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SIMPLE WIGNER-LANGEVIN EQUATION

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Abstract. A new dynamical equation is derived by substituting the Schrödinger-Langevin-Kostin equation into the definition for the Wigner function, it can be called the quantum-classical Wigner-Langevin equation. The proposed equation contains partial derivatives for time and phase space variables of the Wigner function, its coefficients are spatial derivatives of potentials that take into account friction, white noise and external influence. The transition to the classical regime of motion is also discussed.

Keywords: phase space, Wigner-Langevin, quantum-to-classical transition, friction, white noise

1. Introduction and motivations

Quantum dynamics in the phase space, based on the Wigner-Liovile (W-Lv) equation [1], has acquired considerable interest connected with a quantum-to-classical transition phenomenon. Investigation of this transition phenomenon had been realized through the use of the method for continuous coordinate measurement applicable to the quantum Duffing oscillator. The stroboscopic maps for the quantum and classical Duffing oscillators are very similar, Lyapunov exponents being calculated in the context of this method. The quantum system under continuous measurement exhibits qualitatively the same chaotic behavior as the classical system. Lyapunov exponents for the quantum and classical Duffing oscillators in both cases coincide; the continuous measurement transforms the Quantum dynamics into classical dynamics. In details, these results have been discussed in Ref. [2].

The W-Lv equation can be also presented as a continuity equation with classical transport terms and unlimited sum of quantum components, as Taylor series in Planck-constant-even degrees. Below, we will be limited to the first quantum correction proportional to $\hbar^2$. In this case, the temporal evolution of Wigner function can be written as

$$\frac{\partial \rho_w}{\partial t} + \frac{p}{m}\frac{\partial \rho_w}{\partial x} + \left(-\frac{\partial U}{\partial x}\right)\frac{\partial \rho_w}{\partial p} = -\frac{\hbar^2}{24}\frac{\partial^3 U}{\partial x^3}\frac{\partial^3 \rho_w}{\partial p^3}. \quad (1)$$

The left-hand side consists of classical terms and the right-hand side contents a first quantum correction. Here, the Wigner functions is denoted as $\rho_w(x, p, t)$, $\hbar$ is the Planck constant, the classical potential and force, acting on a particle of mass m, are lettered by U, F, respectively. Other variables x, p, t are coordinate, linear momentum and time.

The initial W-Lv equation, shortly mentioned, and its particular case (1) does not include the Langevin friction and Gaussian white noise. The problem consists in generalization of the equation (1) and taking into account the Langevin friction and Gaussian white noise. For the generalization, it is quite possible to use the Schrödinger-Langevin-Kostin equation (Sch-Lg-K) [3,4] in the form (2)

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + (U(x,t) + U_{rand}(x,t) + U_{Lg})\psi, \quad (2)$$

where $\psi = \psi(x,t)$ is a wave function, $U_{rand}(x,t)$ and $U_{Lg}(x,t)$ are the random and

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Langevin potentials. The Langevin potential U_{Lg} is the sum

$$U_{Lg} = \frac{\hbar f}{2im} \ln \frac{\psi}{\psi^*} + W(t),$$
$$W(t) = \frac{\hbar f}{2im} \int \psi^* \ln \frac{\psi}{\psi^*} \psi(dx). \quad (3)$$

Here, f is a frictional coefficient; the force F_{fric} caused by U_{Lg} is

$$F_{fric} = -\frac{\partial U_{Lg}}{\partial x} = -f \frac{J}{\rho'}$$
$$\frac{\partial W(t)}{\partial x} = 0, \quad (4)$$

where ρ is the probability density, J is the probability current density, the relation J/ρ determines the field velocity V.

The first term U_{Lg} in (3) dominates as a descriptive dissipative mechanism. It is known that in scientific literature [5], the second term $W(t)$ is removed or simply ignored. Under differentiatiingn U_{Lg} with respect to a coordinate x with negative sign, the frictional force does not depend on $W(t)$. This important property will be used in going from potential U_{Lg} to the corresponding force in dynamical equations with the Wigner function.

