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Maxwell electrodynamics in media, geometry effect on constitutive relations

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Abstract. The problem of constitutive relations in Maxwell electrodynamics, their possible form and role in physical manifestation of electromagnetic fields, its behavior under the motion of the reference frame and its connection with Special Relativity theory, interplay between electrodynamics constitutive relations and gravity theory, and so on, has a long history. The main accent in our treatment is the known possibility to simulate material media by geometrical methods. This review includes the following items: Riemannian geometry and Maxwell theory; Maxwell equations in Riemannian space and effective media; metrical tensor $g_{\alpha\beta}(x)$ and constitutive relations; inverse constitutive equations; geometric simulation of inhomogeneous media; geometrical simulation of uniform media; geometrical modeling of anisotropic uniform media; the moving medium and anisotropy; geometry effect on material equations in arbitrary linear media; the plane wave in the Lobachevsky space, simulating a special medium; arbitrary metrics, etc.
Keywords: constitutive relations, gravity theory, Lobachevsky space, Maxwell electrodynamics, plane waves, Riemannian geometry

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1. Introduction

We can track interest in the problem of geometry effect on the constitutive relations in Maxwell electrodynamics, for instance, see the references [1-30]. Note that Gordon [2] was first largely interested in trying to describe dielectric media by effective metrics. Gordon tried to use a gravitational field to simulate a dielectric medium. The idea was taken up and developed by Tamm and Mandelstam [3,4], and by many others.

Let us start with the Maxwell equations in Minkowski space for the uniform medium:

$$\mathrm{div}\mathbf{B} = 0, \quad \mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \epsilon\epsilon_0\,\mathrm{div}\mathbf{E} = \rho, \quad \frac{1}{\mu\mu_0}\mathrm{rot}\mathbf{B} = \mathbf{J} + \epsilon\epsilon_0\frac{\partial \mathbf{E}}{\partial t}. \tag{1}$$

Using the constitutive relations

$$\mathbf{H} = \frac{\mathbf{B}}{\mu\mu_0}, \quad \mathbf{D} = \epsilon\epsilon_0\,\mathbf{E} \tag{2}$$

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eqs. (1) can be written with the use of four vectors $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$:

$$\mathrm{div}\, c\mathbf{B} = 0, \quad \mathrm{rot}\,\mathbf{E} = -\frac{\partial c\mathbf{B}}{\partial x^0}, \quad \mathrm{div}\,\mathbf{D} = j^0, \quad \mathrm{rot}\frac{\mathbf{H}}{c} = \frac{\mathbf{J}}{c} + \frac{\partial \mathbf{D}}{\partial x^0} \;\; (x^0 = ct). \tag{3}$$

We represent the electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$ by the antisymmetric tensor H^{ik}, the electric $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ are accounted for by the tensor F^{ik}:

$$(F^{\alpha\beta}) = \left| \begin{array}{cccc} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{array} \right|, \quad (H^{\alpha\beta}) = \left| \begin{array}{cccc} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3/c & H^2/c \\ D^2 & H^3/c & 0 & -H^1/c \\ D^3 & -H^2/c & H^1/c & 0 \end{array} \right|;$$

here we apply the notations

$$E^i = -E_i, \; D^i = -D_i, \; B^i = +B_i, \; H^i = +H_i, \; j^a = (\rho, \mathbf{J}/c).$$

Then eqs. (3) can be presented in tensor form

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \quad \partial_b H^{ba} = j^a. \tag{4}$$

For the vacuum case, the constitutive relations $\mathbf{D} = \epsilon_0 \mathbf{E}, \; \mathbf{H} = \frac{1}{\mu_0}\mathbf{B},$ are read in the tensor form as follows

$$H^{ab}(x) = \epsilon_0 F^{ab}(x), \tag{5}$$

so eqs. (4) contain only one tensor

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \quad \partial_b F^{ba} = \frac{1}{\epsilon_0} j^a. \tag{6}$$

The situation is quite different in the presence of media. Even for the simplest case of the uniform medium, relativizing the above constitutive equations

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu_0 \mu}\mathbf{B}$$

requires the subsidiary (4×4)-matrix with the presumed properties of the 2-rank tensor:

$$\eta^{am} = \sqrt{\epsilon} \left| \begin{array}{cccc} 1/k & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -k \end{array} \right|, \quad k = \frac{1}{\sqrt{\epsilon\mu}}, \quad H^{ab} = \epsilon_0 \eta^{am} \eta^{bn} F_{mn}. \tag{7}$$

We may postulate the class of linear inhomogeneous electromagnetic media characterized by the 4-rank tensor [3,4]:

$$H^{ab}(x) = \epsilon_0 \Delta^{abmn}(x) F_{mn}(x), \tag{8}$$

where the relativistic symmetry presumes the evident symmetry constraints:

$$\Delta^{abmn}(x) = -\Delta^{bamn}(x) = -\Delta^{abnm}(x), \tag{9}$$

and, in general, the tensor $\Delta^{abmn}(x)$ depends upon 36 parameters.

When extending Maxwell's theory to space-time with non-Euclidean geometry, which describes gravity according to General Relativity, one must change previous equations to a more general form. In particular, the vacuum Maxwell equations are given as

$$\nabla_\alpha f_{\beta\gamma} + \nabla_\beta f_{\gamma\alpha} + \nabla_\gamma f_{\alpha\beta} = 0, \quad \nabla_\beta h^{\beta\alpha} = j^\alpha, \quad h_{\alpha\beta} = \epsilon_0 f_{\alpha\beta}, \tag{10}$$

where ∇_β stands for the covariant derivative. In order to distinguish formulas referring to flat and curved models we will use small letters for designating the electromagnetic tensors in curved model, f_{ab} and h^{ab}.

2. Maxwell equations in Riemannian space and effective media

Let us discuss the possibility to consider the vacuum Maxwell equations in a curved space-time as Maxwell equations in a flat space-time but specified for an effective medium, the properties of which are determined by the metrical structure of the initial curved model $g_{\alpha\beta}(x)$. We will restrict ourselves to the case of curved space-time models which are parameterized by some quasi-Cartesian coordinates.

Vacuum Maxwell equations in a Riemannian space-time, parameterized by some quasi-Cartesian coordinates, can be brought to the form [5]

$$\partial_a f_{bc} + \partial_b f_{ca} + \partial_c f_{ab} = 0, \quad \frac{1}{\sqrt{-g}}\partial_b \sqrt{-g}\, f^{ba} = \frac{1}{\epsilon_0} j^a. \qquad (11)$$

Indeed, one can immediately see that after introducing the new variables

$$\sqrt{-g}\, j^a \text{™}\ j^a, \quad f_{ab} \text{™}\ F_{ab}, \quad \epsilon_0 \sqrt{-g}\, g^{am}(x) g^{bn}(x) f_{mn}(x) \text{™}\ H^{ba}, \qquad (12)$$

eqs. (11) in the curved space can be re-written as Maxwell equations for the flat space but in the medium:

$$\partial_a F_{bc} + \partial_b F_{ca} + \partial_c F_{ab} = 0, \quad \partial_b H^{ba} = \frac{1}{\epsilon_0} j^a. \qquad (13)$$

Relations playing the role of constitutive equations are determined by the metrical structure of the geometrical model:

$$H^{\beta\alpha}(x) = \epsilon_0 \left[\sqrt{-g(x)}\, g^{\alpha\rho}(x) g^{\beta\sigma}(x) \right] F_{\rho\sigma}(x). \qquad (14)$$

There exists one special case; namely, if $g(x)$ does not depend on coordinates, then the factor $\sqrt{-g(x)} = \sqrt{-g}$ can be omitted from the above formulas.

