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SPINOR MAXWELL EQUATIONS IN RIEMANNIAN SPACE-TIME AND THE GEOMETRICAL MODELING OF CONSTITUTIVE RELATIONS IN ELECTRODYNAMICS

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Abstract. It is known that vacuum Maxwell equations being considered on the background of any pseudo-Riemannin space-time may be interpreted as Maxwell equations in Minkowski space but specified in some effective medium, which constitutive relations are determined by metric of the curved space-time. In that context, we have considered de Sitter, anti de Sitter, and Schwarzschild models. Also we have studied hyperbolic Lobachevsky and spherical Riemann models, parameterized by coordinates with spherical or cylindric symmetry. We have proved that in all the examined cases, effective tensors and of electric permittivity $\varepsilon_{ij}(x)$ and magnetic permeability $\mu_{ij}(x)$ obey one the same condition: $\varepsilon_{ij}(x)\mu_{jk}(x)=\delta_{ik}$. Expressions for tensors $\varepsilon_{ij}(x)$ and $\mu_{jk}(x)$ are simple, but this simplicity is misleading. For each curved space-time model we are to solve Maxwell equations separately and anew. We have constructed the solutions, applying Maxwell equations in spinor form.

Keywords: constitutive relations, electrodynamics, geometrical modeling, Maxwell equations, Riemannian space-time, spherical and cylindric symmetry, spinor formalism

1. Riemannian geometry and modeling the constitutive relations for special media

It is known that vacuum Maxwell equations being considered on the background of any pseudo-Riemannin space-time may be interpreted as Maxwell equations in Minkowski space but specified in some effective medium, which constitutive relations are determined by metric of the curved space-time [1-44]. Detailed treatment of such a possibility for quasi-Cartesian coordinates was given in Ref. [45].

Let us start with Maxwell equations in a medium when using some curvilinear coordinates (x^σ) and corresponding metric $G_{\alpha\beta}(x)$. So we have

$$\partial_\alpha F_{\beta\gamma}+\partial_\beta F_{\gamma\alpha}+\partial_\gamma F_{\alpha\beta}=0, \quad \frac{1}{\sqrt{-G}}\partial_\beta\sqrt{-G}\,H^{\alpha\beta}=J^\alpha, \tag{1}$$

where $G=\det[G_{\alpha\beta}(x)]$ stands for the determinant of a metric tensor. We are to use two electromagnetic tensors, $F_{\alpha\beta}(x)$ and $H^{\alpha\beta}(x)$, related to each other by means of some constitutive equations. Also, let us assume that a certain Riemannian space-time model is parameterized by formally similar coordinates (x^σ) with metric $g_{\alpha\beta}(x)$. Maxwell vacuum equations in that space-time have the form

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$$\partial_\alpha f_{\beta\gamma} + \partial_\beta f_{\gamma\alpha} + \partial_\gamma f_{\alpha\beta} = 0, \quad \frac{1}{\sqrt{-g}} \partial_\beta \sqrt{-g}\, h^{\beta\alpha} = j^\alpha, \tag{2}$$

$$h_{\alpha\beta}(x) = \varepsilon_0 f_{\alpha\beta}(x), \quad h^{\alpha\beta}(x) = \varepsilon_0 g^{\alpha\rho}(x) g^{\beta\sigma}(x)\, f_{\rho\sigma}(x), \tag{3}$$

in (3) we have specified vacuum constitutive relations; $g = \det[g_{\alpha\beta}(x)]$. The second equation in (2) may be rewritten as

$$\frac{\sqrt{-G}}{\sqrt{-g}} \frac{1}{\sqrt{-G}} \partial_\beta \sqrt{-G} \frac{\sqrt{-g}}{\sqrt{-G}} h^{\beta\alpha} = j^\alpha. \tag{4}$$

Let us define the new variables

$$F_{\alpha\beta}(x) = f_{\alpha\beta}(x), \quad H^{\beta\alpha}(x) = \frac{\sqrt{-g}}{\sqrt{-G}} h^{\beta\alpha}(x), \; J^\alpha(x) = \frac{\sqrt{-g(x)}}{\sqrt{-G(x)}} j^\alpha(x), \tag{5}$$

then Eq. (2) may be understood as Maxwell equations of form (1) in the flat space-time:

