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SPECIFICS OF SYMMETRY CONDITIONS IN GRADIENT ELASTICITY THEORIES

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Abstract. The conditions of symmetry of the common Mindlin types gradient theories of elasticity, which characterize the specificity of these theories in comparison with the classical theory of elasticity, are investigated. We discuss the symmetry conditions of the tensors of the moduli of elasticity of the sixth rank under the permutation of the differentiation indices, which are the consequence of the fact that the second derivatives of the displacement vector do not depend on the order of differentiation. The gradient distortion model and strain gradient model are considered, and the conditions for "variational equivalence" and the differences between these theories from the point of view of symmetry are established. The variational formulation of gradient elasticity of general form and the role of symmetry conditions in the formulation of boundary conditions are investigated. It is shown that for the correct formulation of applied boundary value problems, it is necessary to use the tensors of the elastic moduli of the sixth rank symmetric with respect to the permutation of the last two indices in each triple of indices, even if this symmetry is absent for formally constructed versions of gradient theories.

Keywords: strain gradient theory; elasticity; variational methods; classification of symmetries

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1. Introduction

It is known that in the classical linear elasticity, the Cauchy stress tensor $\boldsymbol{\sigma}$ is linearly related to the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ by equation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, where $\mathbf{C}$ is the fourth-rank tensor of elastic constants. In the general triclinic case, tensor $\mathbf{C}$ has 21 independent materials coefficients, whereas for isotropic materials this number reduces to 2. A more complicated situation takes place for gradient elasticity theories. The contemporary theory of strain gradient elasticity stems from the pioneering works of [1-3], and now it has been widely used for describing the size effects and nonlocal behavior observed in semicrystalline and nanostructures materials, geomaterials, biomaterials [4,5], for modeling the responses of small size cantilever beams and plates, carbon nanotubes and metal nanowires [6-7] in developing homogenization methods accounting for the second-order elastic effects [8,9], and also for bridging atomistic and continuum level simulations [10].

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In the (first) strain gradient elasticity, the Cauchy stress $\boldsymbol{\sigma}$ and double stress $\boldsymbol{\mu}$ in the general case are linearly related to strain $\boldsymbol{\varepsilon}$ and its gradient $\nabla\boldsymbol{\varepsilon}$ (for the definition and physical meaning of tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$, [11,12]. For gradient models of media, extremely many physical independent constants in the general case describe the mechanical properties and this makes it fundamentally impossible to use such models in applied problems. A very large number of materials coefficients emphasizes the richness of the theory, but it also presents obvious difficulties in extracting all those coefficients reliably from available experimental and simulation data as well as in finding solutions for specific theoretical and engineering problems.

Since the original papers by Mindlin and Toupin [1,2] on the general formulation of the isotropic strain gradient elasticity theory, a number of simplified gradient theories have been proposed, see, e.g., [3-8]. These simplified models have been developed assuming the reduced form of the total strain energy density and corresponding form of the high-order constitutive relations. The rational basis for the development of such simplified theories lies in using the symmetry conditions for the tensors of the gradient elastic moduli [13,14] and also in the possible orthogonal decomposition of the strain gradient tensor [15,16], such that some parts of the high-order kinematics in the points of the continuum media can remain or can be excluded. Obviously, the kinematic relations should be considered and implemented together with the physical parameters of the gradient models. As a result, the number of additional parameters in the simplified gradient theories can be reduced that may be important for the experimental validation of the models. The corresponding discussion on physical models and the correct variational formulation of these theories can be found in [17-22].

In this work, we continue to study the symmetry properties of the tensor of gradient elastic moduli, begun in works [23,24]. We shall study the physical significance and variational energy significance of the "order-of-differentiation" symmetry conditions in strain gradient elasticity. This symmetry condition stems from the fact that in elasticity one can interchange the order of differentiation in the components $u_{i,jk}$ of the second displacement gradient tensor $\nabla\nabla\mathbf{u}$, where u is the displacement vector. The energetic significance of this symmetry condition was already discussed in Section 5 of the seminal work of [2] and the recent works [23,24]. Free variational formulations have been commonly employed for deriving the field equations of strain gradient elasticity. However, because of the auxiliary conditions $u_{i,jk} = u_{i,kj}$, one can evidently not handle the variations $\delta u_{i,jk}$ as free, independent variables. Nevertheless, free variational formulations have commonly been used [1,12]. It is interesting and important to assess the variational significance of the order-of-differentiation symmetry condition of strain gradient elasticity.

This work is structured as follows. First, all the symmetry properties of the tensors of the elastic moduli of gradient theories, which are also characteristic of the classical theory of elasticity and gradient elasticity, are investigated successively. The features of the "order-of-differentiation" symmetry condition are defined. Then the structure of the gradient part of the potential energy density and its variation, which determines the structure of the boundary non-classical symmetry conditions, are considered. As a result, the question of whether it is essential to take these conditions into account in the formulation of boundary conditions and the formulation of mathematical models of media as a whole is decided. In the last section, we discuss the physical implications of the symmetry conditions of strain gradient elasticity and also formulate the results of our work which seem to us to be important for gradient elasticity theories and applications.

2. Symmetry conditions in strain gradient elasticity

The constitutive equations in the linear theory of elasticity are written in terms of the tensors of stresses σ_{ij} and strains ε_{ij}:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} . \tag{1}$$

For linear-gradient theory, the constitutive equations in linear gradient elasticity theory in some three-dimensional Cartesian coordinate system can also be written in terms of deformations [18,16]:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + B_{ijklm}\varepsilon_{kl,m} \quad \text{and} \quad \mu_{ijk} = B_{lmijk}\varepsilon_{lm} + A_{ijklmn}\varepsilon_{lm,n} , \tag{2}$$

where σ_{ij}, ε_{ij}, μ_{ijk} and $\varepsilon_{ij,k}$ are the components of tensors $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ and $\nabla\boldsymbol{\varepsilon}$.

On the other hand, the governing relations of gradient elasticity (2) can be written in terms of displacements:

$$\sigma_{ij} = C_{ijkl}u_{k,l} \quad \text{and} \quad \mu_{ijk} = A_{ijklmn}u_{l,mn} . \tag{3}$$

Consider the possible symmetry conditions. Since the issues discussed below are important from the point of view of gradient theories, we will consistently set out all the provisions concerning symmetry, even if they seem to be well known. Let us first consider one of the basic, fundamental symmetry properties for tensors of elastic moduli in the mechanics of media. The strain energy density $U(\boldsymbol{\varepsilon})$ of a homogeneous linear elastic material is given by:

$$U(\boldsymbol{\varepsilon}) = \int_{\mathbf{0}}^{\boldsymbol{\varepsilon}} \sigma_{ij}\, d\varepsilon_{ij}, \quad \text{where} \quad \sigma_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} . \tag{4}$$