3. Metrical tensor and constitutive relations

Let us consider the material equations for electromagnetic fields which are generated by the metrical structure of the space-time model. For an arbitrary metrical tensor $g_{\alpha\beta}(x)$ we may obtain a 3-dimensional form of relation (14):

$$D^i = \epsilon_0 \epsilon^{ik}(x) E_k + \epsilon_0 c \alpha^{ik}(x) B_k, \quad H^i = \epsilon_0 c \beta^{ik}(x) E_k + \mu_0^{-1} (\mu^{-1})^{ik}(x) B_k. \qquad (15)$$

Four dimensionless (3×3)-matrices $\epsilon^{ik}(x)$, $\alpha^{ik}(x)$, $\beta^{ik}(x)$, $(\mu^{-1})^{ik}(x)$ are not independent because they are bilinear functions of only 10 components of the symmetrical tensor $g_{\alpha\beta}(x)$.

After simple calculation, for these tensors one produces expressions

$$\epsilon^{ik}(x) = \sqrt{-g}\left(g^{00}(x) g^{ik}(x) - g^{0i}(x) g^{0k}(x) \right), \quad (\mu^{-1})^{ik}(x) = \frac{1}{2}\sqrt{-g}\, \epsilon_{imn} g^{ml}(x) g^{nj}(x) \epsilon_{ljk},$$

$$\alpha^{ik}(x) = +\sqrt{-g}\, g^{ij}(x) g^{0l}(x) \epsilon_{ljk}, \quad \beta^{ik}(x) = -\sqrt{-g}\, g^{0j}(x) \epsilon_{jil} g^{lk}(x).$$

The tensor $\epsilon^{ik}(x)$ is evidently symmetrical; it is easy to demonstrate the same property for $(\mu^{-1})^{ik}(x)$. Indeed, we have

$$(\mu^{-1})^{ki}(x) = \frac{1}{2}\epsilon_{kmn} g^{ml}(x) g^{nj}(x) \epsilon_{lji},$$

here making changes in mute indices, $m \leftrightarrow j, n \leftrightarrow l$, we get

$$(\mu^{-1})^{ki}(x) = \frac{1}{2}\epsilon_{kjl} g^{jn}(x) g^{lm}(x) \epsilon_{nmi} \epsilon_{imn} g^{lm}(x) g^{jn}(x) \epsilon_{ljk} = (\mu^{-1})^{ik}(x).$$

In the same manner, one can prove the identity $\beta^{ki}(x) = +\alpha^{ik}$:

$$\beta^{ki} = -g^{0j}(x) \epsilon_{jkl} g^{li}(x) = g^{il}(x) g^{0j}(x) \epsilon_{jlk} = +\alpha^{ik}.$$

So, the above tensors obey the following symmetry constraints:

$$\epsilon^{ik}(x) = +\epsilon^{ki}(x), \quad (\mu^{-1})^{ik}(x) = +(\mu^{-1})^{ki}(x), \quad \beta^{ki}(x) = \alpha^{ik};$$

that means the (6×6)-matrix defining constitutive equations is symmetrical

$$\begin{vmatrix} D^i(x) \\ H^i(x) \end{vmatrix} = \begin{vmatrix} \epsilon_0 \epsilon^{ik}(x) & \epsilon_0 c\alpha^{ik}(x) \\ \epsilon_0 c\beta^{ik}(x) & \mu_0^{-1}(\mu^{-1})^{ik}(x) \end{vmatrix} \begin{vmatrix} E_k(x) \\ B_k(x) \end{vmatrix}. \qquad (16)$$

Making $(3+1)$-splitting in the metrical tensor

$$g^{\alpha\beta}(x) = \begin{vmatrix} g^{00} & (g^{0i}) = \bar{g} \\ (g^{i0}) = \bar{g} & (g^{ik}) = g \end{vmatrix}, \quad (\bar{g}^{\times})_{jk} \equiv g^{0l}(x)\epsilon_{ljk} = g^l(x)\epsilon_{ljk}, \qquad (17)$$

tensors $(\epsilon^{ik}), (\alpha^{ik})$, (β^{ik}) can be written in the form

$$\epsilon(x) = \sqrt{-g}\,[g^{00}(x)g(x) - \bar{g}(x)\cdot\bar{g}(x)],$$

$$\alpha(x) = \sqrt{-g}\, g(x)\bar{g}^{\times}(x), \quad \beta(x) = -\sqrt{-g}\,\bar{g}^{\times}(x)g(x). \qquad (18)$$

In doing so, one can produce a more convenient representation for matrix $(\mu^{-1})^{ik}(x)$. Indeed, with the notation $(\tau_i)_{mn} = \epsilon_{imn}$, we get

$$(\mu^{-1})^{ik}(x) = -\frac{1}{2}\sqrt{-g}\ \mathrm{Sp}[\tau_i\, g(x)\tau_k\, g(x)]. \qquad (19)$$

The metrical tensors, which are the most interesting in the context of General relativity [5], have a quasi-diagonal structure, so the effective constitutive relations are simplified:

$$g^{\alpha\beta}(x) = \begin{vmatrix} g^{00} & 0 & 0 & 0 \\ 0 & g^{11} & g^{12} & g^{13} \\ 0 & g^{21} & g^{22} & g^{23} \\ 0 & g^{31} & g^{32} & g^{33} \end{vmatrix}, \quad \alpha(x) = 0, \quad \beta(x) = 0,$$

$$\epsilon(x) = \sqrt{-g}\, g^{00}(x)g(x), \quad (\mu^{ik})(x) = -\frac{1}{2}\sqrt{-g}\ \mathrm{Sp}[\tau_i\, g(x)\tau_k\, g(x)]. \qquad (20)$$

The explicit expressions for tensors $\epsilon^{ik}(x)$ and $(\mu^{-1})^{ik}(x)$ given by (20) are

$$(\epsilon^{ik}) = \sqrt{-g}\, g^{00} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix}, \quad ((\mu^{-1})^{ik}) = \sqrt{-g} \begin{vmatrix} G^{11} & G^{12} & G^{13} \\ G^{21} & G^{22} & G^{23} \\ G^{31} & G^{32} & G^{33} \end{vmatrix}, \qquad (21)$$

where $G^{ik}(x)$ stand for (algebraic) co-factors to the elements $g^{ik}(x)$:

$$G^{ik}(x) = \begin{vmatrix} (g^{22}g^{33} - g^{23}g^{32}) & (g^{31}g^{23} - g^{21}g^{33}) & (g^{21}g^{32} - g^{22}g^{31}) \\ (g^{32}g^{13} - g^{33}g^{12}) & (g^{33}g^{11} - g^{31}g^{13}) & (g^{31}g^{12} - g^{32}g^{11}) \\ (g^{12}g^{23} - g^{13}g^{22}) & (g^{13}g^{21} - g^{11}g^{23}) & (g^{11}g^{22} - g^{12}g^{21}) \end{vmatrix}. \qquad (22)$$

Therefore, two constitutive matrices, $\epsilon(x)$ and $\mu^{-1}(x)$, are not independent and obey the following constraint:

$$\epsilon(x)\mu^{-1}(x) = \frac{-g^{00}g_{00}}{\det g_{ik}} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix} \begin{vmatrix} (g^{22}g^{33} - g^{23}g^{32}) & (g^{31}g^{23} - g^{21}g^{33}) & (g^{21}g^{32} - g^{22}g^{31}) \\ (g^{32}g^{13} - g^{33}g^{12}) & (g^{33}g^{11} - g^{31}g^{13}) & (g^{31}g^{12} - g^{32}g^{11}) \\ (g^{12}g^{23} - g^{13}g^{22}) & (g^{13}g^{21} - g^{11}g^{23}) & (g^{11}g^{22} - g^{12}g^{21}) \end{vmatrix} = -I. \qquad (23)$$

