^{-1}\text{K}^{-1}, \alpha_i = 1.78 \times 10^{-5} \text{ K}^{-1}$$

$$h = 0, \rho = 8.954 \times 10^3 \text{ Kgm}^{-3}, K = 0.383 \times 10^3 \text{ Wm}^{-1}\text{K}^{-1}.$$

Following Bullen [42], the numerical data of granite in an elastic medium is given by

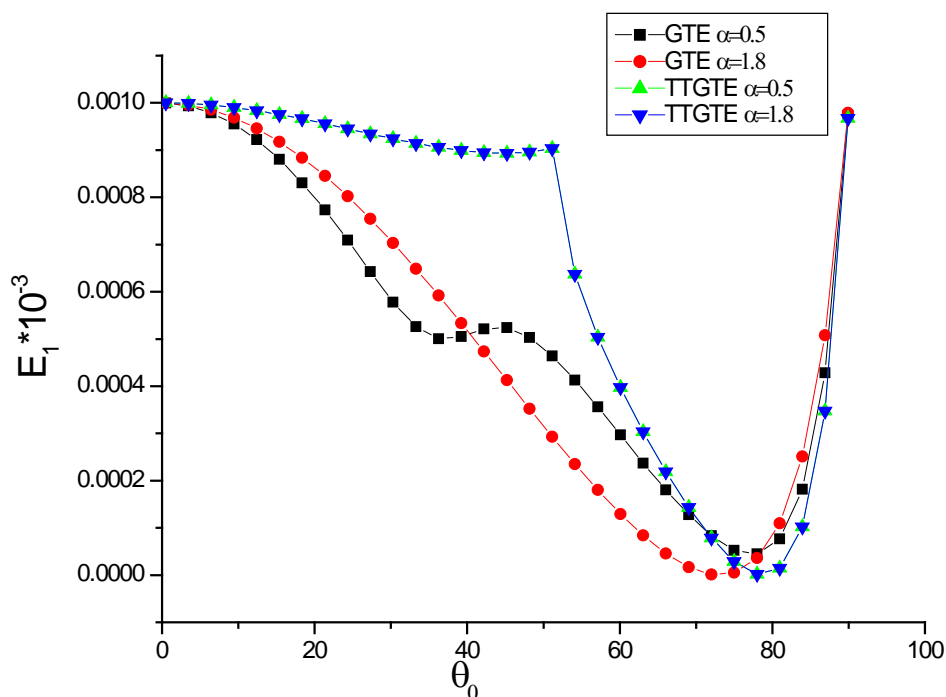
$$\rho^e = 2.65 \times 10^3 \text{ Kgm}^{-3}, \alpha^e = 5.27 \times 10^3 \text{ ms}^{-1}, \beta^e = 3.17 \times 10^3 \text{ ms}^{-1}.$$

The software Matlab 7.0.4 has been used to determine the values of energy ratios  $E_i, i = 1, 2$  and an energy matrix  $E_{ij}, i, j = 1, 2, 3$  defined in the previous section for different values of incident angle ( $\theta_0$ ) ranging from  $0$  to  $89^\circ$  for fixed frequency  $\omega = 2 \times \pi \times 100 \text{ Hz}$ . corresponding to incident P and SV waves, the variation of these energy ratios with respect to the angle of incidence has been plotted in Figs. 2-7 and Figs. 8-13 respectively. In all the figures, the slant and squares correspond to Generalized Thermoelasticity(GTE) and Two Temperature Generalized Thermoelasticity(TTGTE) theories for  $\alpha = 0.5$  whereas horizontal lines and dots correspond to Generalized Thermoelasticity(GTE) and Two Temperature Generalized Thermoelasticity(TTGTE) theories for  $\alpha = 1.8$ .