Our basic aim consists in combination of the Sch-Lg-K equation with the W-Lv equation, in order to formulate the new quantum-classical Wigner-Langevin (W-Lg) equation. The properties of the new W-Lg equation caused by potentials or corresponding forces are briefly discussed.

2. Wigner-Langevin equation with the first quantum correction

To construct the new equation for the density matrix we use Sch-Lg-K (2) for $\psi(x,t)$ and complex-conjugated to Sch-Lg-K (2) with $\psi^*(x',t)$. Multiplying these equations by $\psi^*(x',t), \psi(x,t)$, respectively, and taking the difference, one obtains

$$i\hbar \frac{\partial}{\partial t} \rho(x,x',t) = \left(-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right) + (U(x,t) - U(x',t)) + U_{\Sigma\Sigma}\right) \rho(x,x',t), \quad (5)$$

where $\rho(x,x',t) = \psi(x,t) \cdot \psi^*(x',t)$ is the density matrix,

$$U_{\Sigma\Sigma} = (U_{rand}(x,t) - U_{rand}(x',t)) + (U_{Lg}(x,t) - U_{Lg}(x',t)).$$

Using the relation $\psi = R\exp(iS/\hbar)$, where $R(x,t)$ is the amplitude and $S(x,t)$ is the phase (quantum action) of a wave function, the equation

$$U_{Lg}(x,t) - U_{Lg}(x',t) = \frac{f}{m}[S(x,t) - S(x',t)]$$

is holded.

The variables x, x' can be introduced as combinations

$\frac{(x+x\prime)}{2} = X, \quad x - x' = y, \quad y = \hbar\tau$ or in another way as

$$x = X + \frac{1}{2}y, \quad x' = X - \frac{1}{2}y. \quad (6)$$

Takining into account equation (6), the formula for $\rho(x,x',t)$ can be converted into

$$\rho(x,x',t) = \rho(X + \tfrac{1}{2}y, X - \tfrac{1}{2}y, t) = \psi(X + \tfrac{1}{2}y, t)\psi^*(X - \tfrac{1}{2}y, t). \quad (7)$$

The Wigner function is determined in view of the direct Fourier transform of the density matrix with displayed arguments as

$$\rho_w(X,p,t) = \frac{1}{2\pi\hbar} \int \exp(-\frac{i}{\hbar} py)\rho(X + \tfrac{1}{2}y, X - \tfrac{1}{2}y, t)dy. \quad (8)$$

Here X appears as a parameter, the transforming has been carried out on the variable y. If the Wigner function is the Fourier component, then the reverse Fourier transform gives the following expression

$$\rho(X + \tfrac{1}{2}y, X - \tfrac{1}{2}y, t) = \int exp(ip'y) \cdot \rho_w(X,p',t)dp'. \quad (9)$$

Replacing the variables x and x' in functions U, U_{rand}, U_{Lg} of (5) by the new variables according to (6), the difference of Laplace operators in the new variables acquires the form $\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2}\right) = 2\frac{\partial}{\partial X}\frac{\partial}{\partial \tau}/\hbar$.

The basic equation (5), written in the new variables with the Wigner function of (9), can be represented as

$$\int \left[i\hbar\frac{\partial}{\partial t} + i\frac{\hbar}{m}p'\cdot\frac{\partial}{\partial X} - (U_1 + U_2 + U_3)\right] exp(ip'\tau)\rho_w(X,p',t)\,dp' = 0. \quad (10)$$

Here

$$U_1 = U(X + \tfrac{1}{2}\hbar\tau, t) - U(X - \tfrac{1}{2}\hbar\tau, t),$$
$$U_2 = U_{Lg}(X + \tfrac{1}{2}\hbar\tau, t) - U_{Lg}(X - \tfrac{1}{2}\hbar\tau, t), \quad (11)$$
$$U_3 = U_{rand}(X + \tfrac{1}{2}\hbar\tau, t) - U_{rand}(X - \tfrac{1}{2}\hbar\tau, t).$$

Besides the operation $\frac{\partial}{\partial\tau}$ is applied only to $exp(ip'\tau)$ that gives $ip'exp(ip'\tau)$.