Thus, the metric tensors with the quasi-diagonal structure effectively describe the media with the following constitutive relations (note that the sign minus may be eliminated by changing the notation)

$$\mathbf{D} = -\epsilon_0\,\epsilon(x)\mathbf{E}, \quad \mathbf{B} = \mu_0\mu(x)\mathbf{H}, \quad \mu(x) = -\epsilon(x), \quad (\epsilon^{ik})(x) = \sqrt{-g(x)}\, g^{00}(x) \begin{vmatrix} g^{11}(x) & g^{12}(x) & g^{13}(x) \\ g^{21}(x) & g^{22}(x) & g^{23}(x) \\ g^{31}(x) & g^{32}(x) & g^{33}(x) \end{vmatrix}. \qquad (24)$$

For the general case of an arbitrary metrical tensor, four constitutive tensors may be given explicitly as follows (we use notation $g^i(x) = g^{i0}(x)$)

$$[\epsilon^{ik}(x)] = \sqrt{-g}\, g^{00} \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix} - \sqrt{-g} \begin{vmatrix} g^{1}g^{1} & g^{1}g^{2} & g^{1}g^{3} \\ g^{2}g^{1} & g^{2}g^{2} & g^{2}g^{3} \\ g^{3}g^{1} & g^{3}g^{2} & g^{3}g^{3} \end{vmatrix},$$

$$(\mu^{-1})^{ik}(x) = (\sqrt{-g} \begin{vmatrix} (g^{22}g^{33} - g^{23}g^{32}) & (g^{31}g^{23} - g^{21}g^{33}) & (g^{21}g^{32} - g^{22}g^{31}) \\ (g^{32}g^{13} - g^{33}g^{12}) & (g^{33}g^{11} - g^{31}g^{13}) & (g^{31}g^{12} - g^{32}g^{11}) \\ (g^{12}g^{23} - g^{13}g^{22}) & (g^{13}g^{21} - g^{11}g^{23}) & (g^{11}g^{22} - g^{12}g^{21}) \end{vmatrix}, \tag{25}$$

$$\alpha^{ik}(x) = \sqrt{-g} \begin{vmatrix} (-g^{12}g^{3} + g^{13}g^{2}) & (g^{11}g^{3} - g^{13}g^{1}) & (-g^{11}g^{2} + g^{12}g^{1}) \\ (-g^{22}g^{3} + g^{23}g^{2}) & (g^{21}g^{3} - g^{23}g^{1}) & (-g^{21}g^{2} + g^{22}g^{1}) \\ (-g^{32}g^{3} + g^{33}g^{2}) & (g^{31}g^{3} - g^{33}g^{1}) & (-g^{31}g^{2} + g^{32}g^{1}) \end{vmatrix} = \beta^{ki}(x).$$

4. Inverse constitutive equations

The tensor relationship

$$H^{\rho\sigma}(x) = \epsilon_0 \sqrt{-g(x)}\, g^{\rho\alpha}(x) g^{\sigma\beta}(x) F_{\alpha\beta}(x)$$

in 3-vector form reads

$$D^{i} = \epsilon_0 \epsilon^{ik} E_k + \epsilon_0 c \alpha^{ik} B_k, \quad H^{i} = \epsilon_0 c \beta^{ik} E_k + \frac{1}{\mu_0} (\mu^{-1})^{ik} B_k;$$

let us derive inverse constituents

$$F_{\rho\sigma} = \frac{1}{\epsilon_0} \frac{1}{\sqrt{-g(x)}} g_{\rho\alpha}(x) g_{\sigma\beta}(x) H^{\alpha\beta}, \; E_i = \frac{1}{\epsilon_0} (\epsilon^{-1})_{ik} D^k + \frac{1}{\epsilon_0 c} (\alpha^{-1})_{ik} H^k, \quad B_i = \frac{1}{\epsilon_0 c} (\beta^{-1})_{ik} D^k + \mu_0 \mu_{ik} H^k. \tag{26}$$

One does not need to make new calculations in addition to those given in Section IV. Expressions for the inverse constitutive tensors are

$$(\epsilon^{-1})_{ik} = \frac{1}{\sqrt{-g(x)}} [g_{00}(x) g_{ik}(x) - g_i(x) g_k(x)], \quad \mu_{ik} = \frac{1}{2} \frac{1}{\sqrt{-g(x)}} \epsilon_{imn} g_{ml}(x) g_{nj}(x) \epsilon_{ljk},$$

$$(\alpha^{-1})_{ik} = + \frac{1}{\sqrt{-g(x)}} g_{ij}(x) g_l(x) \epsilon_{ljk}, \quad (\beta^{-1})_{ik} = - \frac{1}{\sqrt{-g(x)}} g_j(x) \epsilon_{jil}\, g_{lk}(x), \tag{27}$$

with symmetry

$$(\epsilon^{-1})_{ik}(x) = +(\epsilon^{-1})_{ki}(x), \quad \mu_{ik}(x) = +\mu_{ki}(x), \quad (\beta^{-1})_{ki}(x) = +(\alpha^{-1})_{ik}. \tag{28}$$

Therefore, the (6×6)-matrix, determining the inverse material equations, is symmetrical as well

$$\begin{vmatrix} E_k \\ B_k \end{vmatrix} = \begin{vmatrix} \epsilon_0^{-1} (\epsilon^{-1})_{kl} & \epsilon_0^{-1} c^{-1} (\alpha^{-1})_{kl} \\ \epsilon_0^{-1} c^{-1} (\beta^{-1})_{kl} & \mu_0 \mu_{kl} \end{vmatrix} \begin{vmatrix} D^l \\ H^l \end{vmatrix}. \tag{29}$$

5. Geometrical modeling of inhomogeneous media

Let us consider a special form of the diagonal metrical tensor

$$g_{\alpha\beta} = \begin{vmatrix} a^2(x) & 0 & 0 & 0 \\ 0 & -b^2(x) & 0 & 0 \\ 0 & 0 & -b^2(x) & 0 \\ 0 & 0 & 0 & -b^2(x) \end{vmatrix}. \tag{30}$$

In this case, relations (24) take the form

$$\mathbf{D} = -\epsilon_0\, \epsilon(x) \mathbf{E}, \quad \mathbf{B} = \mu_0 \mu(x) \mathbf{H}, \quad \mu(x) = -\epsilon(x) = \frac{b(x)}{a(x)}. \tag{31}$$

For the diagonal anisotropic metrics

$$g_{\alpha\beta} = \begin{vmatrix} a^2(x) & 0 & 0 & 0 \\ 0 & -b_1^2(x) & 0 & 0 \\ 0 & 0 & -b_2^2(x) & 0 \\ 0 & 0 & 0 & -b_3^2(x) \end{vmatrix}, \qquad (32)$$

relations (24) lead to

$$\mathbf{D} = -\epsilon_0\, \epsilon(x)\mathbf{E}, \quad \mathbf{B} = \mu_0 \mu(x)\mathbf{H}, \quad \mu(x) = -\epsilon(x) = \begin{vmatrix} b_2 b_3 / a b_1 & 0 & 0 \\ 0 & b_3 b_1 / a b_2 & 0 \\ 0 & 0 & b_1 b_2 / a b_3 \end{vmatrix}. \qquad (33)$$

6. Geometrical modeling of uniform media

Let us consider one special form of the metrical tensor

$$g_{\alpha\beta} = \begin{vmatrix} a^2 & 0 & 0 & 0 \\ 0 & -b^2 & 0 & 0 \\ 0 & 0 & -b^2 & 0 \\ 0 & 0 & 0 & -b^2 \end{vmatrix}, \qquad (34)$$

where a^2 and b^2 are numerical parameters. This is a special case mentioned in connection with eq. (14): if $g(x)$ does not depend on coordinates, then the factor $\sqrt{-g}$ can be omitted from the formulas. In this way we get the following constitutive equations:

$$(\epsilon^{ik}) = \frac{1}{a^2 b^2} \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad ((\mu^{-1})^{ik}) = \frac{1}{b^4} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad (35)$$

or

$$D^i = -\frac{\epsilon_0}{a^2 b^2} E_i, \quad H^i = \frac{1}{\mu_0 b^4} B_i, \qquad (36)$$

whence it follows

$$b^2 = \sqrt{\mu}, \quad a^2 = \frac{1}{\epsilon} \frac{1}{\sqrt{\mu}}. \qquad (37)$$

The corresponding metrical tensor (34) can be presented as follows

$$g_{\alpha\beta} = \frac{1}{\sqrt{\epsilon}} \begin{vmatrix} 1/\sqrt{\epsilon\mu} & 0 & 0 & 0 \\ 0 & -\sqrt{\epsilon\mu} & 0 & 0 \\ 0 & 0 & -\sqrt{\epsilon\mu} & 0 \\ 0 & 0 & 0 & -\sqrt{\epsilon\mu} \end{vmatrix}. \qquad (38)$$

Such possibilities will exist in all other cases considered below.