The tensor of elastic constants C_{ijkl} in (4) can be restricted by assuming that $U(\boldsymbol{\varepsilon})$ is independent of the integration path followed in the strain space, and hence by requiring:

$$\frac{\partial^2 U}{\partial \varepsilon_{kl}\partial \varepsilon_{ij}} = \frac{\partial^2 U}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} \;\Rightarrow\; C_{ijkl} = C_{klij} . \tag{5}$$

Similarly, the strain-gradient energy density $W(\nabla\boldsymbol{\varepsilon})$ of linear elastic material is given by:

$$W(\nabla\boldsymbol{\varepsilon}) = \int_{\mathbf{0}}^{\nabla\boldsymbol{\varepsilon}} \mu_{ijk}\, d\varepsilon_{ij,k}, \quad \text{where} \quad \mu_{ijk} = \frac{\partial W}{\partial \varepsilon_{ij,k}} = C_{ijklmn}\varepsilon_{lm,n} ,$$

and then, the tensor C_{ijklmn} can also be restricted by assuming that $W(\nabla\boldsymbol{\varepsilon})$ is path-independent, and hence by requiring:

$$\frac{\partial^2 W}{\partial \varepsilon_{lm,n}\partial \varepsilon_{ij,k}} = \frac{\partial^2 W}{\partial \varepsilon_{ij,k}\partial \varepsilon_{lm,n}} \;\Rightarrow\; C_{ijklmn} = C_{lmnijk} . \tag{6}$$

Similar expressions can be written for the governing relations (3). Following (4)-(6) we can conclude, that the components A_{ijklmn}, B_{ijklm} and C_{ijkl} of the elastic stiffness tensors **A**, **B** and **C** obey the potentiality:

$$C_{ijkl} = C_{klij}, \quad A_{ijklmn} = A_{lmnijk} . \tag{7}$$

It is easy to show that for a tensor of the fifth rank, the symmetry associated with potentiality does not impose any additional properties. Note one more property of the gradient theory of deformations, which is related to the symmetry of the strain symmetry conditions:

$$C_{ijkl} = C_{ijlk}, \quad B_{ijklm} = B_{jiklm} = B_{ijlkm}, \quad A_{ijklmn} = A_{ijkmln} . \tag{8}$$

For centrosymmetric materials (i.e., for those with point groups involving an inversion center as one of their symmetry elements), the fifth-rank tensor ***B*** vanishes. As a result, the number of independent material coefficients is reduced and the constitutive relations (4) become uncoupled

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{and} \quad \mu_{ijk} = A_{ijklmn}\varepsilon_{lm,n} . \tag{9}$$

In what follows, for the sake of simplicity, we will study the symmetry properties of the sixth rank tensor A and will consider only centrosymmetric materials. We assume that both tensors C and A in (9) obey the conditions (7). Then the stresses $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ also satisfy similar symmetry conditions:

$$\sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \mu_{ijk} = \mu_{jik} .$$

Substituting $\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2$ into the governing relations (9), we find:

$$\sigma_{ij} = C_{ijkl}u_{k,l} \quad \text{and} \quad \mu_{ijk} = A_{ijklmn}u_{l,mn} . \tag{10}$$

Let us point out one more property of symmetry, which takes place only for the gradient theory of elasticity. It is obvious that in expression (10) the tensor of the second third rank $u_{i,jk}$ is symmetric with respect to the permutation of the indices of differentiation *j*, *k*, $u_{i,jk} = u_{i,kj}$. Moreover, the requirement of this symmetry is a necessary and sufficient condition for the existence of continuous first derivatives of the displacement vector, which, by definition, are arguments of the potential energy density in the variational formulation of gradient models. Therefore, such a requirement is fully justified. Otherwise, non-integrability of $u_{i,jk}$ takes place, which corresponds to the kinematics of media with defect fields. Due to this symmetry, which we will call the symmetry in the order of differentiation, only symmetric components of the tensor of elastic moduli of the sixth rank in the last pair of indices in the first and second triples of indices can be present in the constitutive equation for the tensor of the third rank μ_{ijk}:

$$A^{s}_{ijklmn} : A_{ijklmn} = A_{ikjlmn}. \quad A_{ijklmn} = A_{ikjlnm} . \tag{11}$$

Consequently, only the components of tensor A_{ijklmn} that are symmetric in the order of differentiation are physically essential. Indeed, tensors of the sixth rank are asymmetric in the indicated indices:

$$A^{a}_{ijklmn} : A_{ijklmn} = -A_{ikjlmn}. \quad A_{ijklmn} = -A_{ikjlnm} \tag{12}$$

can be arbitrary in (10). By virtue of the convolution, taking into account the symmetry of the tensor $u_{i,jk}$, the sum of all terms corresponding to the components A^{a}_{ijklmn} in (10) is identically equal to zero, $\mu_{ijk} = A^{a}_{ijklmn}u_{l,mn} = 0$.

It is obvious that for the gradient component of the potential energy density $w_g\left(u_{i,jk}\right)$ recorded in displacements, only the components of the tensor of elastic moduli of the sixth rank A^{s}_{ijklmn} are significant (energetically significant), and the components A^{a}_{ijklmn} are energetically insignificant:

$$2w_g\left(u_{i,jk}\right) = \mu_{ijk}u_{i,jk} = A_{ijklmn}u_{i,jk}u_{l,mn} = A^{s}_{ijklmn}u_{i,jk}u_{l,mn} . \tag{13}$$

Moreover, for any values A^{a}_{ijklmn} and for any $u_{i,jk}$, the following identity takes place

$$A^{a}_{ijklmn}u_{i,jk}u_{l,mn} \equiv 0 . \tag{14}$$

Therefore, equality (14) is a necessary condition for the absence of symmetry in the order of differentiation.

On the other hand, let us consider the quadratic form corresponding to the density of the gradient part of the potential energy recorded for distortions. We have:

$$u_{i,jk}A_{ijklmn}u_{l,mn}=u_{i,jk}\frac{1}{4}(A_{ijklmn}+A_{ikjlmn}+A_{ijklnm}+A_{ikjlnm})u_{l,mn}\,.$$

Consequently, for the potential energy written for integrable distortions, only the symmetric components under permutation of the differentiation indices of the sixth rank tensor are essential:

$$\hat{A}_{ijklmn}=\frac{1}{4}(A_{ijklmn}+A_{ikjlmn}+A_{ijklnm}+A_{ikjlnm})\,.$$

Consider a sixth-order tensor ***B***. In Levi-Civita index notations it can be represented by the components B_{ijklmn}. Assume that for any tensors of the third rank $u_{i,jh}$ symmetric with respect to the second and third indices the following equality holds:

$$u_{i,jk}B_{ijklmn}u_{l,mn}=0\,.$$

Then we state that the validity of the previously written equation implies that:

$$B_{ijklmn}=-B_{ijklnm}\,.$$

This last equality is a sufficient condition for $u_{i,jk}B_{ijklmn}u_{l,mn}=0$.