If the terms U_2, U_3 are equal to naught, then the friction and noise can be excluded from the analysis, so equation (10) changes to simple equation (1). Multiplying (10) by $exp(ip\tau)\,/\,(2\pi)$ and integrating over τ, consider

$$\frac{1}{(2\pi)}\int \exp(i(p' - p)\tau)d\tau = \delta(p' - p)$$

The quantum-classical W-Lg equation can be found from (10)

$$\left(\frac{\partial}{\partial t} + \frac{p}{m}\frac{\partial}{\partial X}\right)\rho_w(X,p,t) = \frac{1}{i\hbar}\int (U_1 + U_2 + U_3)\exp(\mathrm{i}(\mathrm{p}' - \mathrm{p})\tau)\rho_w(X,p,t)d\tau dp'. \quad (12)$$

The right side of W-Lg (12) can be split into three integrals I_i in accordance with $U_i, i = 1,2,3$. Every term U_i is expanded into Taylor series in powers of $h\tau$. As a consequence, we have the expressions for U_i with the accuracy of cubic term $(h\tau)^3$

$$U_1 = \frac{\partial U(X,t)}{\partial X}\hbar\tau + \frac{1}{24}\frac{\partial^3 U(X,t)}{\partial X^3}(\hbar\tau)^3 + \cdots,$$
$$U_2 = \frac{f}{m}\left(\frac{\partial S(X,t)}{\partial X}\hbar\tau + \frac{1}{24}\frac{\partial^3 S(X,t)}{\partial X^3}(\hbar\tau)^3 + \cdots\right). \quad (13)$$

For formulating U_2 we have used (3) and the known relation between ψ-function and quantum action; the second term $W(t)$ can be ignored (see the note above), and then we have replaced x, x' by new variables $X + \frac{1}{2}\hbar\tau, X + \frac{1}{2}\hbar\tau$. Using the expansion of U_3 in powers of $\hbar\tau$, one can at once to receive the simply expression

$$U_3 = -\hbar\tau F_{rand}(t). \quad (14)$$

The next step suggests that the expansions are substituted by I_1, I_2, I_3, respectively; it is necessary to take into account integrating over variable τ

$$\frac{1}{(2\pi)}\int \tau\exp(i(p' - p)\tau)d\tau = -i\frac{\partial}{\partial p'}\delta(p' - p),$$
$$\frac{1}{(2\pi)}\int \tau^3\exp(i(p' - p)\tau)d\tau = i\frac{\partial^3}{\partial p'^3}\delta(p' - p).$$

Then, integrating over variable p' and using the rule of transfer of the derivative of $\delta(p' - p)$ to the Wigner function ρ_w, we have reached the final results:

$$I_1 = \frac{\partial U}{\partial X}\frac{\partial \rho_w}{\partial p} - \frac{\hbar^2}{24}\frac{\partial^3 U}{\partial X^3}\frac{\partial^3 \rho_w}{\partial p^3},$$
$$I_2 = \frac{f}{m}\left[\frac{\partial S}{\partial X}\frac{\partial \rho_w}{\partial p} - \frac{\hbar^2}{24}\frac{\partial^3 S}{\partial X^3}\frac{\partial^3 \rho_w}{\partial p^3}\right], \quad (15)$$
$$I_3 = -F_{rand}(t)\frac{\partial}{\partial p}\rho_w.$$

Here $F_{rand}(t) = -\partial U_{rand}/\partial X$ is caused by the random white noise; in addition, the first derivative $\partial S/\partial x$ defines the linear momentum p.