7. Geometrical modeling of anisotropic media

Let us extend the previous analysis and consider the metrical tensor

$$g_{\alpha\beta} = \begin{vmatrix} a^2 & 0 & 0 & 0 \\ 0 & -b_1^2 & 0 & 0 \\ 0 & 0 & -b_2^2 & 0 \\ 0 & 0 & 0 & -b_3^2 \end{vmatrix}, \qquad (39)$$

where a^2, b_i^2 are arbitrary numerical parameters. The constitutive equations generated by this geometry have the form

$$D^i = \epsilon_0 \epsilon^{ik} E_k\,, \quad H^i = \mu_0^{-1} \mu^{ik} B_k\,,$$

$$(\epsilon^{ik}) = a^{-2} \begin{vmatrix} -b_1^{-2} & 0 & 0 \\ 0 & -b_2^{-2} & 0 \\ 0 & 0 & -b_3^{-2} \end{vmatrix}, \quad (\mu^{ik}) = \begin{vmatrix} b_2^{-2}b_3^{-2} & 0 & 0 \\ 0 & b_3^{-2}b_1^{-2} & 0 \\ 0 & 0 & b_1^{-2}b_2^{-2} \end{vmatrix}, \tag{40}$$

or differently

$$D^1 = -\frac{\epsilon_0 E_1}{a^2 b_1^2}, D^2 = -\frac{\epsilon_0 E_2}{a^2 b_2^2}, D^3 = -\frac{\epsilon_0 E_3}{a^2 b_3^2}, \qquad H^1 = \frac{B_1}{\mu_0 b_2^2 b_3^2}, H^2 = \frac{B_2}{\mu_0 b_3^2 b_1^2}, H^3 = \frac{B_3}{\mu_0 b_1^2 b_2^2}.$$

Being compared these equations with the physical ones

$$D^1 = -\epsilon_0 \epsilon_1 E_1, D^2 = -\epsilon_0 \epsilon_2 E_2, D^3 = -\epsilon_0 \epsilon_3 E_3, \qquad H^1 = \frac{B_1}{\mu_0 \mu_1}, H^2 = \frac{B_2}{\mu_0 \mu_2}, H^3 = \frac{B_3}{\mu_0 \mu_3},$$

one obtains

$$\epsilon_1 = \frac{1}{a^2 b_1^2}, \quad \epsilon_2 = \frac{1}{a^2 b_2^2}, \quad \epsilon_3 = \frac{1}{a^2 b_3^2}, \quad \mu_1 = b_2^2 b_3^2, \quad \mu_2 = b_3^2 b_1^2, \quad \mu_3 = b_1^2 b_2^2. \tag{41}$$

It follows that

$$\frac{\mu_1}{\epsilon_1} = \frac{\mu_2}{\epsilon_2} = \frac{\mu_3}{\epsilon_3} = (a^2 b_1^2 b_2^2 b_3^2) = -g, \quad -g = \sqrt{\frac{\mu_1^2 + \mu_2^2 + \mu_3^2}{\epsilon_1^2 + \epsilon_2^2 + \epsilon_2^2}}, \qquad \frac{\mu_i}{\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2}} = \frac{\epsilon_i}{\sqrt{\epsilon_1^2 + \epsilon_2^2 + \epsilon_2^2}}. \tag{42}$$

The latter means that one can use four independent parameters ϵ, μ, n_i:

$$\epsilon_i = \epsilon n_i\,, \quad \mu_i = \mu n_i\,, \quad \mathbf{n}^2 = 1. \tag{43}$$

One can express b_i^2 in terms of μ_i:

$$b_1^2 = \sqrt{\frac{\mu_2 \mu_3}{\mu_1}} = \sqrt{\mu} \sqrt{\frac{n_2 n_3}{n_1}}, \quad b_2^2 = \sqrt{\frac{\mu_3 \mu_1}{\mu_2}} = \sqrt{\mu} \sqrt{\frac{n_3 n_1}{n_2}}, \quad b_3^2 = \sqrt{\frac{\mu_1 \mu_2}{\mu_3}} = \sqrt{\mu} \sqrt{\frac{n_1 n_2}{n_3}}. \tag{44}$$

In its turn, from $a^2 b_1^2 b_2^2 b_3^2 = \mu / \epsilon$ it follows

$$a^2 = \frac{\mu}{\epsilon} \frac{1}{b_1^2 b_2^2 b_3^2} = \frac{1}{\epsilon \sqrt{\mu}} \frac{1}{\sqrt{n_1 n_2 n_3}}. \tag{45}$$

The formulas (44), (45) provide us with the anisotropic metrical tensor

$$g_{ab}(x) = \frac{1}{\sqrt{\epsilon}} \begin{vmatrix} \frac{1}{\sqrt{\epsilon\mu}} \frac{1}{\sqrt{n_1 n_2 n_3}} & 0 & 0 & 0 \\ 0 & -\sqrt{\epsilon\mu} \sqrt{\frac{n_2 n_3}{n_1}} & 0 & 0 \\ 0 & 0 & -\sqrt{\epsilon\mu} \sqrt{\frac{n_3 n_1}{n_2}} & 0 \\ 0 & 0 & 0 & -\sqrt{\epsilon\mu} \sqrt{\frac{n_1 n_2}{n_3}} \end{vmatrix}. \tag{46}$$

8. Moving medium and anisotropy

A more complicated example of an effective anisotropic medium is provided by the constitutive equations for the uniform media in a moving reference frame. Following the initial Minkowski investigation [1], this problem was considered by many authors. The starting point is that, in Minkowski approach to electrodynamics, the constitutive relations explicitly depend on the 4-velocity of the reference frame motion under a medium. Gordon [2], Tamm, and Mandelstam [3,4] noticed that for a moving observer, the constitutive relations can be expressed with the help of effective metric as follows:

$$H^{ab}(x) = \epsilon_0 \Delta^{abmn} F_{mn}, \; \Delta^{abmn} = \epsilon_0 \frac{1}{\sqrt{\mu}}[g^{am} + (\epsilon\mu - 1)u^a u^m]\frac{1}{\sqrt{\mu}}[g^{bn} + (\epsilon\mu - 1)u^b u^n], \tag{47}$$

where $g^{ab} = \mathrm{diag}(+1,-1,-1,-1)$. Corresponding constitutive 3-dimensional tensors are (let us use the notation $\epsilon\mu - 1 = \gamma$)

$$\epsilon^{ik} = \frac{1}{\mu}\begin{vmatrix} (-1+\gamma u^1u^1 - \gamma u^0u^0) & \gamma u^1u^2 & \gamma u^1u^3 \\ \gamma u^1u^2 & (-1+\gamma u^2u^2 - \gamma u^0u^0) & \gamma u^2u^3 \\ \gamma u^3u^1 & \gamma u^3u^2 & (-1+\gamma u^3u^3 - \gamma u^0u^0) \end{vmatrix},$$