The correct necessary and sufficient characterization of the tensor ***B*** verifying the condition $u_{i,jk}B_{ijklmn}u_{l,mn}=0$ is the following one:

$$B_{ijklmn}+B_{ijklnm}+B_{ikjnlmn}+B_{ikjlnm}=0\,.$$

This condition can be proven to be necessary simply by choosing the third-order tensor in $u_{i,jk}B_{ijklmn}u_{l,mn}=0$ such that $u_{l,mn}=u_{l,nn}=1$ while all others $u_{i,jk}=0$.

It is also trivially sufficient as every tensor symmetric with respect to the second and third-order indices can be represented as a linear combination of the considered ones.

The following statement holds.

Lemma. In the gradient theory of distortions, for the correct variational formulation of the gradient model, it is sufficient that the tensor of moment stresses is symmetric in the last pair of indices

Proof. For tensors of the third rank μ_{ijk} and $u_{i,jk}$, there are decompositions into symmetric and asymmetric components with respect to the last indices (symmetry and asymmetry in the order of differentiation):

$$\mu_{ijk}=\frac{1}{2}\left(\mu_{ijk}+\mu_{ikj}\right)+\frac{1}{2}\left(\mu_{ijk}-\mu_{ikj}\right)=\hat{\mu}_{ijk}+\tilde{\mu}_{ijk}\,,$$
$$u_{i,jk}=\frac{1}{2}\left(u_{i,jk}+u_{i,kj}\right)+\frac{1}{2}\left(u_{i,jk}-u_{i,kj}\right)=\hat{u}_{i,jk}+\tilde{u}_{i,jk}\,. \qquad (15)$$

We consider the kinematics of media without defect fields, which is determined by the symmetric component of the second-rank tensor of the distortion gradient $\hat{u}_{i,jk}$. To formulate the kinematics of such a medium in variational modeling, it is necessary to introduce the kinematic connections:

$$\tilde{u}_{i,jk}=0\,. \qquad (16)$$

Consider the gradient part of the potential energy density $w_g\left(u_{i,jk}\right)$ and introduce the corresponding extended Lagrange functional $\overline{w}_g\left(u_{i,jk}\right)$, taking into account constraints (16), using the Lagrange multiplier technique:

$$\overline{w}_g\left(u_{i,jk}\right)=\mu_{ijk}\delta u_{i,jk}+\lambda_{ijk}\delta\tilde{u}_{i,jk}\,, \qquad (17)$$

where λ_{ijk} is the Lagrange multiplier tensor.

We transform the variational form (13), taking into account the expansions (15):

$$2\overline{w}_g\left(u_{i,jk}\right)=\mu_{ijk}\delta u_{i,jk}+\lambda_{ijk}\delta\tilde{u}_{i,jk}=(\hat{\mu}_{ijk}+\tilde{\mu}_{ijk})(\delta\hat{u}_{i,jk}+\delta\tilde{u}_{i,jk})+\lambda_{ijk}\delta\tilde{u}_{i,jk}=$$
$$=\hat{\mu}_{ijk}\delta\hat{u}_{i,jk}+\tilde{\mu}_{ijk}\delta\tilde{u}_{i,jk}+\lambda_{ijk}\delta\tilde{u}_{i,jk}=\hat{\mu}_{ijk}\delta u_{i,jk}+\tilde{\mu}_{ijk}\delta\tilde{u}_{i,jk}+\lambda_{ijk}\delta\tilde{u}_{i,jk}= \quad (18)$$
$$=\hat{\mu}_{ijk}\delta u_{i,jk}+(\tilde{\mu}_{ijk}+\lambda_{ijk})\delta\tilde{u}_{i,jk}\ .$$

In the last equality (18), in the first term, all $\delta u_{i,jk}$ are free variations. We assume that the tensor of the Lagrange multipliers is determined from the conditions $\tilde{\mu}_{ijk}+\lambda_{ijk}=0$. Then from (18), we have:

$$2\overline{w}_g\left(u_{i,jk}\right)=\hat{\mu}_{ijk}\delta u_{i,jk}\ .$$

The lemma is proved.

3. Properties of symmetry defined by deformation symmetry conditions and by the order of differentiation

Let us consider the symmetry conditions and establish the related structure of the tensors of the elastic moduli of the sixth rank. The general form of tensors of the sixth rank for an isotropic body can be constructed as an expansion in a system of 15 linearly independent basis tensors of the sixth rank e^{s}_{ijklmn}:

$$A_{ijklmn}=a_\alpha e^{\alpha}_{ijklmn}=a_1e^{1}_{ijklmn}+a_2e^{2}_{ijklmn}+...+a_{15}e^{15}_{ijklmn}, \quad (19)$$

where

$$e^{1}_{ijklmn}=\delta_{ij}\delta_{kl}\delta_{mn};\quad e^{2}_{ijklmn}=\delta_{ij}\delta_{km}\delta_{ln};\quad e^{3}_{ijklmn}=\delta_{ij}\delta_{kn}\delta_{lm};$$
$$e^{4}_{ijklmn}=\delta_{ik}\delta_{jl}\delta_{mn};\quad e^{5}_{ijklmn}=\delta_{ik}\delta_{jm}\delta_{ln};\quad e^{6}_{ijklmn}=\delta_{ik}\delta_{jn}\delta_{lm};$$
$$e^{7}_{ijklmn}=\delta_{il}\delta_{jk}\delta_{mn};\quad e^{8}_{ijklmn}=\delta_{il}\delta_{jn}\delta_{km};\quad e^{0}_{ijklmn}=\delta_{il}\delta_{jm}\delta_{kn};$$
$$e^{10}_{ijklmn}=\delta_{im}\delta_{jk}\delta_{ln};\quad e^{11}_{ijklmn}=\delta_{im}\delta_{jl}\delta_{kn};\quad e^{12}_{ijklmn}=\delta_{im}\delta_{jn}\delta_{kl};$$
$$e^{13}_{ijklmn}=\delta_{in}\delta_{jk}\delta_{lm};\quad e^{14}_{ijklmn}=\delta_{in}\delta_{jl}\delta_{km};\quad e^{15}_{ijklmn}=\delta_{in}\delta_{jm}\delta_{kl}\ .$$

Let us take into account the symmetry condition (7) for the tensor of the sixth rank, $A_{ijklmn}=A_{lmnijk}$. As a result, we obtain the following additional relations for the expansion coefficients (19):

$$a_1=a_{13},\ \ a_2=a_6,\ \ a_4=a_{10},\ \ a_{12}=a_{14}$$

and reduces the number of independent gradient elastic moduli to 11

$$A_{ijklmn}=a_1(e^{1}_{ijklmn}+e^{13}_{ijklmn})+a_2(e^{2}_{ijklmn}+e^{6}_{ijklmn})+$$
$$+a_3e^{3}_{ijklmn}+a_4(e^{4}_{ijklmn}+e^{10}_{ijklmn})+a_5e^{5}_{ijklmn}+a_7e^{7}_{ijklmn}+$$
$$+a_8e^{8}_{ijklmn}+a_9e^{9}_{ijklmn}+a_{11}e^{11}_{ijklmn}+a_{12}(e^{12}_{ijklmn}+e^{14}_{ijklmn})+a_{15}e^{15}_{ijklmn}\ .$$

Consequently, the number of basis tensors decreased to 11.