The substitution of all I_i into (15) allows rewrite (12) in the convenient form

$$\left[\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial X}\right]\rho_w(X,p,t)+\Phi(X,t)\frac{\partial}{\partial p}\rho_w(X,p,t)=-\frac{\hbar^2}{24}\frac{\partial^3}{\partial X^3}U_\Sigma\frac{\partial^3}{\partial p^3}\rho_w(X,p,t), \quad (16)$$

where

$$U_\Sigma = \left[U(X,t)+\frac{f}{m}S(X,t)\right],$$

$$\Phi(X,t)=-\frac{\partial U(X,t)}{\partial X}-\frac{f}{m}\frac{\partial S(X,t)}{\partial X}+F_{rand}(t).$$

The additional terms in this equation arise due to friction and random white noise; they complicate the original W-Lv. Equation (16) can be named as quantum-classical W-Lg equation. This equation can be rewritten if the potentials are replaced by forced F, F_{rand}, F_{fric}.

Note. Equations (8) and (16) were written for the Wigner function in the form $W(X,p,t)$ according to the limits $\tau \to 0$ and $X \to x$; they can be rewritten using $\rho_w(x,p,t)$.

3. Supplement: forces and relations

In the classical limit, the W-Lg equation is written as

$$\left(\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial X}+\Phi(X,t)\frac{\partial}{\partial p}\right)\rho_w(X,p,t)=0. \quad (17)$$

If in the expression for Φ, the force sum

$$-\frac{f}{m}\frac{\partial S(X,t)}{\partial X}+F_{rand}(t)$$

is equal to naught, then equation (17) assumes a simpler form

$$\left(\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial X}+F\frac{\partial}{\partial p}\right)\rho_w(X,p,t)=0. \quad (18)$$

Equaton (18) represents a classical statistical equation; it describes the evolution of classical distribution function which depends on coordinate and linear momentum.

Now we discuss the quantum-classical dynamical regime when the force sum (friction and random) is as before equal to zero. In this particular case one can rewrite the coefficient before the derivative $\frac{\partial^3}{\partial p^3}$ as

$$\frac{\hbar^2}{24}\frac{\partial^2}{\partial X^2}(F-F_{rand}).$$

The W-Lg equation can be written as follows

$$\left[\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial X}+F\frac{\partial}{\partial p}-\frac{\hbar^2}{24}\frac{\partial^2}{\partial X^2}(F-F_{rand})\frac{\partial^3}{\partial p^3}\right]\rho_w(X,p,t)=0.$$

Another special condition is of no less interest; it consists in

$$F_{fric}+F=0. \quad (19)$$

Can this equality hold for a quantum Duffing oscillator? So far this remains an open problem and more research is needed. However if condition (19) fulfilled, then the quantum correction disappears and W-Lg equation becomes classical

$$\left[\frac{\partial}{\partial t}+\frac{p}{m}\frac{\partial}{\partial X}+F_{rand}\frac{\partial}{\partial p}\right]\rho_w(X,p,t)=0. \quad (20)$$

Therefore, in the context of the W-Lg equation it can be expected that the quantum-to-classical transition takes place similarly as in [2] and the dynamical system behaves as in statistical classical mechanics. If the balance conditions (19) has been disturbed, then the quantum properties arise again and system becomes by quantum-classical one.

4. Conclusion

In this article, W-Lv and Sch-Lg-K equations are combined into the unified W-Lg equation. The new W-Lg equation allows investigate the dynamical effects in systems with friction and random white noise; it consists of classical transport terms and the quantum correction. The coefficients in the W-Lg equations themselves represent spatial derivatives of the classical

Langevin and random white noise potentials. The dynamical equalities for the forces caused by potentials show the possibility of switching from the quantum mode of motion to the classical one if an external force is proportional to the third power of coordinate as in Duffing oscillator. The classical part of the W-Lg contains also interesting information about the properties of dynamical systems.

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