$$(\mu^{-1})^{ik} = \frac{1}{\mu}\begin{vmatrix} (1-\gamma u^2u^2 - \gamma u^3u^3) & \gamma u^1u^2 & \gamma u^1u^3 \\ \gamma u^1u^2 & (1-\gamma u^3u^3 - \gamma u^1u^1) & \gamma u^2u^3 \\ \gamma u^3u^1 & \gamma u^3u^2 & (1-\gamma u^1u^1 - \gamma u^2u^2) \end{vmatrix}, \tag{48}$$

$$\alpha^{ik} = \frac{1}{\mu}\begin{vmatrix} 0 & -\gamma u^0u^3 & +\gamma u^0u^2 \\ +\gamma u^0u^3 & 0 & -\gamma u^0u^1 \\ -\gamma u^0u^2 & +\gamma u^0u^1 & 0 \end{vmatrix}, \quad \beta^{ik} = \frac{1}{\mu}\begin{vmatrix} 0 & +\gamma u^0u^3 & -\gamma u^0u^2 \\ -\gamma u^0u^3 & 0 & +\gamma u^0u^1 \\ +\gamma u^0u^2 & -\gamma u^0u^1 & 0 \end{vmatrix}.$$

Let us deduce 3-dimensional form of these relations. For the vector D^i we have

$$D^1 = \frac{\epsilon_0}{\mu}\Big[(-1+\gamma u^1u^1 - \gamma u^0u^0)E_1 + \gamma u^1u^2E_2 + \gamma u^1u^3E_3\Big] + \frac{\epsilon_0 c}{\mu}(-\gamma u^0u^3B_2 + \gamma u^0u^2B_3),$$

$$D^2 = \frac{\epsilon_0}{\mu}\Big[+\gamma u^1u^2E_1 + (-1+\gamma u^2u^2 - \gamma u^0u^0)E_2 + \gamma u^2u^3E_3\Big] + \frac{\epsilon_0 c}{\mu}(\gamma u^0u^3B_1 - \gamma u^0u^1B_3),$$

$$D^1 = \frac{\epsilon_0}{\mu}\Big[+\gamma u^1u^3E_1 + \gamma u^2u^3E_2 + (-1+\gamma u^3u^3 - \gamma u^0u^0)E_3\Big] + \frac{\epsilon_0 c}{\mu}(-\gamma u^0u^2B_1 + \gamma u^0u^1B_2),$$

and further, we obtain

$$D^1 = -\frac{\epsilon_0}{\mu}E_1 + \frac{\epsilon_0\gamma}{\mu}[-u^0u^0E_1 + (u^1E_1 + u^2E_2 + u^3E_3)u^1] + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^2B_3 - u^3B_2),$$

$$D^2 = -\frac{\epsilon_0}{\mu}E_2 + \frac{\epsilon_0\gamma}{\mu}[-u^0u^0E_2 + (u^1E_1 + u^2E_2 + u^3E_3)u^2] + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^3B_1 - u^1B_3),$$

$$D^3 = -\frac{\epsilon_0}{\mu}E_3 + \frac{\epsilon_0\gamma}{\mu}[-u^0u^0E_3 + (u^1E_1 + u^2E_2 + u^3E_3)u^3] + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^1B_2 - u^2B_1).$$

Using the notations

$$V = v/c, \quad u^0 = \frac{1}{\sqrt{1-V^2}}, \quad u^i = \frac{V^i}{\sqrt{1-V^2}}$$

the previous relations can be written as follow

$$D^1 = -\frac{\epsilon_0}{\mu}E_1 + \frac{\epsilon_0\gamma}{\mu}\frac{[-E_1 + (V^1E_1 + V^2E_2 + V^3E_3)V^1]}{1-V^2} + \frac{\epsilon_0 c\gamma}{\mu}\frac{(V^2B_3 - V^3B_2)}{1-V^2},$$

$$D^2 = -\frac{\epsilon_0}{\mu}E_2 + \frac{\epsilon_0\gamma}{\mu}\frac{[-E_1 + (V^1E_1 + V^2E_2 + V^3E_3)V^2]}{1-V^2} + \frac{\epsilon_0 c\gamma}{\mu}\frac{(V^3B_1 - V^1B_3)}{1-V^2},$$

$$D^3 = -\frac{\epsilon_0}{\mu}E_3 + \frac{\epsilon_0\gamma}{\mu}\frac{[-E_3 + (V^1E_1 + V^2E_2 + V^3E_3)V^3]}{1-V^2} + \frac{\epsilon_0 c\gamma}{\mu}\frac{(V^1B_2 - V^2B_1)}{1-V^2};$$

in the vector form, they can be rewritten as

$$\mathbf{D} = \frac{\epsilon_0}{\mu}\mathbf{E} + \frac{\epsilon_0\gamma}{\mu}\frac{\mathbf{E} - (\mathbf{VE})\mathbf{V}}{1-V^2} + \frac{\epsilon_0 c\gamma}{\mu}\frac{\mathbf{V}\times\mathbf{B}}{1-V^2}. \tag{49}$$

Similarly, we obtain three relations for H^i:

$$H_1 = \frac{1}{\mu_0\mu}[(1-\gamma u^2u^2 - \gamma u^3u^3)B_1 + \gamma u^1u^2B_2 + \gamma u^1u^3B_3] + \frac{\epsilon_0 c}{\mu}(\gamma u^0u^3E_2 - \gamma u^0u^2E_3),$$

$$H_2 = \frac{1}{\mu_0\mu}[\gamma u^1u^2B_1 + (1-\gamma u^3u^3 - \gamma u^1u^1)B_2 + \gamma u^2u^3B_3] + \frac{\epsilon_0 c}{\mu}(\gamma u^0u^3E_2 - \gamma u^0u^2E_3),$$

$$H_3 = \frac{1}{\mu_0\mu}[\gamma u^3u^1B_1 + \gamma u^3u^2B_3 + (1-\gamma u^1u^1 - \gamma u^2u^2)B_3] + \frac{\epsilon_0 c}{\mu}(\gamma u^0u^2E_1 - \gamma u^0u^1E_2),$$

and then

$$H_1 = \frac{1}{\mu_0\mu}B_1 + \frac{\gamma}{\mu_0\mu}(-u^2u^2B_1 - u^3u^3B_1 + u^1u^2B_2 + u^1u^3B_3) + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^3E_2 - u^2E_3),$$

$$H_2 = \frac{1}{\mu_0\mu}B_2 + \frac{\gamma}{\mu_0\mu}(+u^1u^2B_1 - u^3u^3B_2 - u^1u^1B_2 + u^2u^3B_3) + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^3E_2 - u^2E_3),$$

$$H_3 = \frac{1}{\mu_0\mu}B_3 + \frac{\gamma}{\mu_0\mu}(+u^3u^1B_1 + u^3u^2B_2 - u^1u^1B_3 - u^2u^2B_3) + \frac{\epsilon_0 c\gamma}{\mu}u^0(u^2E_1 - u^1E_2);$$

these relations are written in the vector form as follows:

$$\mathbf{H} = \frac{1}{\mu_0\mu}\mathbf{B} + \frac{\gamma}{\mu_0\mu}\frac{\mathbf{V}\times(\mathbf{V}\times\mathbf{B}))}{1-V^2} + \frac{\epsilon_0 c\gamma}{\mu}\frac{\mathbf{V}\times\mathbf{E}}{1-V^2}. \quad (50)$$

Relations (49), (50) provide us with the vector form of constitutive relations for the uniform medium moving with velocity $\mathbf{V}$. We may conclude that the motion is effectively equivalent to an anisotropic medium.