Additional consideration of symmetry in the order of differentiation (11), $A_{ijklmn}=A_{lmnijk}=A_{ijklnm}$ leads to additional connections between the elastic moduli in the last equality:

$$a_1=a_4=a_{10}=a_{13},\ \ a_2=a_3=a_5=a_6,\ \ a_8=a_9,\ \ a_{11}=a_{12}=a_{14}=a_{15}\ .$$

This reduces the number of independent basis tensors to five and the sixth rank elastic modulus tensor is determined by five physical constants $a_1,\ a_2, a_7, a_8, a_{11}$:

$$\begin{aligned} A_{ijklmn} &= a_1\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln}\right) \\ &+a_2\left(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm} + \delta_{ij}\delta_{kn}\delta_{lm} + \delta_{ik}\delta_{jm}\delta_{ln}\right) + a_7\delta_{il}\delta_{jk}\delta_{mn} \\ &+a_8\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km}\right) + a_{11}\left(\delta_{im}\delta_{jl}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right). \end{aligned} \tag{20}$$

The structure of the elastic moduli of the sixth rank (20) corresponds to the distortion gradient model followed from Mindlin Form I [2,3]. On the other hand, if, considering relations (19), we require the fulfillment of both potentiality conditions (7) and the strain gradient condition $A_{ijklmn} = A_{ijkmln}\left(= A_{jiklmn}\right)$, then the following connections between the modules are additionally established:

$a_1 = a_2,\ \ a_4 = a_5 = a_7,\ \ a_8 = a_{11},\ \ a_9 = a_{12} = a_{15}$.

As a result, the number of modules will also decrease to five, but the structure of the tensor of elastic moduli will be different:

$$\begin{aligned} A_{ijklmn} &= a_1\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}\right) + a_3\delta_{ij}\delta_{kn}\delta_{lm} \\ &+a_4\left(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn}\right) + a_8\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jl}\delta_{kn}\right) \\ &+a_9\left(\delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right). \end{aligned} \tag{21}$$

The structure of the moduli of elasticity of the sixth rank (21) corresponds to the strain gradient model followed from Mindlin Form II. It is obvious that in (20), as compared with (21), the expansion is carried out in terms of other basis tensors, since these expansions contain different fives of the elastic moduli a_1, a_3, a_4, a_8, a_9.

Let us now consider the potential energy densities constructed according to the two considered types of symmetry: the symmetry conditions on the order of differentiation (20) and the symmetry conditions connected with the symmetry of the strain tensor (21). For the elastic moduli (20) of the Mindlin form I, we find the expression for the moment tensor:

$$\begin{aligned} \mu_{ijk} &= A_{ijklmn}u_{l,mn} = a_1\left(\delta_{ij}u_{k,mm} + \delta_{jk}u_{m,mi} + \delta_{ik}u_{j,mm} + \delta_{jk}u_{n,ni}\right) + \\ &+ a_2\left(\delta_{ij}u_{n,nk} + \delta_{ik}u_{m,mj} + \delta_{ij}u_{m,mk} + \delta_{ik}u_{n,nj}\right) + a_7\delta_{jk}u_{i,mm} + \\ &+ 2a_8\left(u_{i,jk}\right) + 2a_{11}\left(u_{j,ik} + u_{k,ij}\right) = \\ &= a_1\left(\delta_{ij}\Delta u_k + 2\delta_{jk}\theta_{,i} + \delta_{ik}\Delta u_j\right) + 2a_2\left(\delta_{ij}\theta_{,k} + \delta_{ik}\theta_{,j}\right) + \\ &+ a_7\delta_{jk}\Delta u_i + 2a_8u_{i,jk} + 2a_{11}\left(u_{j,ik} + u_{k,ij}\right). \end{aligned} \tag{22}$$

For the elastic moduli (21) of the Mindlin form II, we will have a different form of moment stresses:

$$\begin{aligned} \sigma_{ijk} &= A_{ijklmn}u_{l,mn} = a_1\left(\delta_{ij}u_{k,mm} + \delta_{jk}u_{l,li} + \delta_{ij}u_{l,kl} + \delta_{ik}u_{l,lj}\right) + a_3\delta_{ij}u_{l,lk} + \\ &+ a_4\left(\delta_{ik}u_{j,mm} + \delta_{jk}u_{l,il} + \delta_{ik}u_{l,jl} + \delta_{jk}u_{i,mm}\right) + a_8\left(u_{i,jk} + u_{j,ik}\right) + \\ &+ a_9\left(u_{k,ij} + u_{j,ki} + u_{i,kj} + u_{k,ji}\right) = \\ &= a_1\left(\delta_{ij}\Delta u_k + \delta_{jk}\theta_{,i} + \delta_{ij}\theta_{,k} + \delta_{ik}\theta_{,j}\right) + a_3\delta_{ij}\theta_{,k} + \\ &+ a_4\left(\delta_{ik}\Delta u_j + \delta_{jk}\theta_{,i} + \delta_{ik}\theta_{,j} + \delta_{jk}\Delta u_i\right) + (a_8 + a_9)\left(u_{i,jk} + u_{j,ik}\right) + 2a_9u_{k,ij}. \end{aligned} \tag{23}$$

From a comparison of the moment stress tensors (22) and (23), it follows that the models constructed using the symmetry properties in the order of differentiation and the symmetry properties associated with the symmetry in the first two indices are different. For the distortion gradient model (Mindlin Form I), the moment tensor is symmetric in the last two indices, while for the strain gradient model (Mindlin Form II), the moment tensor has a "pairing" symmetry - symmetry in the first two indices.

The two types of symmetry under study are fundamentally different, because symmetry in the order of differentiation is fundamental and is associated with the existence of a gradient defect-free medium when the symmetry conditions for the tensor of the second derivatives of displacements are necessary and sufficient conditions for the continuity of distortions.

The violation of this symmetry leads to the appearance of energetically insignificant "extra" modules in the tensor of the sixth rank, which do not affect the form of potential energy. Nevertheless, it is not yet clear whether only energetically significant components of the sixth rank tensor with the indicated symmetry property should be retained when we construct the solutions.

Symmetry associated with pairing (symmetry in the first indices in the first and second triples of indices) is introduced in the construction of the gradient theory of deformations and is associated with the symmetry of the strain tensor. This is a different symmetry.