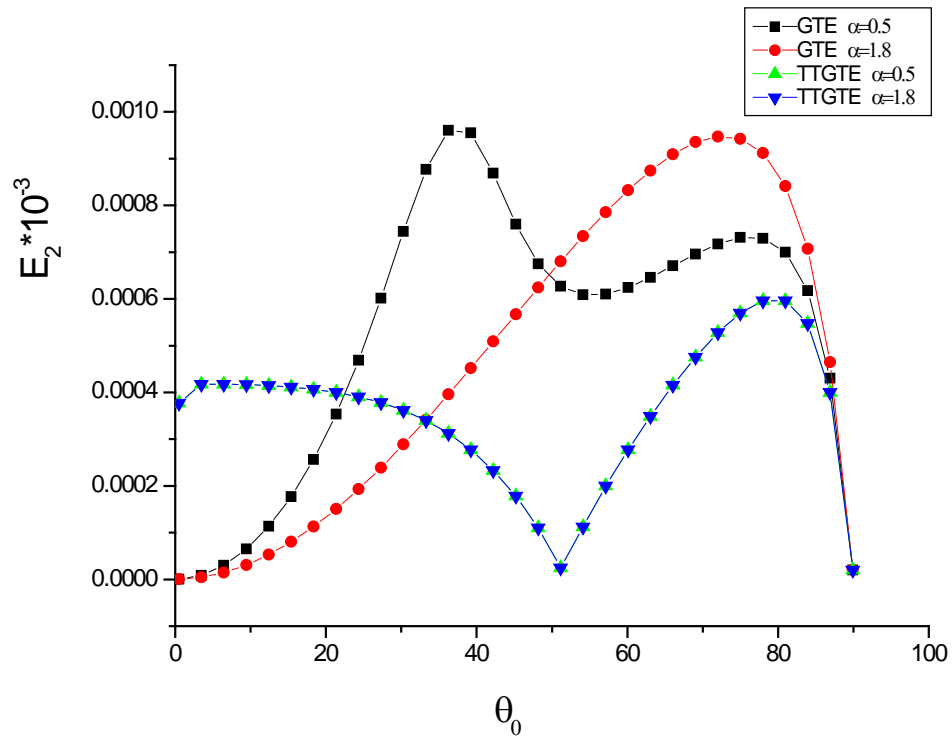
**Incident P wave.** It is clear from Fig. 2 that the values of energy ratio  $E_1$  first decrease from  $0$  to  $80^\circ$  and increases rapidly in  $80^\circ \leq \theta_0 \leq 90^\circ$  for both GTE and TTGTE theory and both fractional orders. Figure 3 shows that for GTE and both fractional orders  $E_2$  increases in the range  $0 \leq \theta_0 \leq 60^\circ$  and then decreases for  $60^\circ \leq \theta_0 \leq 90^\circ$ . Also for TTGTE,  $E_2$  attains minimum at  $\theta_0 = 50^\circ$ . Figure 4 indicates that for  $\alpha = 1.8$  (GTE and TTGTE) and  $\alpha = 0.5$  (TTGTE), values of  $E_{11}$  are nearly equal to zero whereas for  $\alpha = 0.5$  (GTE),  $E_{11}$  increases for  $0 \leq \theta_0 \leq 50^\circ$  and decreases for  $50^\circ \leq \theta_0 \leq 90^\circ$ . Figure 5 depicts that  $E_{22}$  for  $\alpha = 1.8$  (TTGTE) increases smoothly in the range  $0 \leq \theta_0 \leq 20^\circ$  and then decreases for  $20^\circ \leq \theta_0 \leq 90^\circ$ . For  $\alpha = 0.5, 1.8$  (GTE),  $E_{22}$  attains value nearly equal to 3 in the range  $0 \leq \theta_0 \leq 60^\circ$  and decreases rapidly and attains a minimum in the range  $60^\circ \leq \theta_0 \leq 90^\circ$ . Also for  $\alpha = 0.5$  (TTGTE), it attains the minimum value. Figure 6 depicts the same behavior and variation for  $E_{33}$  as  $E_2$  with difference in magnitude values for both the theories (GTE and TTGTE) and both fractional orders. Magnitude values in case of  $E_2$  are higher than  $E_{33}$ . Figure 7 shows that values of  $E_{RR}$  are minimum for  $\alpha = 0.5$  (TTGTE) than other theories. It is noticed that the sum of the values of energy ratios  $E_1, E_2, E_{11}, E_{22}, E_{33}$  and  $E_{RR}$  is found to be exactly unity at each value of  $\theta$ . which proves the law of conservation of energy at the interface.