9. Constitutive equations for an arbitrary linear medium

Above, we started with Maxwell equations in vacuum and changed them to the Riemannian space-time. At that, the vacuum material equations $H_{\alpha\beta} = \epsilon_0 F_{\alpha\beta}$, due to the presence of metrical tensor $g^{\rho\alpha}(x)$, gave us the modified material equations

$$H^{\rho\sigma}(x) = \sqrt{-g}\, g^{\rho\alpha}(x) g^{\sigma\beta}(x)\epsilon_0 F_{\alpha\beta}(x). \quad (51)$$

As a first generalization, let us start with Maxwell equations in the uniform media. In this case, the material equations for the uniform media

$$H_{\alpha\beta}(x) = \epsilon_0 \eta_\alpha^{\ a}\eta_\beta^{\ b} F_{ab}(x) \quad (52)$$

will take the form

$$H^{\rho\sigma}(x)) = \sqrt{-g}\, g^{\rho\alpha}(x) g^{\sigma\beta}(x)\epsilon_0 \eta_\alpha^{\ a}\eta_\beta^{\ b} F_{ab}(x). \quad (53)$$

With the notation $\hat{F}_{\alpha\beta}(x) = \eta_\alpha^{\ a}\eta_\beta^{\ b} F_{ab}(x)$, they can be rewritten as follows

$$H^{\rho\sigma}(x)(x) = \sqrt{-g}\, g^{\rho\alpha}(x) g^{\sigma\beta}(x)\epsilon_0 \hat{F}_{\alpha\beta}(x), \quad \hat{F}_{\alpha\beta}(x) = \begin{vmatrix} 0 & \epsilon F_{0i} \\ \epsilon F_{i0} & F_{ik}/\mu \end{vmatrix}. \quad (54)$$

One should not make any additional calculation, it suffices to make one formal change $F_{\alpha\beta}(x) \longrightarrow \hat{F}_{\alpha\beta}(x)$, so the constitutive equations take the form

$$D^i = \epsilon_0\epsilon\epsilon^{ik}(x)E_k + \epsilon_0\epsilon c\alpha^{ik}(x)B_k, \quad H^i = \epsilon_0\epsilon c\beta^{ik}(x)E_k + \frac{1}{\mu_0\mu}(\mu^{-1})^{ik}(x)B_k. \quad (55)$$

These relations provide us with the material equations for the uniform media modified by Riemannian geometry of the background space-time.

It is easy to make another extension: let us start with an anisotropic medium in Minkowski space

$$D_i = \epsilon_0 \epsilon_{(0)kl} E_l , \quad H_i = \mu_0^{-1} \mu_{(0)kl}^{-1} B_k ; \tag{56}$$

these equations will be modified as

$$D^i = \epsilon_0 \epsilon^{ik}(x) \epsilon_{(0)kl} E_l + \epsilon_0 c \alpha^{ik}(x) \mu_{(0)kl}^{-1} B_l , \quad H^i = \epsilon_0 c \beta^{ik}(x) \epsilon_{(0)kl} E_l + \mu_0^{-1} (\mu^{-1})^{ik}(x) \mu_{(0)kl} B_l . \tag{57}$$

Now we make the final extension: constitutive equations for an arbitrary linear medium, determined by 4-rank tensor $H_{\alpha\beta}(x) = \epsilon_0 \Delta_{\alpha\beta}{}^{ab} F_{ab}(x)$, will be transformed by the Riemannian geometry into the following ones

$$H^{\rho\sigma}(x)(x) = \sqrt{-g}\, g^{\rho\alpha}(x) g^{\sigma\beta}(x) \epsilon_0 \Delta_{\alpha\beta}{}^{ab} F_{ab}(x), \tag{58}$$

or in the 3-dimensional form

$$D^i = \epsilon_0 \left[\epsilon^{ik}(x) \epsilon_{(0)kl} + \alpha^{ik}(x) \beta_{(0)kl} \right] E_l + \epsilon_0 c \left[\epsilon^{ik}(x) \alpha_{(0)kl} + \alpha^{ik}(x) \mu_{(0)kl}^{-1} \right] B_l ,$$

$$H^i = \epsilon_0 c \left[\beta^{ik}(x) \epsilon_{(0)kl} + (\mu^{-1})^{ik}(x) \beta_{(0)kl} \right] E_l + \frac{1}{\mu_0} \left[\beta^{ik}(x) \alpha_{(0)kl} + (\mu^{-1})^{ik}(x) \mu_{(0)kl}^{-1} \right] B_l .$$

In the index-free form, they can be written as

$$\mathbf{D} = \epsilon_0 \left[\epsilon(x) \epsilon_{(0)} + \alpha(x) \beta_{(0)} \right] \mathbf{E} + \epsilon_0 c \left[\epsilon(x) \alpha_{(0)} + \alpha(x) \mu_{(0)}^{-1} \right] \mathbf{B},$$

$$\mathbf{H} = \epsilon_0 c \left[\beta(x) \epsilon_{(0)} + \mu^{-1}(x) \beta_{(0)} \right] \mathbf{E} + \frac{1}{\mu_0} \left[\beta(x) \alpha_{(0)} + \mu^{-1}(x) \mu_{(0)}^{-1} \right] \mathbf{B}. \tag{59}$$

These formulas can be presented symbolically as follow

$$\hat{\epsilon} = \epsilon(x) \epsilon_{(0)} + \alpha(x) \beta_{(0)} , \quad \hat{\alpha} = \epsilon(x) \alpha_{(0)} + \alpha(x) \mu_{(0)}^{-1} ,$$

$$\hat{\beta} = \beta(x) \epsilon_{(0)} + \mu^{-1}(x) \beta_{(0)} , \quad \hat{\mu}^{-1} = \beta(x) \alpha_{(0)} + \mu^{-1}(x) \mu_{(0)}^{-1} . \tag{60}$$

If the initial constitutive equations have only diagonal blocks $\epsilon_{(0)}, \alpha_{(0)} = 0, \beta_{(0)} = 0, \mu_{(0)}^{-1}$, relations (60) become simpler

$$\hat{\epsilon} = \epsilon(x) \epsilon_{(0)} , \quad \hat{\alpha} = +\alpha(x) \mu_{(0)} , \quad \hat{\beta} = \beta(x) \epsilon_{(0)} , \quad \hat{\mu}^{-1} = \mu(x) \mu_{(0)}^{-1} . \tag{61}$$

10. Plane wave in the Lobachevsky space

Let us study a simple example, the solutions of plane wave type for the Maxwell equations in 3-dimensional Lobachevsky space, with the use of the horospherical coordinates. We start with equations

$$(I) \quad \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \quad (II) \quad \frac{1}{\sqrt{-g}} \partial_\beta \sqrt{-g}\, F^{\beta\alpha} = \epsilon_0^{-1} j^\alpha , \tag{62}$$

where $g(x) = \det[g_{\alpha\beta}(x)] < 0$. Eqs. (I) are equivalent to

$$(123) \quad \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0, \quad (012) \quad \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0,$$

$$(023) \quad \partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0, \quad (031) \quad \partial_0 F_{31} + \partial_3 F_{10} + \partial_1 F_{03} = 0. \tag{63}$$

In its turn, eqs. (II) in the detailed form are written as

$$\frac{1}{\sqrt{-g}} \partial_1 \sqrt{-g}\, F^{10} + \frac{1}{\sqrt{-g}} \partial_2 \sqrt{-g}\, F^{20} + \frac{1}{\sqrt{-g}} \partial_3 \sqrt{-g}\, F^{30} = \epsilon_0^{-1} j^0 ,$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \sqrt{-g}\, F^{01} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^2} \sqrt{-g}\, F^{12} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^3} \sqrt{-g}\, F^{31} = \epsilon_0^{-1} j^1 ,$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \sqrt{-g}\, F^{02} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^1} \sqrt{-g}\, F^{12} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^3} \sqrt{-g}\, F^{23} = \epsilon_0^{-1} j^2 ,$$