Consider again tensors of elastic moduli for the Mindlin form I with the properties $\left(A_{ijklmn} = A_{lmnijk} = A_{ijklnm}\right)$ and the Mindlin form II with properties $\left(A^*_{ijklmn} = A^*_{lmnijk} = A^*_{ijkmln}\right)$. Accordingly, for the Mindlin form I, the density of the gradient part of the potential energy is written in terms of the tensor of elastic moduli, which have the property of symmetry in the order of differentiation (11), (20):

$$w(u_{l,mn}) = \tfrac{1}{2} u_{i,jk} A^s{}_{ijklmn} u_{l,mn}\,. \tag{24}$$

For the Mindlin Form II, the density potential energy of the stain gradient model is written from another tensor of gradient properties of the sixth rank A^*_{ijklmn} that do not have properties (11):

$$w^*(\varepsilon_{lm,n}) = \tfrac{1}{2} \varepsilon_{ij,k} A^*_{ijklmn} \varepsilon_{lm,n}\,. \tag{25}$$

The two written forms of Mindlin's theory (24) and (25) are equivalent in the sense that each of them is written in terms of five physical constants (a_i are the moduli in the Mindlin Form I and a_i^* are the moduli in the Mindlin Form II) and the coefficients a_i and a_i^* can be expressed in terms of each other if, for example, we rewrite the quadratic form (24) in the terms of strains taking into account the identity $u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i}$. As a result of such identification, it is possible to obtain the same shapes for the potential energy density for any trial displacement field.

Remark. The identity $u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i}$ is considered to be known; nevertheless, we give proof of it. To do this, we first use the obvious expansion of the distortion tensor $u_{i,j}$ and write down the corresponding expression for the derivative of the distortion tensor:

$$u_{i,j} = \varepsilon_{ij} - \omega_a Э_{ija}, \quad u_{i,jk} = \varepsilon_{ij,k} - \omega_{a,k} Э_{ija}, \tag{26}$$

where $\omega_a = -(1/2) u_{p,q} Э_{pqa}$, $Э_{pqa}$ is the Levi-Civita tensor

Now we exclude the gradient tensor of the rotation vector $\omega_{a,k}$ from the second expression in (26) using the compatibility equation (the integrability condition):

$$u_{i,jk} Э_{jkr} = \varepsilon_{ij,k} Э_{jkr} - \omega_{a,k} Э_{ija} Э_{jkr} = 0\,.$$

If we take into account the properties of the Levi-Civita tensor $Э_{ija}Э_{jkr} = (\delta_{ak}\delta_{ir} - \delta_{ar}\delta_{ik})$, then the last equality allows us to obtain:

$$\omega_{a,k} = -\varepsilon_{kp,q}Э_{pqa}. \tag{27}$$

Substitution of $\omega_{a,k}$ with the help of (27) into expression (26) gives the required identity:

$$u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{kp,q}Э_{pqa}Э_{ija} = \varepsilon_{ij,k} + \varepsilon_{kp,q}(\delta_{pi}\delta_{qj} - \delta_{pj}\delta_{qi}) = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{kj,i}.$$

Let us return to the potential energy density and show that the Mindlin Form I can be rewritten in terms of strains in the Mindlin Form II, which makes it possible to formally express in the general case the modulus tensor of the Mindlin I form in terms of the moduli tensor of the Mindlin Form II Indeed, consider the Mindlin Form I:

$$w_V^I = A^I_{ijklmn}u_{i,jk}u_{l,mn}$$

and rewrite it taking into account the relations:

$$u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{kj,i}$$

$$u_{l,mn} = \varepsilon_{lm,n} + \varepsilon_{nl,m} - \varepsilon_{nm,l}.$$

As a result, we obtain an expression for the potential energy density expressed in terms of the elastic moduli of the Mindlin Form I, but written in terms of the derivatives of the strain tensor components, i.e. in the Mindlin Form II decomposition:

$$\begin{aligned} w^I &= A^I_{ijklmn}u_{i,jk}u_{l,mn} = A^I_{ijklmn}(\varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{kj,i})(\varepsilon_{lm,n} + \varepsilon_{nl,m} - \varepsilon_{nm,l}) = \\ &= A^I_{ijklmn}(\varepsilon_{ij,k}\varepsilon_{lm,n} + \varepsilon_{ki,j}\varepsilon_{lm,n} - \varepsilon_{kj,i}\varepsilon_{lm,n} + \varepsilon_{ij,k}\varepsilon_{nl,m} + \varepsilon_{ki,j}\varepsilon_{nl,m} - \varepsilon_{kj,i}\varepsilon_{nl,m} + \\ &- \varepsilon_{ij,k}\varepsilon_{nm,l} - \varepsilon_{ki,j}\varepsilon_{nm,l} + \varepsilon_{kj,i}\varepsilon_{nm,l}) = \\ &= A^I_{ijklmn}(\varepsilon_{ij,k}\varepsilon_{lm,n} + \varepsilon_{ki,j}\varepsilon_{lm,n} - \varepsilon_{kj,i}\varepsilon_{lm,n} + \varepsilon_{ij,k}\varepsilon_{nl,m} + \varepsilon_{ki,j}\varepsilon_{nl,m} - \varepsilon_{kj,i}\varepsilon_{nl,m} + \\ &- \varepsilon_{ij,k}\varepsilon_{nm,l} - \varepsilon_{ki,j}\varepsilon_{nm,l} + \varepsilon_{kj,i}\varepsilon_{nm,l}) = \\ &= [(A^I_{ijklmn} + A^I_{ikjlmn} - A^I_{kjilmn}) + (A^I_{ijkmnl} + A^I_{jkimnl} - A^I_{kjimnl}) - (A^I_{ijknml} + A^I_{jkinml} - A^I_{kjinml})]\varepsilon_{ij,k}\varepsilon_{lm,n}. \end{aligned}$$

Comparing the resulting form with the quadratic form written in terms of the moduli of the Mindlin Form II, $A^{II}_{ijklmn}\varepsilon_{ij,k}\varepsilon_{lm,n}$ we find, in the general case, the connection between the components of the moduli tensor of the Mindin Form I and the components of the moduli tensor of the Mindlin Form II:

$$A^{II}_{ijklmn} = (A^I_{ijklmn} + A^I_{ikjlmn} - A^I_{kjilmn}) + (A^I_{ijkmnl} + A^I_{jkimnl} - A^I_{kjimnl}) - (A^I_{ijknml} + A^I_{jkinml} - A^I_{kjinml}).$$

In conclusion of the section, we note that the apparent equivalence of quadratic forms (24) and (25) does not provide complete equivalence, because these forms are constructed using tensors of elastic moduli with different symmetry properties, which do not provide a formal identical transition from one symmetry to another.