**Incident SV wave.** From Figure 8, it is evident that for both fractional orders  $E_1$  increases smoothly in the range  $0 \leq \theta_0 \leq 40^\circ$  and then rapidly decreases at  $\theta_0 = 40^\circ$  and attains

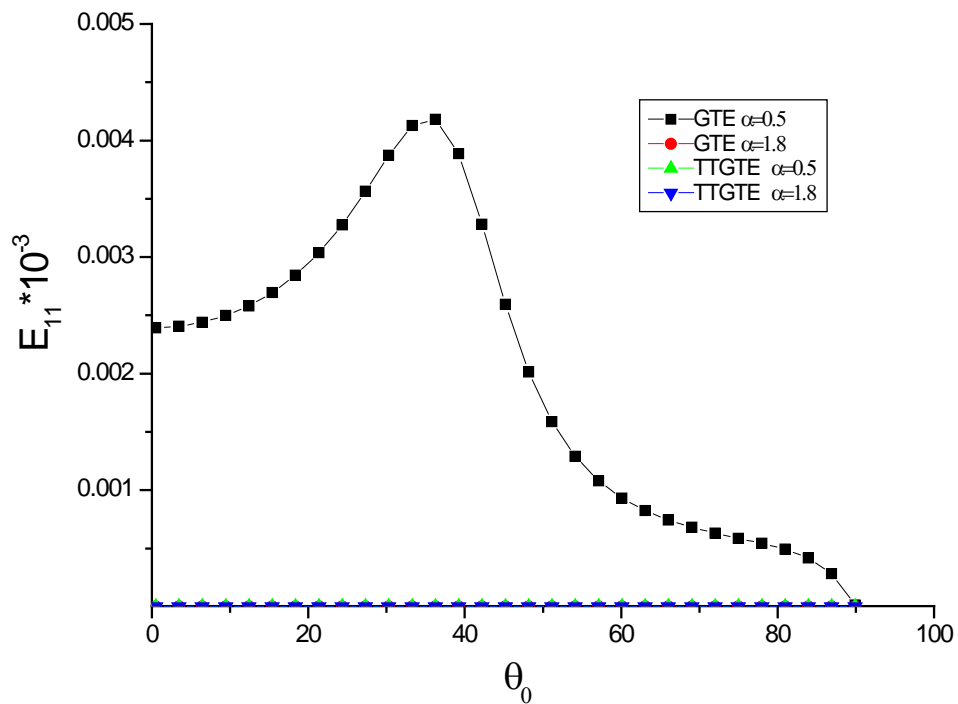
the minimum value elsewhere for both GTE and TTGTE theories. Figure 9 shows that  $E_2$  first decreases in the range  $0 \leq \theta_0 \leq 30^\circ$ , fluctuates in  $30^\circ \leq \theta_0 \leq 50^\circ$  and increases smoothly for  $50^\circ \leq \theta_0 \leq 90^\circ$ . From Fig. 10 and Fig. 11, it is noticed that  $E_{11}$  and  $E_{22}$  depict the same behavior and variation for both theories and fractional orders as Fig. 4 shows for  $E_{11}$ , but magnitude values of  $E_{22}$  are higher than  $E_{11}$ . Figure 12 indicates that  $E_{33}$  decreases for  $0 \leq \theta_0 \leq 40^\circ$  attains the minimum value at  $\theta_0 = 40^\circ$ , again increases and finally decreases to 0. Figure 13 indicates that for  $\alpha = 0.5$  (GTE) and  $\alpha = 0.5, 1.8$  (TTGTE),  $E_{RR}$  attains minimum value nearly to 0, while for  $\alpha = 1.8$  (GTE),  $E_{RR}$  increases for  $0 \leq \theta_0 \leq 30^\circ$  then decreases rapidly and attains minimum value elsewhere. In all the figures a very small change occurs for  $\alpha = 0.5, 1.8$  (TTGTE). Therefore the curves corresponding to  $\alpha = 0.5, 1.8$  (TTGTE) coincide.