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \quad \frac{1}{\sqrt{-G}} \partial_\beta \sqrt{-G}\, H^{\alpha\beta} = J^\alpha; \tag{6}$$

the relation between two electromagnetic tensors is described by the formula

$$H^{\alpha\beta}(x) = \varepsilon_0 \frac{\sqrt{-g(x)}}{\sqrt{-G(x)}} g^{\alpha\rho}(x) g^{\beta\sigma}(x)\, F_{\rho\sigma}(x). \tag{7}$$

In that context, we will consider space-time models with event horizon; all of them have a metric of one the same structure. We restrict ourselves to spherically symmetric case, for definiteness we keep in mind de Sitter, anti de Sitter, and Schwarzschild models:

$$dS^2 = \varphi dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - \varphi^{-1} dr^2, \quad dS_0^2 = dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - dr^2. \tag{8}$$

Because the metric determinant of the models is the same, $G(x) = g(x)$, the effective constitutive relations (7) become simpler

$$H^{\alpha\beta}(x) = \varepsilon_0\, g^{\alpha\rho}(x) g^{\beta\sigma}(x) F_{\rho\sigma}(x). \tag{9}$$

Taking into account the metric tensor(numerating coordinates as $x^\alpha = (t, \theta, \phi, r)$):

$$g^{\beta\alpha} = \begin{vmatrix} 1/\varphi & 0 & 0 & 0 \\ 0 & -1/r^2 & 0 & 0 \\ 0 & 0 & -1/r^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -\varphi \end{vmatrix}, \tag{10}$$

we get the constitutive relations modified by the Riemannian metric:

$$H^{0j} = \varepsilon_0 g^{00} g^{jj} F_{0j},$$

$$H^{01} = \frac{1}{\varphi}\left(-\frac{\varepsilon_0}{r^2}, \quad F_{01}\right), \quad H^{02} = \frac{1}{\varphi}\left(-\frac{\varepsilon_0}{r^2 \sin^2\theta} F_{02}\right), \quad H^{03} = -\varepsilon_0 F_{03},$$

$$H^{ij} = \varepsilon_0 g^{ii} g^{jj} F_{ij},$$

$$H^{23} = \varphi\left(\frac{\varepsilon_0}{r^2 \sin^2\theta} F_{23}\right), \quad H^{31} = \varphi\left(\frac{\varepsilon_0}{r^2} F_{31}\right), \quad H^{12} = \left(\frac{\varepsilon_0}{r^4 \sin^4\theta} F_{12}\right).$$

The last formulas can be rewritten differently:

$$D^\theta = \frac{1}{\varphi} \varepsilon_0 \frac{E_\theta}{r^2}, \quad D^\phi = \frac{1}{\varphi} \varepsilon_0 \frac{E_\phi}{r^2 \sin^2\theta}, \quad D^r = \varepsilon_0 E_r;$$

$$H^\theta = \varphi \frac{1}{\mu_0} \frac{B_\theta}{r^2 \sin^2\theta}, \quad H^\phi = \varphi \frac{1}{\mu_0} \frac{B_\phi}{r^2}, \quad H^r = \frac{1}{\mu_0} \frac{B_r}{r^4 \sin^4\theta}; \tag{11}$$

where we have used the identity $c^2 = 1/\varepsilon_0\mu_0$ and the definitions for two tensors:

$$(F_{ab}) = \begin{vmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & +cB^2 \\ E^2 & +cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & +cB^1 & 0 \end{vmatrix}, \quad (H^{ab}) = \begin{vmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3/c & +H^2/c \\ D^2 & +H^3/c & 0 & -H^1/c \\ D^3 & -H^2/c & +H^1/c & 0 \end{vmatrix}. \tag{12}$$