Indeed, let us consider the definitions of the tensor A_{ijklmn} using the equality (20), and the definition of the tensor A^*_{ijklmn} using the equality (21), and construct a tensor of gradient elastic moduli $A^\#_{ijklmn}$ that simultaneously possess symmetry in the order of differentiation and symmetry in pairing $\left(A^\#_{ijklmn} = A^\#_{lmnijk} = A^\#_{ijklnm} = A^\#_{ijkmln}\right)$. We get:

$$\begin{aligned} A^\#_{ijklmn} = a_1 &\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}\right. \\ &\left.+\delta_{ij}\delta_{kn}\delta_{lm} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn}\right) \\ &+a_8\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{il}\delta_{jn}\delta_{km} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right). \end{aligned}$$

Obviously, the quadratic form that determines the potential energy density:

$$w^{\#} = \tfrac{1}{2}\varepsilon_{ij,k} A^{\#}_{ijklmn} \varepsilon_{lm,n}$$

is identical in the indicated sense to forms (24) and (25), but defines a model with complete symmetry, different from the two models mentioned above. For this model, completely correct from the point of view of symmetry properties, the third-rank moment tensor is symmetric in the first and second pairs of indices. Indeed, this model automatically satisfies the requirements of the symmetry theorem for the moment tensors and is simultaneously a stain gradient model. However, note that this theory imposes more stringent requirements on the gradient properties, which is reflected in the number of corresponding basis tensors and modules.

Next, we will consider the problem of the importance of fulfillment of the symmetry conditions in the order of differentiation as necessary conditions for the absence of defect fields in the gradient theory of elasticity. The following question arises, whether it is necessary to strictly achieve the fulfillment of these conditions in applied models if the tensor of modules of the sixth rank does not satisfy these conditions by construction or the presence of asymmetric components does not affect the solution of applied problems due to the fact that antisymmetric components are energetically insignificant (14).

4. Symmetry conditions in the order of differentiation and correctness criterion

We represent a tensor of the sixth rank in the form of a decomposition:

$$C_{ijkmnl} = C^{++}_{ijkmnl} + C^{+-}_{ijkmnl} + C^{-+}_{ijkmnl} + C^{--}_{ijkmnl} \tag{28}$$

$$\begin{cases} C^{++}_{ijkmnl} = (C_{ijkmnl} + C_{ikjmnl} + C_{ijkmln} + C_{ikjmln})/4 \\ C^{+-}_{ijkmnl} = (C_{ijkmnl} + C_{ikjmnl} - C_{ijkmln} - C_{ikjmln})/4 \\ C^{-+}_{ijkmnl} = (C_{ijkmnl} - C_{ikjmnl} + C_{ijkmln} - C_{ikjmln})/4 \\ C^{--}_{ijkmnl} = (C_{ijkmnl} - C_{ikjmnl} - C_{ijkmln} + C_{ikjmln})/4 \end{cases} \tag{29}$$

By the formulas (29) we introduce a tensor basis in the space of six rank tensors for the four distinguished orthogonal subspaces that take into account the symmetry in pairs of differentiation indices. For example, one of the basis tensors of the subspace C^{++}_{ijkmnl} is e^{++}_{ijkmnl}:

$$e^{++}_{ijkmnl} = (\delta_{ij}\delta_{km}\delta_{nl} + \delta_{ik}\delta_{jm}\delta_{nl} + \delta_{ij}\delta_{km}\delta_{nl} + \delta_{ik}\delta_{jm}\delta_{nl})/4 .$$

It is formed through the basic tensors $\overset{1}{e}_{ijkmnl}$, $\overset{4}{e}_{ijkmnl}$ given in definition (19), $e^{++}_{ijkmnl} = (\overset{1}{e}_{ijkmnl} + \overset{4}{e}_{ijkmnl})/2$. On the other side, one of the basis tensors of the subspace C^{+-}_{ijkmnl} is the tensor e^{+-}_{ijkmnl}, $e^{+-}_{ijkmnl} = (\delta_{ij}\delta_{km}\delta_{nl} + \delta_{ik}\delta_{jm}\delta_{nl} - \delta_{ij}\delta_{km}\delta_{nl} - \delta_{ik}\delta_{jm}\delta_{nl})/4$ etc.

The differentiation indices are the last two indices in each of the triples of indices in the sixth rank tensors under consideration. Orthogonality means that the convolution over any differentiation indices is zero if they are not the same.

The proof of the orthogonality of basis tensors from different subspaces follows directly from the identity:

$$C^{++}_{ijkmnl} C^{+-}_{ijkmnl} = (C_{ijkmnl} + C_{ikjmnl} + C_{ijkmln} + C_{ikjmln})(C_{ijkmnl} + C_{ikjmnl} - C_{ijkmln} - C_{ikjmln})/16 \equiv 0 ,$$

the validity of which can be verified after the implementation of the convolution operation.

By direct convolution of expansion (28) for tensor C_{ijkmnl} with a tensor of the sixth rank $R_{i,jk}R_{m,nl}$, one can verify that the potential energy of curvatures depends only on the moduli included in the definition of the tensor C^{++}_{ijkmnl}. Consequently, the modules that make up the tensors $C^{+-}_{ijkmnl}, C^{-+}_{ijkmnl}, C^{--}_{ijkmnl}$ are not limited by any relations. Note that, as shown earlier, among the components of the tensor C_{ijkmnl}, only the components of C^{++}_{ijkmnl} are physically and energetically significant.

Consider the variation of the gradient part of the potential energy:

$$\begin{aligned}\delta w_g &= \int_V \sigma_{ijk}\delta u_{i,jk}dV = \int_V \sigma_{ijk,jk}\delta u_i dV - \int_F [\sigma_{ijk,k}n_j\delta u_i - \sigma_{ijk}n_k\delta u_{i,j}]dF = \\ &= -\int_F [\sigma_{ijk,k}n_j + (\sigma_{ijk}n_k)_{,p}\delta^*_{jp} + 2H\sigma_{ijk}n_k n_j]\delta u_i dF + \int_F (\sigma_{ijk}n_k)n_j\delta(u_{i,p}n_p)dF + \\ &+ \oint [(\sigma_{ijk}n_k)\nu_j)]\delta u_i ds\,,\end{aligned} \tag{30}$$

where $H = -(1/2)(n_i)_{,j}\delta^*_{ij}, \quad \delta^*_{ij} = \delta_{ij} - n_i n_j$.

Writing (30), we took into account the following equality [25]:

$$\int_F (\sigma_{ijk}n_k\delta R_i)_{,p}\delta^*_{jp}dF = \oint (\sigma_{ijk}n_k)\nu_j\delta R_i ds - \int_F 2H\sigma_{ijk}n_k n_j\delta R_i dF, \quad H = -(1/2)(n_i)_{,j}\delta^*_{ij}.$$

Equality (30) can be rewritten completely in displacements, taking into account the relations of Hooke's law $\sigma_{ijk} = C_{ijklmn}u_{l,mn}$:

$$\begin{aligned}\delta w_g &= \int_V C_{ijklmn}u_{l,mnjk}\delta u_i dV - \int_F [C_{ijklmn}u_{l,mnk}n_j\delta u_i - C_{ijklmn}u_{l,mn}n_k\delta u_{i,j}]dF = \\ &= -\int_F [C_{ijklmn}u_{l,mnk}n_j + (C_{ijklmn}u_{l,mn}n_k)_{,p}\delta^*_{jp} + 2HC_{ijklmn}u_{l,mn}n_k n_j]\delta u_i dF + \\ &+ \int_F (C_{ijklmn}u_{l,mn}n_k)n_j\delta(u_{i,p}n_p)dF + \oint [(C_{ijklmn}u_{l,mn}n_k)\nu_j)]\delta u_i ds\,.\end{aligned} \tag{31}$$

Let us construct a correctness condition for the variational form of the potential energy density of gradient models. We say that the gradient model is correct if the boundary conditions do not include physically and energetically insignificant components of the elastic moduli A^a_{ijklmn}, (8) and, accordingly, the moment tensor $\tilde{\mu}_{ijk}$ asymmetric in the last indices (15). Recall that the components of the tensors of elastic moduli A_{ijklmn} are called physically and energetically insignificant if they satisfy the identities:

$A^a_{ijklmn}u_{l,mn} = 0$, $A^a_{ijklmn}u_{i,jk}u_{l,mn} \equiv 0$.