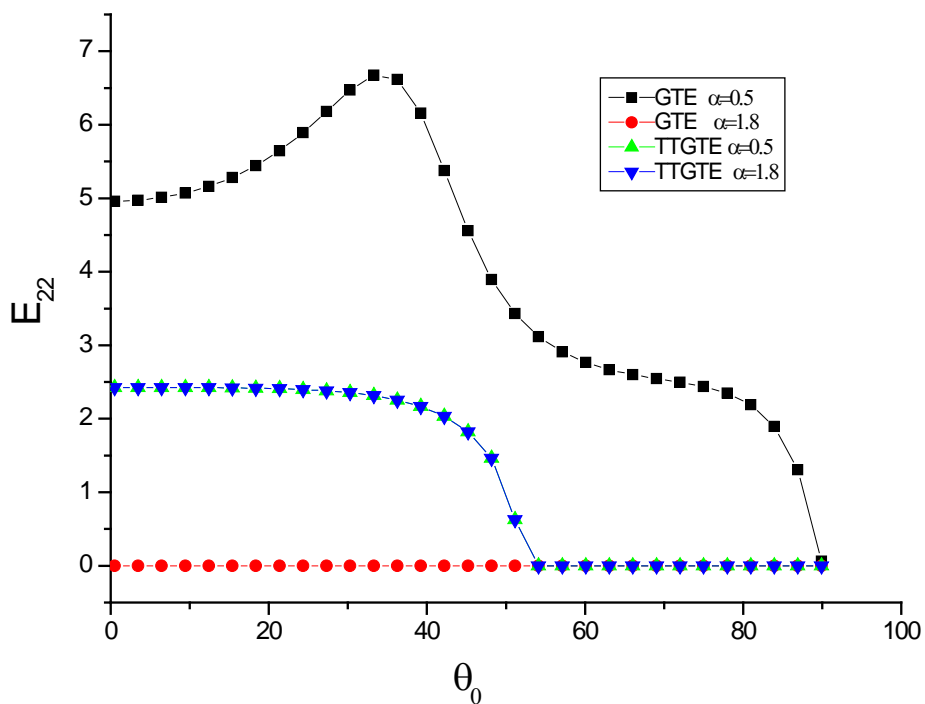
**Fig. 2.** Variation of energy ratio  $E_1$  w.r.t. angle of incidence  $\theta_0$  for P wave



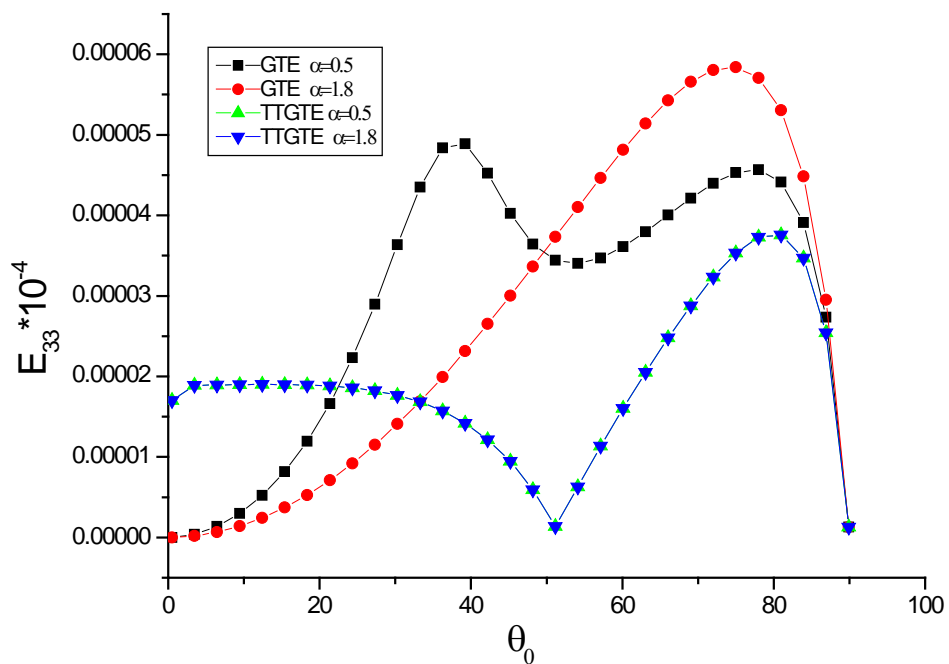
**Fig. 3.** Variation of energy ratio  $E_2$  w.r.t. angle of incidence  $\theta_0$  for P wave