The constitutive relations (11) can be presented with the help of electric permittivity and magnetic permeability tensors:

$$D^i(x) = \varepsilon_0\varepsilon_{ij}(x)E^{(j)}(x), \quad H^i(x) = \frac{1}{\mu_0}\mu_{ij}(x)B^{(j)}(x),$$

$$[\varepsilon_{ij}(r)] = \begin{vmatrix} \varphi^{-1}(r) & 0 & 0 \\ 0 & \varphi^{-1}(r) & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad [\mu_{ij}(r)] = \begin{vmatrix} \varphi(r) & 0 & 0 \\ 0 & \varphi(r) & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{13}$$

where

$$E^{(j)} = \frac{E_j}{h_j}, \quad B^{(1)} = \frac{B_1}{h_2h_3}, \quad B^{(2)} = \frac{B_3}{h_3h_1}, \quad B^{(3)} = \frac{B_3}{h_1h_2}, \tag{14}$$

and h_j are determined by Minkowski metric (7) in the spherical coordinates:

$$dS_0^2 = dt^2 - h_1d\theta^2 - h_2d\phi^2 - h_3dr^2. \tag{15}$$

Simplicity of relations (13) is misleading; in fact, for each of curved space-time models we are to solve Maxwell equations separately and anew. Let us specify the effective constitutive equations for four models:

Minkowski, $r \in (0, +\infty)$,

$$\varepsilon = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

de Sitter, $r \in (0, +1)$,

$$\varepsilon = \begin{vmatrix} (1-r^2)^{-1} & 0 & 0 \\ 0 & (1-r^2)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1-r^2) & 0 & 0 \\ 0 & (1-r^2) & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

anti de Sitter, $r \in (0, +\infty)$,

$$\varepsilon = \begin{vmatrix} (1+r^2)^{-1} & 0 & 0 \\ 0 & (1+r^2)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1+r^2) & 0 & 0 \\ 0 & (1+r^2) & 0 \\ 0 & 0 & 1 \end{vmatrix};$$

Schwarzschild, $r \in (1, +\infty)$,

$$\varepsilon = \begin{vmatrix} (1-1/r)^{-1} & 0 & 0 \\ 0 & (1-1/r)^{-1} & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mu = \begin{vmatrix} (1-1/r) & 0 & 0 \\ 0 & (1-1/r) & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Direct comparison is possible only for Minkowski and anti de Sitter models, due to the same region for the radial coordinates.

Such an interpretation is possible for other curved spaces as well. Let us discuss hyperbolic Lobachevsky and spherical Riemann models being compared with

Minkowski one:

$$dS_0^2 = dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - dr^2,$$
$$dS^2 = dt^2 - \sinh^2 r\, d\theta^2 - \sinh^2 r \sin^2\theta\, d\phi^2 - dr^2, \tag{16}$$
$$dS^2 = dt^2 - \sin^2 r\, d\theta^2 - \sin^2 r \sin^2\theta\, d\phi^2 - dr^2.$$

For Lobachevsky space we have

$$H^{0j} = \frac{\sinh^2 r}{r^2}\varepsilon_0 g^{jj} F_{0j}, \quad H^{03} = -\frac{\sinh^2 r}{r^2}\varepsilon_0 F_{03},$$
$$H^{01} = -\frac{\sinh^2 r}{r^2}\frac{\varepsilon_0}{\sinh^2 r} F_{01}, H^{02} = -\frac{\sinh^2 r}{r^2}\frac{\varepsilon_0}{\sinh^2 r \sin^2\theta} F_{02},$$
$$H^{ij} = \frac{\sinh^2 r}{r^2}\varepsilon_0 g^{ii} g^{jj} F_{ij}, \quad H^{12} = \frac{\sinh^2 r}{r^2}\frac{\varepsilon_0}{\sinh^4 r \sin^4\theta} F_{12},$$
$$H^{23} = \frac{\sinh^2 r}{r^2}\frac{\varepsilon_0}{\sinh^2 r \sin^2\theta} F_{23}, H^{31} = \frac{\sinh^2 r}{r^2}\frac{\varepsilon_0}{\sinh^2 r} F_{31},$$