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \sqrt{-g}\, F^{03} - \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^1} \sqrt{-g}\, F^{31} + \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^2} \sqrt{-g}\, F^{23} = \epsilon_0^{-1} j^3 . \tag{64}$$

We will specify these equations in the absence of any sources, and in the horospherical coordinates of the hyperbolic model H_3:

$$u_1 = re^{-z}\cos\phi, \quad u_2 = re^{-z}\sin\phi, \quad u_3 = \sinh z + \frac{1}{2}r^2 e^{-z} = \frac{1}{2}[e^{+z} + (r^2-1)e^{-z}],$$

$$u_0 = \cosh z + \frac{1}{2}r^2 e^{-z} = \frac{1}{2}[e^{+z} + (r^2+1)e^{-z}], \quad u_1^2 + u_2^2 + u_3^2 - u_0^2 = -1, \quad u_0 = +\sqrt{1+u_1^2+u_2^2+u_3^2} > 0,$$

$$dS^2 = c^2dt^2 - e^{-2z}dr^2 - e^{-2z}r^2 d\phi^2 - dz^2; \tag{65}$$

here all the coordinates are dimensionless. In the limit of vanishing curvature, the given coordinates reduce to cylindrical ones. Using the notations $x^0, r = x^1, \phi = x^2, z = x^3$, eqs. (63) and (64) take the form

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0, \quad \partial_0 F_{12} + \partial_1 F_{20} + \partial_2 F_{01} = 0,$$

$$\partial_0 F_{23} + \partial_2 F_{30} + \partial_3 F_{02} = 0, \quad \partial_0 F_{31} + \partial_3 F_{10} + \partial_1 F_{03} = 0, \tag{66}$$

$$\frac{1}{r}\partial_r rF^{10} + \partial_\phi F^{20} + e^{2z}\partial_z e^{-2z}F^{30} = 0, \quad \partial_0 F^{01} - \partial_\phi F^{12} + e^{2z}\partial_z e^{-2z}F^{31} = 0,$$

$$\partial_0 F^{02} + \frac{1}{r}\partial_r rF^{12} - e^{2z}\partial_z e^{-2z}F^{23} = 0, \quad \partial_0 F^{03} - \frac{1}{r}\partial_r rF^{31} + \partial_\phi F^{23} = 0. \tag{67}$$

Let us impose the simplicity restrictions

$$F_{01} = 0, \quad F_{02} = F_{0\phi}(r,z) \neq 0, \quad F_{03} = 0, \tag{68}$$

then eqs. (66), (67) become simpler

$$\partial_r F_{23} + \partial_z F_{12} = 0, \partial_0 F_{12} + \partial_r F_{20} = 0, \partial_0 F_{23} + \partial_z F_{02} = 0, \tag{60}$$

$$F_{31} = 0, \quad \partial_0 F^{02} + \frac{1}{r}\partial_r rF^{12} - e^{2z}\partial_z e^{-2z}F^{23} = 0. \tag{70}$$

Note that in (69), the first equation follows from 2-nd and 3-rd ones, so only three equations are independent

$$\partial_0 F_{12} = \partial_r F_{02}, \quad \partial_0 F_{23} = -\partial_z F_{02}, \tag{71}$$

$$\partial_0^2 g^{22} F_{02} + \frac{1}{r}\partial_r r g^{11} g^{22}\partial_0 F_{12} - e^{2z}\partial_z e^{-2z} g^{22} g^{33}\partial_0 F_{23} = 0. \tag{72}$$

Differentiating the third equation with respect to x_0 and taking into account (71), we get

$$-\frac{e^{2z}}{r^2}\partial_0^2 F_{02} + \frac{1}{r}\partial_r \frac{e^{4z}}{r}\partial_r F_{02} + \frac{e^{2z}}{r^2}\partial_z^2 F_{02} = 0. \tag{73}$$

Let us search for solutions of eq. (73) in the form of a plane wave

$$F_{02} = E(r)\cos(k_0 x^0 - kz + \beta), \tag{74}$$

where the dimensionless quantities are used

$$k_0 = \frac{\omega\rho}{c}, \quad k = \pm\frac{\omega\rho}{c} = \pm k_0, \tag{75}$$

then eq. (73) leads to

$$\frac{k_0^2 e^{2z}}{r^2}E + \frac{e^{4z}}{r}\frac{d}{dr}\frac{1}{r}\frac{d}{dr}E - \frac{k_0^2 e^{2z}}{r^2}E = 0, \tag{76}$$

or

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}E = 0, \tag{77}$$

so we arrive at two simple solutions

$$E = \text{const}\, r^2, \quad E = \text{const}. \tag{78}$$

Eq. (77) coincides with the similar equation arising in the flat space; correspondingly its solutions coincide with the known ones in the Minkowski space. The most interesting is the second solution $E = \text{const}$, so that

$$F_{02} = E\cos(k_0 x^0 - kz + \beta). \tag{79}$$

From (71) we readily find F_{12} and F_{23}:

$$\partial_0 F_{12} = \partial_r F_{02}, \quad \partial_0 F_{23} = -\partial_z F_{02} \;\Rightarrow\; F_{12} = 0, \quad F_{23} = E\frac{k}{k_0}\cos(k_0 x^0 - kz + \beta). \tag{80}$$

The following 4-potential corresponds to electromagnetic field (79), (80)

$$A_\phi = A_2 = \frac{E}{k_0}\sin(k_0 x^0 - kz + \beta). \tag{81}$$

It exactly coincides with the expression of the 4-potential for the simplest cylindrical wave in Minkowski space. Taking $\beta = 0, \pi/2$, we get two linearly independent solutions.

The above solution obtained in horospherical coordinates can be translated to quasi-Cartesian coordinates in H_3:

$$(x^1, x^2, x^3) = (r, \phi, z) \;\Rightarrow\; y^i = \frac{u_i}{u_0} = (q_1, q_2, q_3),$$

$$y^1 = \frac{2r\cos\phi}{e^{2z} + r^2 + 1}, \quad y^2 = \frac{2r\sin\phi}{e^{2z} + r^2 + 1}, \quad y^3 = \frac{e^{2z} + r^2 - 1}{e^{2z} + r^2 + 1}. \tag{82}$$

Allowing for the following identities

$$A_i(y) = \frac{\partial x^j}{\partial y^i} A_j = \frac{\partial \phi}{\partial y^i} A_\phi(x), \quad \frac{y^2}{y^1} = \tan\phi, \quad \frac{1}{\cos^2\phi} = \frac{(y^1)^2 + (y^2)^2}{(y^1)^2},$$

$$\frac{(y^1)^2 + (y^2)^2}{(y^1)^2}\frac{\partial\phi}{\partial y^1} = -\frac{y^2}{(y^1)^2} \;\Rightarrow\; \frac{\partial\phi}{\partial y^1} = -\frac{y^2}{(y^1)^2 + (y^2)^2},$$

$$\frac{(y^1)^2 + (y^2)^2}{(y^1)^2}\frac{\partial\phi}{\partial y^2} = \frac{1}{y^1} \;\Rightarrow\; \frac{\partial\phi}{\partial y^2} = \frac{y^1}{(y^1)^2 + (y^2)^2}, \tag{83}$$

we arrive at (for simplicity we use notation $y^j = q_j$)

$$A_1(q) = -\frac{q_2}{q_1^2 + q_2^2}A\sin(\omega t - kz + \beta), \quad A_2(q) = \frac{q_1}{q_1^2 + q_2^2}A\sin(\omega t - kz + \beta), \quad A_3(q) = 0. \tag{84}$$