Let us write equality (31) first only in terms of physically and energetically significant (essential) components of the tensor of sixth rank models C^{++}_{ijkmnl}:

$$\begin{aligned}\delta w^s_g &= \int_V C^{++}_{ijkmnl}u_{l,mnjk}\delta u_i dV - \\ &- \int_F [C^{++}_{ijkmnl}u_{l,mnk}n_j + (C^{++}_{ijkmnl}u_{l,mn}n_k)_{,p}\delta^*_{jp} + 2HC^{++}_{ijkmnl}u_{l,mn}n_k n_j]\delta u_i dF + \\ &+ \int_F (C^{++}_{ijkmnl}u_{l,mn}n_k)n_j\delta(u_{i,p}n_p)dF + \oint [(C^{++}_{ijkmnl}u_{l,mn}n_k)\nu_j)]\delta u_i ds\,.\end{aligned} \tag{32}$$

On the other hand, we keep in (31) the complete expansion for the tensor of elastic moduli of the sixth rank (28). In this case, we get:

$$\delta w_g^{full} = \int_V C^{++}_{ijklmn} u_{l,mnjk} \delta u_i dV -$$
$$- \int_F [(C^{++}_{ijklmn} + C^{-+}_{ijklmn}) u_{l,mnk} n_j + ((C^{++}_{ijklmn} + C^{-+}_{ijklmn}) u_{l,mn} n_k)_{,p} \delta^*_{jp} + 2H C^{++}_{ijklmn} u_{l,mn} n_k n_j] \delta u_i dF + \quad (33)$$
$$+ \int_F ((C^{++}_{ijklmn} + C^{-+}_{ijklmn}) u_{l,mn} n_k) n_j \delta(u_{i,p} n_p) dF + \oint [((C^{++}_{ijklmn} + C^{-+}_{ijklmn}) u_{l,mn} n_k) \nu_j)] \delta u_i ds \,.$$

Comparing the variational forms (32) and (33), we find:

$$\delta w_g^{full} - \delta w_g^s = -\int_F [C^{-+}_{ijklmn} u_{l,mnk} n_j + (C^{-+}_{ijklmn} u_{l,mn} n_k)_{,p} \delta^*_{jp\,j}] \delta u_i dF +$$
$$+ \oint [C^{-+}_{ijklmn} u_{l,mn} n_k \nu_j] \delta u_i ds \,. \quad (34)$$

Equation (34) shows that the formulation of the boundary value problem without taking into account the symmetry of the tensor of elastic moduli with respect to the differentiation indices can lead to significant errors due to the possible appearance in the boundary conditions of physically and energetically insignificant components of the elastic moduli, which can take any values.

Thus, to eliminate parasitic terms in the static boundary conditions, the following correctness condition should be required:

$$B = \delta w_g^{full} - \delta w_g^s = -\int_F [C^{-+}_{ijklmn} u_{l,mnk} n_j + (C^{-+}_{ijklmn} u_{l,mn} n_k)_{,p} \delta^*_{jp\,j}] \delta u_i dF +$$
$$+ \oint [C^{-+}_{ijklmn} u_{l,mn} n_k \nu_j] \delta u_i ds = 0 \,. \quad (35)$$

Following (35), it can be argued that errors are associated with uncompensated energetically insignificant components C^{-+}_{ijkmnl} and can formally appear only under boundary conditions for stresses on the body surface and ribs.

For greater clarity, we write down the variational form (31) in terms of the double stresses μ_{ijk}:

$$\delta w_g = -\int_V \mu_{ijk,jk} \delta R_i dV +$$
$$= \int_F [\mu_{ijk,k}) n_j + [(\mu_{ijk} n_k)]_{,p} \delta^*_{jp} + 2H \mu_{ijk} n_k n_j] \delta R_i dF - \int_F (\mu_{ijk} n_k) n_j \delta(R_{i,p} n_p) dF -$$
$$- \oint [(\mu_{ijk} n_k) \nu_j)] \delta R_i ds \,.$$

In the last equality, we represent the tensor of moments in the form of a sum of symmetric and asymmetric tensors with respect to the last pair of indices $\mu_{ijk} = \hat{\mu}_{ijk} + \tilde{\mu}_{ijk}$ (15). After transformations of the convolution, we get:

$$\delta w_g = -\int_V \hat{\mu}_{ijk,jk} \delta R_i dV +$$
$$= \int_F [(\hat{\mu}_{ijk} + \tilde{\mu}_{ijk})_{,k} n_j + [(\hat{\mu}_{ijk} + \tilde{\mu}_{ijk}) n_k)]_{,p} \delta^*_{jp} + 2H \hat{\mu}_{ijk} n_k n_j] \delta R_i dF - \quad (36)$$
$$- \int_F (\hat{\mu}_{ijk} n_k) n_j \delta(R_{i,p} n_p) dF - \oint [(\hat{\mu}_{ijk} + \tilde{\mu}_{ijk}) n_k \nu_j)] \delta R_i ds \,.$$

Let us take into account that physically insignificant moments $\tilde{\mu}_{ijk}$ should not enter into the expression for the variation of the density of the gradient part of the potential energy, so as not to introduce, generally speaking, arbitrarily large errors in the boundary conditions. For simplicity consider the case when the surface of the body is formed by planes. In this case $H = 0$ and the expression (36) takes on a simpler form:

$$\delta w_g = -\int_V \hat{\mu}_{ijk,jk}\delta R_i dV +$$
$$+\int_F [(\hat{\mu}_{ijk}+\tilde{\mu}_{ijk})_{,k} n_j + [(\hat{\mu}_{ijk}+\tilde{\mu}_{ijk})n_k)]_{,p}\delta^*_{jp}]\delta R_i dF -$$
$$-\int_F (\hat{\mu}_{ijk}n_k)n_j\delta(R_{i,p}n_p)dF - \oint [(\hat{\mu}_{ijk}+\tilde{\mu}_{ijk})n_k v_j)]\delta R_i ds = \quad (37)$$
$$= -\int_V \hat{\mu}_{ijk,jk}\delta R_i dV + \int_F [\hat{\mu}_{ijk,k}n_j + (\hat{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta R_i dF - \oint [\hat{\mu}_{ijk}n_k v_j)]\delta R_i d +$$
$$+\int_F [\tilde{\mu}_{ijk,k}n_j + (\tilde{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta R_i dF - \oint [\tilde{\mu}_{ijk}n_k v_j)]\delta R_i ds \ .$$