or differently

$$H^{01} = -\frac{\varepsilon_0}{r^2} F_{01}, \quad H^{02} = -\frac{\varepsilon_0}{r^2 \sin^2\theta} F_{02}, \quad H^{03} = -\frac{\sinh^2 r}{r^2}\varepsilon_0 F_{03},$$
$$H^{23} = \frac{\varepsilon_0}{r^2 \sin^2\theta} F_{23}, \quad H^{31} = \frac{\varepsilon_0}{r^2} F_{31}, \quad H^{12} = \frac{r^2}{\sinh^2 r}\frac{\varepsilon_0}{r^4 \sin^4\theta} F_{12}, \tag{17}$$

which may be rewritten in terms of effective tensors as follows:

$$D^i(x) = \varepsilon_0 \varepsilon_{ij}(x) E^{(j)}(x), \quad H^i(x) = \frac{1}{\mu_0}\mu_{ij}(x) B^{(j)}(x),$$
$$[\varepsilon_{ij}(r)] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sinh^2 r/r^2 \end{vmatrix}, \quad [\mu_{ij}(r)] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2/\sinh^2 r \end{vmatrix}. \tag{18}$$

For spherical Riemann space we have similar results with one change:
$\sinh^2 r,\ r \in (0, +\infty) \Rightarrow \sin^2 r,\ r \in (0, \pi)$.

Let us consider examples with cylindric symmetry. For flat Minkowski space we have (numerating coordinates as $x^\alpha = (t, r, \phi, z)$)

$$G_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}; \tag{19}$$

in spherical Riemann space an analogous metric is

$$g_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\sin^2 r & 0 \\ 0 & 0 & 0 & -\cos^2 z \end{vmatrix}, \quad r \in (0, \frac{\pi}{2}), \quad z \in (-\frac{\pi}{2}, +\frac{\pi}{2}). \tag{20}$$

Generated by this metric constitutive relations have the form

$$H^{\alpha\beta} = \varepsilon_0 \frac{\sin r \cos z}{r} g^{\alpha\rho} g^{\beta\sigma} F_{\rho\sigma}, \tag{21}$$

that is

$$H^{0j} = \frac{\sin r \cos z}{r} \varepsilon_0 g^{jj} F_{0j}, \quad H^{03} = -\frac{\sin r \cos z}{r} \frac{\varepsilon_0}{\cos^2 z} F_{03},$$

$$H^{01} = -\frac{\sin r \cos z}{r} \varepsilon_0 F_{01}, \quad H^{02} = -\frac{\sin r \cos z}{r} \frac{\varepsilon_0}{\sin^2 r} F_{02},$$

$$H^{ij} = \frac{\sin r \cos z}{r} \varepsilon_0 g^{ii} g^{jj} F_{ij}, \quad H^{12} = \frac{\sin r \cos z}{r} \frac{\varepsilon_0}{\sin^2 r} F_{12},$$

$$H^{23} = \frac{\sin r \cos z}{r} \frac{\varepsilon_0}{\sin^2 r \cos^2 z} F_{23}, \quad H^{31} = \frac{\sin r \cos z}{r} \frac{\varepsilon_0}{\cos^2 r} F_{31}.$$

This result may be presented with the help of two effective tensors as follows

$$D^i(x) = \varepsilon_0 \varepsilon_{ij}(x) E^{(j)}(x), \quad [\varepsilon_{ij}(r,z)] = \begin{vmatrix} \frac{\sin r \cos z}{r} & 0 & 0 \\ 0 & \frac{r \cos z}{\sin r} & 0 \\ 0 & 0 & \frac{\sin r}{r \cos z} \end{vmatrix},$$

$$H^i(x) = \frac{1}{\mu_0} \mu_{ij}(x) B^{(j)}(x), \quad [\mu_{ij}(r,z)] = \begin{vmatrix} \frac{r}{\sin r \cos z} & 0 & 0 \\ 0 & \frac{\sin r}{r \cos z} & 0 \\ 0 & 0 & \frac{r \cos z}{\sin r} \end{vmatrix}, \tag{22}$$