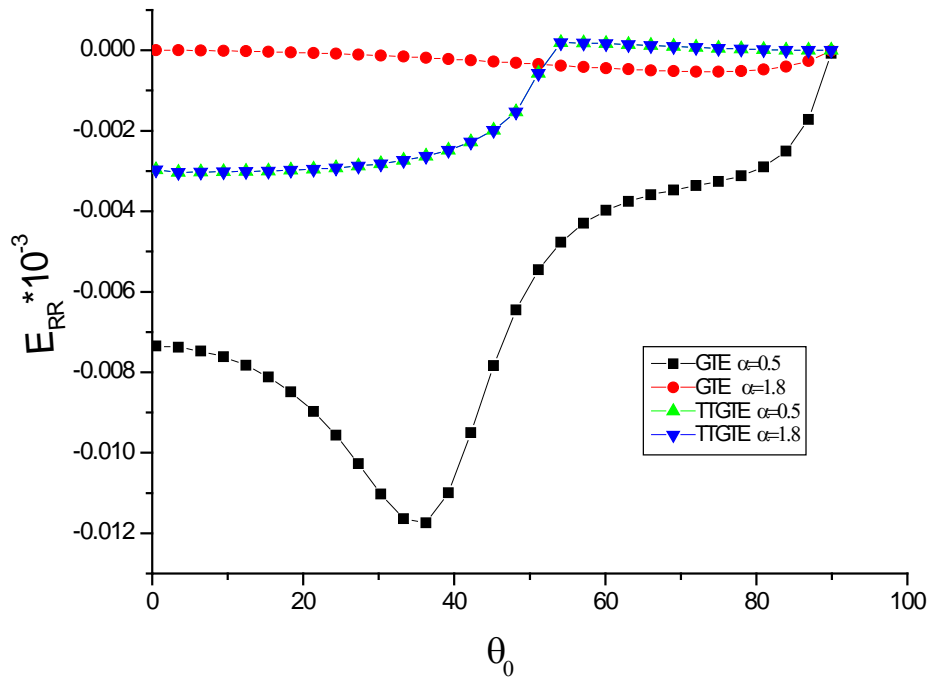
**Fig. 4.** Variation of energy ratio  $E_{11}$  w.r.t. angle of incidence  $\theta_0$  for P wave



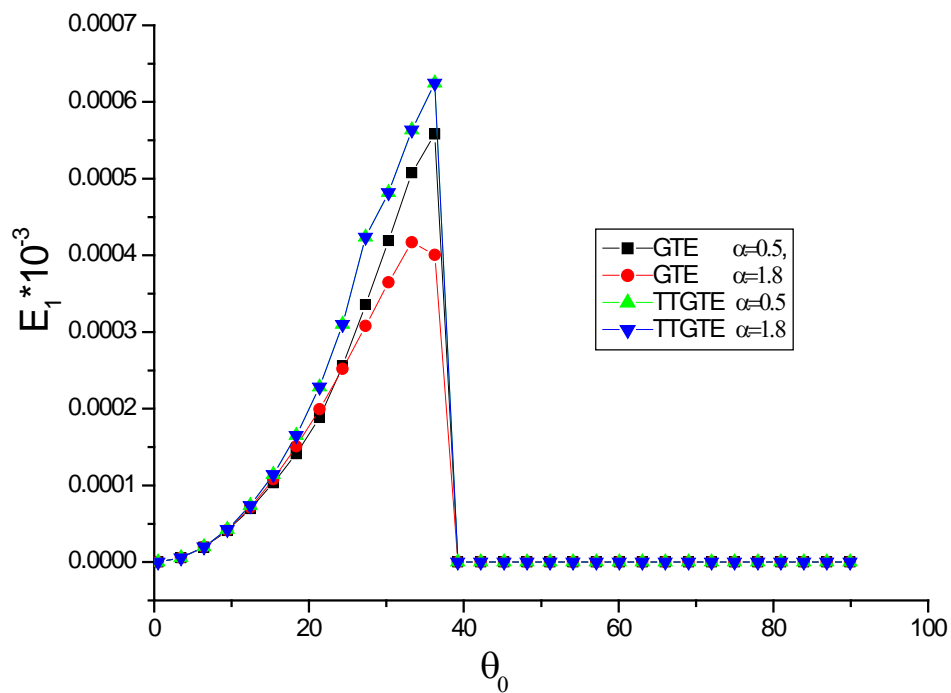
**Fig. 5.** Variation of energy ratio  $E_{22}$  w.r.t. angle of incidence  $\theta_0$  for P wave