Allowing for the formulas

$$e^z = \frac{\sqrt{1 - q_1^2 - q_2^2 - q_3^2}}{1 - q_3} \;\Rightarrow\; z = \ln\frac{\sqrt{1 - q_1^2 - q_2^2 - q_3^2}}{1 - q_3}, \tag{85}$$

we obtain the following expressions for these solutions:

$$A_1(q) = -\frac{q_2}{q_1^2 + q_2^2}A\sin\Big(\omega t - k\ln\frac{\sqrt{1 - q_1^2 - q_2^2 - q_3^2}}{1 - q_3} + \beta\Big),$$

$$A_2(q) = \frac{q_1}{q_1^2 + q_2^2}A\sin\Big(\omega t - k\ln\frac{\sqrt{1 - q_1^2 - q_2^2 - q_3^2}}{1 - q_3} + \beta\Big), \quad A_3(q) = 0. \tag{86}$$

It should be noted that the very similar and trivial form of solutions in horospherical coordinates turns out to be much more complicated in quasi-Cartesian coordinates. Let us specify the electromagnetic tensor for that solution (compare it with (79) and (80))

$$\varphi = \Big(\omega t - k\ln\frac{\sqrt{1 - q^2}}{1 - q_3} + \beta\Big), \quad q^2 = q_1^2 + q_2^2 + q_3^2,$$

$$E_1 = F_{01} = -\frac{q_2 \omega}{q_1^2 + q_2^2} A\cos\varphi, \quad E_2 = F_{02} = \frac{q_1 \omega}{q_1^2 + q_2^2} A\cos\varphi, \quad E_3 = F_{03} = 0, \quad -B_3 = F_{12} = Ak\frac{1}{1-q^2}\cos\varphi,$$

$$-B_1 = F_{23} = Ak\frac{q_1}{(q_1^2+q_2^2)}\frac{(1-q_3-q_1^2-q_2^2)}{(1-q_3)(1-q^2)}\cos\varphi = Ak\frac{q_1}{(q_1^2+q_2^2)}F\cos\varphi,$$

$$-B_2 = F_{31} = Ak\frac{q_2}{(q_1^2+q_2^2)}\frac{(1-q_3-q_1^2-q_2^2)}{(1-q_3)(1-q^2)}\cos\varphi = Ak\frac{q_2}{(q_1^2+q_2^2)}F\cos\varphi, \tag{87}$$

here the notation is used

$$F = \frac{(1-q_3-q_1^2-q_2^2)}{(1-q_3)(1-q^2)}.$$

For this wave, the electric vector is oriented along the circle in the plane $1-2$; whereas the magnetic vector has a projection on axis 3 and radial direction. The density of energy flow $\mathbf{S} = \mathbf{E} \times \mathbf{B}$ is described by the formulas

$$S_1 = E_2 B_3 = -A^2 k\omega \frac{q_1}{q_1^2+q_2^2}\frac{1}{1-q^2}\frac{1+\cos 2\varphi}{2}, \quad S_2 = -E_1 B_3 = -A^2 k\omega \frac{q_2}{q_1^2+q_2^2}\frac{1}{1-q^2}\frac{1+\cos 2\varphi}{2},$$

$$S_3 = E_1 B_2 - E_2 B_1 = \frac{A^2 \omega k}{(q_1^2+q_2^2)} F \frac{1+\cos 2\varphi}{2}; \tag{88}$$

therefore, the vector $\mathbf{S}$ has the structure $\mathbf{S} = \mathbf{S}_{\parallel} + \mathbf{S}\perp$. Let us detail the surface of a fixed phase

$$\omega t - k \ln \frac{\sqrt{1-q_1^2-q_2^2-q_3^2}}{1-q_3} = \lambda = \omega t_0 ,$$

whence it follows

$$\frac{1-q_1^2-q_2^2-q_3^2}{(1-q_3)^2} = a^2(t), \quad a^2(t) = e^{2\omega(t-t_0)}. \tag{89}$$

In the canonical form, it is recognized as an ellipsoid equation

$$\left(q_3 - \frac{a^2}{1+a^2}\right)^2 + \frac{q_1^2}{1+a^2} + \frac{q_2^2}{1+a^2} = \frac{1}{(1+a^2)^2}. \tag{90}$$

In the moment $t = t_0$, the parameter $a = 1$ and we have

$$\left(q_3 - \frac{1}{2}\right)^2 + \frac{q_1^2}{2} + \frac{q_2^2}{2} = \frac{1}{4} \quad \longrightarrow \quad q_3 \to +1.$$

When $a \to 0$ $(t \to -\infty)$, we obtain $q_3^2 + q_1^2 + q_2^2 = 1$. When $a \to \infty$ $(t \to +\infty)$, we will have

$$\left(q_3 - 1\right)^2 + \frac{q_1^2}{\infty^2} + \frac{q_2^2}{\infty^2} = \frac{1}{\infty^4}.$$

The constructed solution can be considered as an exact solution for the Maxwell equations in Minkowski space-time but in the special effective medium.

11. Arbitrary metrical tensor, additional consideration

In the general case, four (3×3) constitutive tensors contain $6+6+9=21$ independent functions, but the metrical tensor $g^{\alpha\beta}(x)$ is determined only by 10 functions, so we must assume the existence of additional constraints on constitutive tensors. In the first place, note evident identity

$$\mathrm{Sp}\,\alpha = \mathrm{Sp}\,\beta = 0. \tag{91}$$

Besides, we can see two identities

$$\mathbf{g}\beta = 0 \quad \Rightarrow \quad (g^{01}, g^{02}, g^{03}) \begin{vmatrix} \beta^{11} & \beta^{12} & \beta^{13} \\ \beta^{21} & \beta^{22} & \beta^{23} \\ \beta^{31} & \beta^{32} & \beta^{33} \end{vmatrix} = 0, \tag{92}$$

$$\alpha \mathbf{g}^{+} = 0 \quad \Rightarrow \quad \begin{vmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ \alpha^{31} & \alpha^{32} & \alpha^{33} \end{vmatrix} \begin{vmatrix} g^{01} \\ g^{02} \\ g^{03} \end{vmatrix} = 0; \tag{93}$$

which are equivalent to each other. To proceed, let us rewrite expressions for $\epsilon^{ik}(x)$ and $(\mu^{-1})^{ik}(x)$ as follows

$$\frac{1}{g^{00}(x)} \Big(\frac{\epsilon^{ik}(x)}{\sqrt{-G(x)}} + (\mathbf{g}(x) \cdot \mathbf{g}(x))^{ik} \Big) = g^{ik}(x), \quad \frac{(\mu^{-1})^{ik}(x)}{\sqrt{-G(x)}} = \det(g^{ik}(x))\,(g^{ik}(x))^{-1} = \frac{1}{\det(g_{ik}(x))}(g_{ik}(x))$$

$$\Rightarrow \quad \mu^{ik}(x)\sqrt{-G(x)} = \det(g_{ik}(x))\, g^{ik}(x); \tag{94}$$

whence we derive the equation

$$\frac{1}{g^{00}(x)} \Big(\frac{\epsilon^{ik}(x)}{\sqrt{-G(x)}} + (\mathbf{g}(x) \cdot \mathbf{g}(x))^{ik} \Big) = \frac{\mu^{ik}(x)\sqrt{-G(x)}}{\det(g_{ik}(x))}, \tag{95}$$

or differently

$$\epsilon^{ik}(x) + \sqrt{-G(x)}[\mathbf{g}(x) \cdot \mathbf{g}(x)]^{ik} = -g^{00}(x) G(x) \frac{\mu^{ik}(x)}{\det(g_{ik}(x))}. \tag{96}$$

This relationship provides us with the needed additional constraint on $\epsilon^{ik}(x)$ and $\mu^{ik}(x)$. For the special form of metrical tensor

$$\mathbf{g}(x) = 0, \quad G(x) = g_{00} \det[g_{ij}(x)]; \tag{97}$$

we obtain the known result

$$\epsilon^{ik}(x) = -\frac{1}{g_{00}(c)} G(x) \frac{1}{\det[g_{ik})x)]} \mu^{ik}(x) = -\mu^{ik}(x). \tag{98}$$

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