referring to cylindric coordinates in Minkowski space, the electromagnetic components are

$$E^{(j)} = \frac{E_j}{h_j}, \quad B^{(1)} = \frac{B_1}{h_2 h_3}, \quad B^{(2)} = \frac{B_3}{h_3 h_1}, \quad B^{(3)} = \frac{B_3}{h_1 h_2},$$

$$dt^2 - dr^2 - r^2 d\phi^2 - dz^2 = dt^2 - h_1 dr^2 - h_2 d\phi^2 - h_3 dz^2.$$

In the case of hyperbolic Lobachevsky space, we have similar results with the evident modifications

$$D^i(x) = \varepsilon_0 \varepsilon_{ij}(x) E^{(j)}(x), \quad [\varepsilon_{ij}(r,z)] = \begin{vmatrix} \frac{\sinh r \cosh z}{r} & 0 & 0 \\ 0 & \frac{r \cosh z}{\sinh r} & 0 \\ 0 & 0 & \frac{\sinh r}{r \cosh z} \end{vmatrix},$$

$$H^i(x) = \frac{1}{\mu_0} \mu_{ij}(x) B^{(j)}(x), \quad [\mu_{ij}(r,z)] = \begin{vmatrix} \frac{r}{\sinh r \cosh z} & 0 & 0 \\ 0 & \frac{\sinh r}{r \cosh z} & 0 \\ 0 & 0 & \frac{r \cosh z}{\sinh r} \end{vmatrix}.$$

For all the listed examples, effective tensors obey one the same restriction:

$$\varepsilon_{ij}(x)\ \mu_{jk}(x) = \delta_{ik}.$$

2. Spinor form of Maxwell equations

To introduce spinor notations, it is convenient to start with the ordinary Dirac equation

$$(i\gamma^a\partial_a - m)\Psi = 0, \quad \gamma^a = \begin{vmatrix} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{vmatrix}, \Psi = \begin{vmatrix} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{vmatrix}, \ \{\alpha, \dot{\alpha}\} = 1, 2\ ; \tag{23}$$

$$\sigma^a = (I, \sigma^j), \quad \bar{\sigma}^a = (I, -\sigma^j).$$

In the 2-spinor form we have two equations

$$i\sigma^a\partial_a\xi = m\eta, \quad i\bar{\sigma}^a\partial_a\eta = m\xi. \tag{24}$$

Let us attach spinor indices to Pauli matrices: $\sigma^a = (\sigma^a)_{\dot{\beta}\alpha}$, $\bar{\sigma}^a = (\bar{\sigma}^a)^{\beta\dot{\alpha}}$, then Eq. (24) read

$$i(\sigma^a\partial_a)_{\dot{\beta}\alpha}\ \xi^\alpha = m\eta_{\dot{\beta}}, \quad i(\bar{\sigma}^a\partial_a)^{\beta\dot{\alpha}}\ \eta_{\dot{\alpha}} = m\xi^\beta. \tag{25}$$

Electromagnetic tensor is equivalent to a pair of symmetrical second rang spinors: $F_{mn} \longleftrightarrow \{\xi^{\alpha\beta}, \eta_{\dot{\alpha}\dot{\beta}}\}$; correspondingly, the eight Maxwell equations are presented as follows:

$$(\sigma^a\partial_a)_{\dot{\rho}\alpha}\ \xi^{\alpha\beta} = (\sigma^b)_{\dot{\rho}\alpha}\varepsilon^{\alpha\beta}J_b, \quad (\bar{\sigma}^a\partial_a)^{\rho\dot{\alpha}}\ \eta_{\dot{\alpha}\dot{\beta}} = (\bar{\sigma}^b)^{\rho\dot{\alpha}}\varepsilon_{\dot{\alpha}\dot{\beta}}\ J_b\ ; \tag{26}$$

the second equation is conjugate to the first one. In (26) we use spinor metrical matrices:

$$(\varepsilon_{\alpha\beta}) = i\sigma^2, \quad (\varepsilon^{\alpha\beta}) = -i\sigma^2; \quad (\varepsilon_{\dot{\alpha}\dot{\beta}}) = i\sigma^2, \quad (\varepsilon^{\dot{\alpha}\dot{\beta}}) = -i\sigma^2. \tag{27}$$