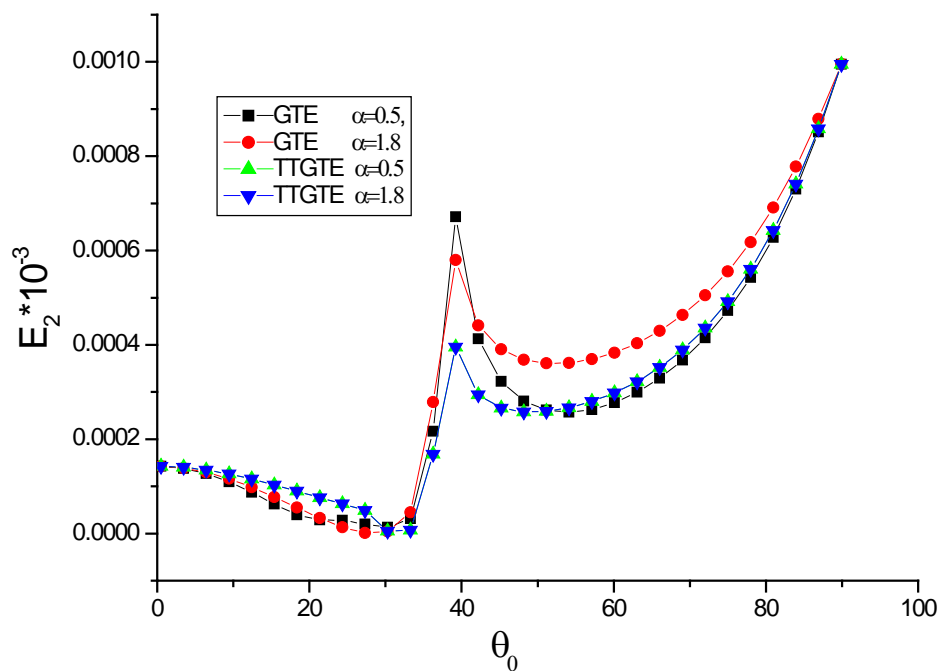
**Fig. 6.** Variation of energy ratio  $E_{33}$  w.r.t. angle of incidence  $\theta_0$  for P wave



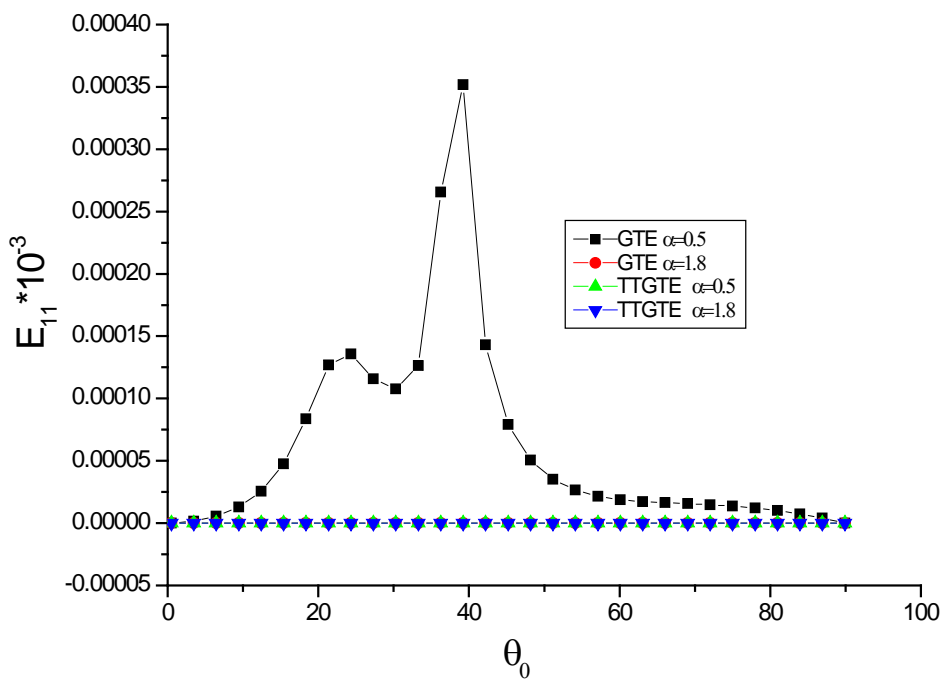
**Fig. 7.** Variation of energy ratio  $E_{RR}$  w.r.t. angle of incidence  $\theta_0$  for P wave