In equality (37), the last row contains terms that can give erroneous components in the boundary conditions for gradient elasticity, because it contains physically and energetically insignificant moments. In the correct formulation of boundary value problems of gradient elasticity, these terms should be absent. Therefore, the correctness conditions have the form:

$$\int_F [\tilde{\mu}_{ijk,k}n_j + (\tilde{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta R_i dF - \oint [\tilde{\mu}_{ijk}n_k v_j)]\delta R_i ds = 0 \ . \quad (38)$$

Consider the first term in expression (38) for the common case when $H \neq 0$

$$\int_F [\tilde{\mu}_{ijk,k}n_j + (\tilde{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta u_i dF =$$
$$= \int_F [\tilde{\mu}_{ijk,k}n_j + (\tilde{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta u_i dF = \int_F [\tilde{\mu}_{ijk,p}(n_j\delta_{kp} + n_k\delta^*_{jp}) + \tilde{\mu}_{ijk}n_{k,p}\delta^*_{jp}]\delta u_i dF =$$
$$= \int_F [\tilde{\mu}_{ijk,p}(n_j\delta^*_{kp} + n_k\delta^*_{jp}) + \tilde{\mu}_{ijk,p}(n_j n_k)n_p + \tilde{\mu}_{ijk}n_{k,p}\delta^*_{jp}]\delta u_i dF = \quad (39)$$
$$= \int_F [\tilde{\mu}_{ijk,p}(n_j\delta^*_{kp} + n_k\delta^*_{jp})\delta u_i dF + \int_F [\tilde{\mu}_{ijk,p}(n_j n_k)n_p + \tilde{\mu}_{ijk}n_{k,p}\delta^*_{jp}]\delta u_i dF \ .$$

Obviously, the first term in the last line of (39) is equal to zero due to the convolution of the symmetric $(n_j\delta^*_{kp} + n_k\delta^*_{jp})$ and antisymmetric $\tilde{\mu}_{ijk,p}$ tensors. In the case when the surface of the body is formed by a family of planes, the second term is also equal to zero, because $n_{k,p} = 0$. Consequently, if the surface is formed by planes, then there are no erroneous terms containing uncompensated $\tilde{\mu}_{ijk}$ in the static boundary condition because the first term in (38) is equal to zero.

In the general case, however, equality (39) shows that erroneous terms in the classical boundary condition (with a variation of δu_i) can also appear in the term characterizing the curvature of the surface through the plane divergence of the normal vector:

$$\int_F [\tilde{\sigma}_{ijk}n_{k,p}\delta^*_{jp}]\delta u_i dF \neq 0 \ .$$

Equations (35), (39) allow us to formulate the following statement:

1. For any gradient theory of elasticity, the correct static boundary conditions on the force vector t_i and static conditions for meniscus forces $f_i = C_{ijklmn}u_{l,mn}n_k v_j$ should be formulated using only the gradient elastic moduli tensor C^{++}_{ijklmn}, which is symmetric with respect to the order of differentiation (i.e., symmetric with respect to the last indices in each triple), even if these elastic moduli do not have these properties for the model under consideration:

$$(C_{ijkl}u_{k,l} - C^{++}_{ijklmn}u_{l,mnk})n_j + (C^{++}_{ijklmn}u_{l,mn}n_k)_{,p}\delta^*_{jp} + \\ + 2HC_{ijklmn}u_{l,mn}n_k n_j = t^0_i, \quad x_i \in F = F_1 \cup F_2 ... \cup F_i \ , \tag{40}$$

$$C^{++}_{ijklmn}u_{l,mn}n_k v_j = f^0_i, \quad x_i \in S = s_1 \cup s_2 ... \cup s_i \ . \tag{41}$$

If the surface of the body is formed by planes, then the relation (40) is fulfilled identically, but the condition (41) should be kept in mind.

2. If the boundary static conditions are formulated in forces and moments, then, following conditions (38), both the boundary condition for the vector of forces on the surface and the boundary condition for the vector of forces of the meniscus on the contours should be written in terms of the symmetric components of the moment tensor relative to the last indices $\hat{\mu}_{ijk}$.

Remark. Note that in the integral over the surface:

$$\int_F [\tilde{\mu}_{ijk,k}n_j + (\tilde{\mu}_{ijk}n_k)_{,p}\delta^*_{jp}]\delta u_i dF \tag{42}$$

the convolution $\int_F [\sigma\tilde{\mu}_{ijk,p}(n_j\delta^*_{kp} + n_k\delta^*_{jp})\delta u_i dF = 0$ can be equal to zero only by a combination of both terms in the general case.

This remark seems important if we take into account that the second term in (42) gives a correction to the boundary conditions formulated for semiclassical force conditions by taking into account the moment force factors.

However, there is the set of gradient media models [25], for which moment corrections $(\sigma_{ijk}n_k)_{,p}\delta^*_{jp}$ in the boundary conditions are absent and static boundary conditions are written in the forms of conditions of classical elasticity through quasi-classical stresses (total stresses) τ_{ij}:

$$\tau_i = \tau_{ij}n_j, \ \ \tau_{ij} = \sigma_{ij} - \mu_{ijk,k} .$$

In this case, the expressions $\tilde{\mu}_{ijk,k}n_j \neq 0$ appear in the boundary conditions. They are nonzero and can be arbitrary. Therefore, the requirement of symmetry in the order of differentiation (symmetry of the moment tensor with respect to the last indices) becomes essential, and the correct force condition, written taking into account the indicated symmetry property, has the form

$$\tau_i = (\sigma_{ij} - \hat{\mu}_{ijk,k})n_j = t^0_i ,$$

t^0_i is a known function on the surface of the body.

5. Conclusions

The analysis of the symmetry conditions for the tensor components of the generalized elastic moduli of the gradient elasticity is given, the symmetry conditions are established, which are characteristic only for the gradient models. It is shown that the symmetry conditions may not be taken into account when we construct the solution by the direct method of minimizing the generalized Lagrange functional, because the discussed specific symmetry conditions are energetically insignificant and are not formally included in the definition of the potential energy density due to convolution procedure. A nontrivial result has been established that shows that energetically insignificant components of the tensor of gradient elastic moduli can lead to an erroneous form of static boundary conditions and conditions on the contours – intersection lines of piecewise smooth surfaces in the formal formulation of the mathematical model of gradient elasticity without taking into account additional symmetry conditions. The procedure that allows always us to obtain the correct boundary conditions for arbitrary variants of gradient elasticity theories is proposed.

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