To prove equivalence of spinor form (26) to the ordinary Maxwell equation in vector form, let us apply notations without spinor indices. To this end, we take into account identities

$$(\xi^{\alpha\beta}) = \Sigma^{mn}F_{mn}\sigma^2, \quad (\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn}F_{mn}\sigma^2,$$

$$\Sigma^{mn} = \frac{1}{4}(\bar{\sigma}^m\sigma^n - \bar{\sigma}^n\sigma^m), \quad \bar{\Sigma}^{mn} = \frac{1}{4}(\sigma^m\bar{\sigma}^n - \sigma^n\bar{\sigma}^m). \tag{28}$$

Then Eq. (26) can be rewritten as

$$\sigma^a\partial_a\ \Sigma^{mn}F_{mn} = -\sigma^b J_b, \qquad \bar{\sigma}^a\partial_a\ \bar{\Sigma}^{mn}F_{mn} = -\bar{\sigma}^b J_b. \tag{29}$$

Now we are to take into account the identities

$$\Sigma^{mn}F_{mn} = \sigma^1(F_{01} - iF_{23}) + \sigma^2(F_{02} - iF_{31}) + \sigma^3(F_{03} - iF_{12}),$$

$$\bar{\Sigma}^{mn}F_{mn} = \sigma^1(-F_{01} - iF_{23}) + \sigma^2(-F_{02} - iF_{31}) + \sigma^3(-F_{03} - iF_{12});$$

with the designations

$$F_{01} = -E^1, F_{02} = -E^2, F_{03} = -E^3, F_{23} = B^1, F_{31} = B^2, F_{12} = B^3 \tag{30}$$

they read

$$\Sigma^{mn}F_{mn} = -\sigma^1(E^1 + iB^1) - \sigma^1(E^2 + iB^2) - \sigma^1(E^3 + iB^3) = -\sigma^j a_j,$$

$$\bar{\Sigma}^{mn}F_{mn} = \sigma^1(E^1 - iB^1) + \sigma^1(E^2 - iB^2) + \sigma^1(E^3 - iB^3) = +\sigma^j b_j. \tag{31}$$

So we get

$$(\xi^{\alpha\beta}) = \Sigma^{mn}F_{mn}\sigma^2 = \begin{vmatrix} -i(a_1 - ia_2) & ia_3 \\ ia_3 & +i(a_1 + ia_2) \end{vmatrix},$$

$$(\eta_{\dot{\alpha}\dot{\beta}}) = -\bar{\Sigma}^{mn}F_{mn}\sigma^2 = \begin{vmatrix} -i(b_1 - ib_2) & ib_3 \\ ib_3 & i(b_1 + ib_2) \end{vmatrix}.$$

Taking into account (31), Eq.(29) can be presented in the form

$$\sigma^a\partial_a\ \sigma^j a_j = \sigma^b J_b, \quad \bar{\sigma}^a\partial_a\ \sigma^j b_j = -\bar{\sigma}^b J_b, \tag{32}$$

or

$$(\partial_0 + \sigma^l\partial_l)\,(\sigma^k a_k) = J_0 + \sigma^j J_j, \quad (\partial_0 - \sigma^l\partial_l)\,(\sigma^k b_k) = -J_0 + \sigma^j J_j. \tag{33}$$

From (33) we derive

$$\sigma^n \partial_0 a_n + (\delta_{lk} + i\omega_{nlk}\sigma^n)\partial_l a_k \,, = J_0 + \sigma^n J_n,$$
$$\sigma^n \partial_0 b_n - (\delta_{lk} + i\omega_{nlk}\sigma^n)\partial_l b_k \,. = -J_0 + \sigma^n J_n \,.$$