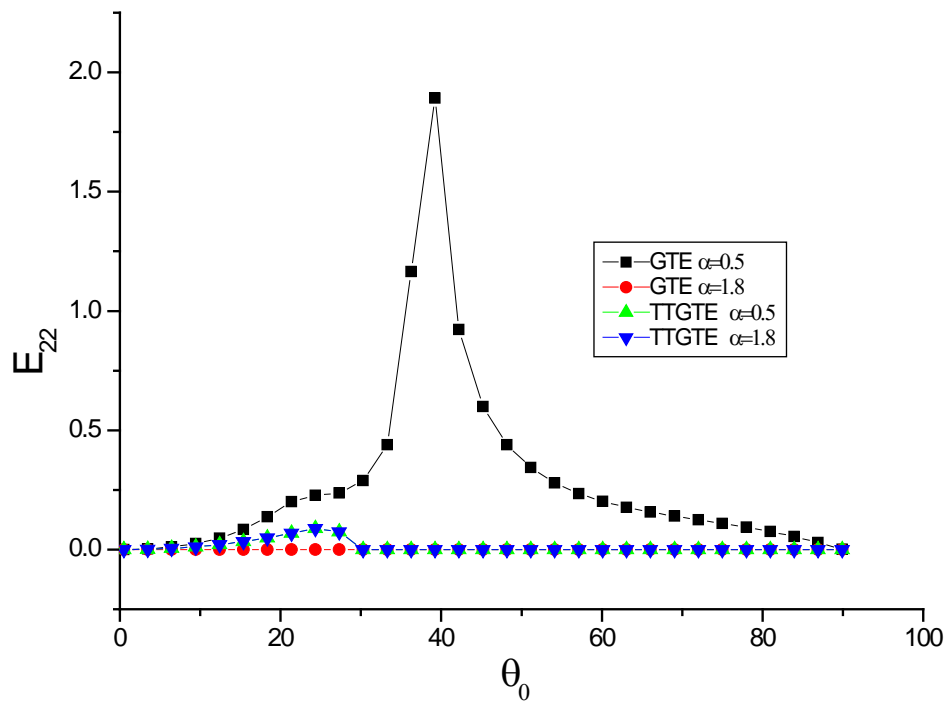
**Fig. 8.** Variation of energy ratio  $E_1$  w.r.t. angle of incidence  $\theta_0$  for SV wave



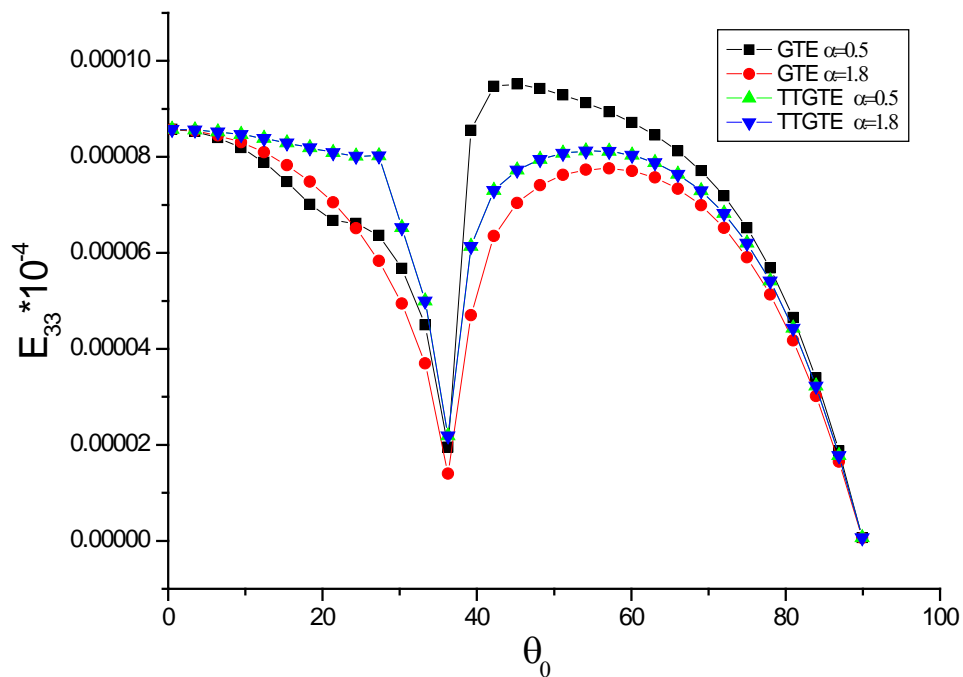
**Fig. 9.** Variation of energy ratio  $E_2$  w.r.t. angle of incidence  $\theta_0$  for SV wave



**Fig. 10.** Variation of energy ratio  $E_{11}$  w.r.t. angle of incidence  $\theta_0$  for SV wave

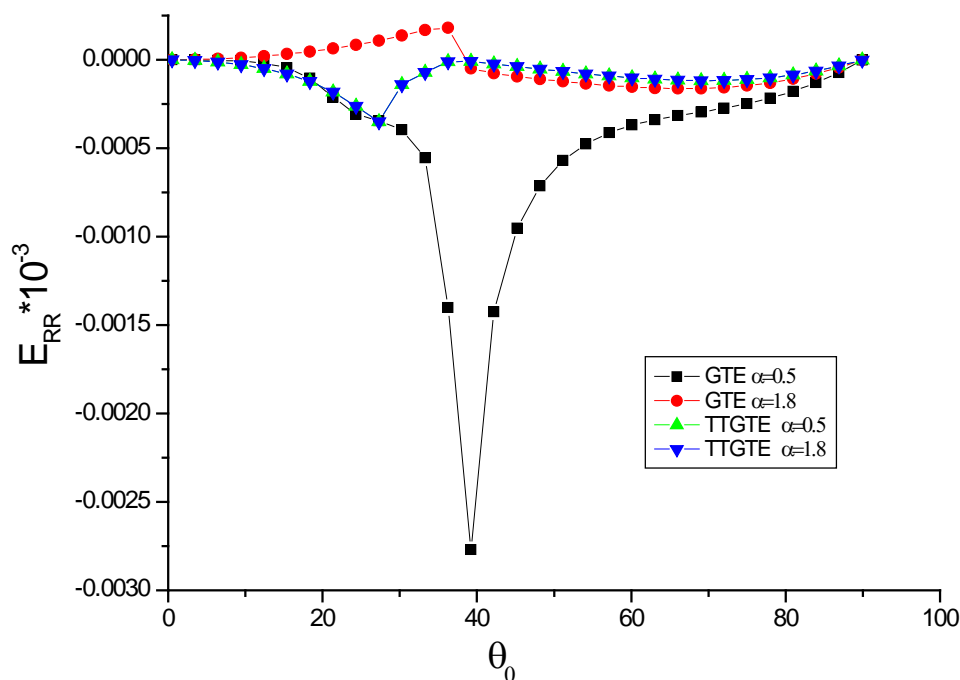


**Fig. 11.** Variation of energy ratio  $E_{22}$  w.r.t. angle of incidence  $\theta_0$  for SV wave



**Fig. 12.** Variation of energy ratio  $E_{33}$  w.r.t. angle of incidence  $\theta_0$  for SV wave





**Fig. 13.** Variation of energy ratio  $E_{RR}$  w.r.t. angle of incidence  $\theta_0$  for SV wave

## 7. Conclusion

In the present article, the phenomenon of reflection and transmission of obliquely incident elastic wave at the interface between an elastic solid half-space and two-temperature generalized thermoelastic solid half-space with fractional order derivative has been studied. The three waves in thermoelastic solid medium are identified and explained through different wave equations in terms of displacement potentials. The energy ratios of different reflected and refracted waves to that of incident wave are computed numerically and presented graphically with respect to the angle of incidence for fractional order  $\alpha = 0.5$  and  $\alpha = 1.8$  for both GTE and TTGTE theory. For fractional order  $\alpha = 0.5$ , the values of energy ratios are higher than  $\alpha = 1.8$ . From numerical results, we conclude that the effect of angle of incidence and fractional order and two-temperature theory on the energy ratios of the reflected and refracted waves are significant. The sum of all energy ratios of the reflected waves, refracted waves, and interference between refracted waves is verified to be always unity which ensures the law of conservation of incident energy at the interface.

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