Therefore, we have four equations

$$\partial_l a_l = J_0 \,, \quad \partial_0 a_n + i\omega_{nlk}\partial_l a_k = J_n \,, \quad \partial_l b_l = J_0 \,, \quad \partial_0 b_n - i\omega_{nlk}\partial_l b_k = J_n \,,$$

or differently

$$(1) \qquad \partial_l (E^l + iB^l) = J_0 \,,$$
$$(2) \qquad \partial_0 (E^l + iB^l) + i\omega_{nlk}\partial_l (E^k + iB^k) = J_n \,,$$
$$(1') \qquad \partial_l (E^l - iB^l) = J_0 \,,$$
$$(2') \qquad \partial_0 (E^l - iB^l) - i\omega_{nlk}\partial_l (E^k - iB^l) = J_n \,.$$

Summing and subtracting equations within pairs as shown, we obtain

$$1+1', \quad \partial_l E^l = J_0 \,, \quad 1-1', \quad \partial_l B^l = 0;$$
$$2+2', \quad \partial_0 E^n - \omega_{nlk}\partial_l B^k = J_k \,, \quad 2-2', \quad \partial_0 B^n + \omega_{nlk}\partial_l E^k = 0;$$

that can be identified with Maxwell equations in vector form:

$$\operatorname{div}\mathbf{E} = J^0, \quad \operatorname{div}\mathbf{B} = 0, \quad \operatorname{rot}\mathbf{B} = \partial_0 \mathbf{E} + \mathbf{J}, \quad \operatorname{rot}\mathbf{E} = -\partial_0 \mathbf{B}, \tag{34}$$

where

$$\mathbf{E} = (E^n), \mathbf{B} = (B^n), \quad J^0 = J_0 \,, \mathbf{J} = (J^n) = (-J_n).$$

3. Separating the variables in de Sitter model

Covariant Maxwell equations in spinor form can be obtained by the same method, which is used for generalizing the Dirac equation [45]:

$$i\sigma^\alpha(x)\left[\partial_\alpha + \Sigma_\alpha(x)\right]\xi(x) = m\eta(x), \quad i\bar{\sigma}^\alpha(x)\left[\partial_\alpha + \bar{\Sigma}_\alpha(x)\right]\eta(x) = m\xi(x). \tag{35}$$

So, the generally covariant Maxwell equations are

$$i\sigma^\alpha(x)\left[\partial_\alpha + \Sigma_\alpha(x)\otimes I + I\otimes\Sigma_\alpha(x)\right]\xi(x) = \sigma^\beta(x)(-i\sigma^2)J_\beta(x),$$
$$i\bar{\sigma}^\alpha(x)\left[\partial_\alpha + \bar{\Sigma}_\alpha(x)\otimes I + I\otimes\bar{\Sigma}_\alpha(x)\right]\eta(x) = \sigma^\beta(x)(+i\sigma^2)J_\beta(x), \tag{36}$$

where we use notations[45]:

$$\sigma^\alpha(x) = \sigma^a e^\alpha_{(a)}(x), \quad \bar{\sigma}^\alpha(x) = \bar{\sigma}^a e^\alpha_{(a)}(x),$$
$$\Sigma_\alpha(x) = \frac{1}{2}\Sigma^{ab} e^\beta_{(a)}\nabla_\alpha(e_{(b)\beta}), \quad \bar{\Sigma}_\alpha(x) = \frac{1}{2}\bar{\Sigma}^{ab} e^\beta_{(x)}\nabla_\alpha(e_{(b)\beta}), \tag{37}$$
$$\Sigma^{ab} = \frac{1}{4}(\,\bar{\sigma}^a\sigma^b - \bar{\sigma}^b\,\sigma^a\,), \quad \bar{\Sigma}^{ab} = \frac{1}{4}(\sigma^a\,\bar{\sigma}^b - \sigma^b\,\bar{\sigma}^a).$$

As in Minkowski space, the second equation is conjugate to the first, so it suffices to study only the first one (further we follow the case in the absence of electric currents)

$$\sigma^\alpha(x)\left[\partial_\alpha + \Sigma_\alpha(x)\otimes I + I\otimes\Sigma_\alpha(x)\right]\xi(x) = 0. \tag{38}$$

In (38), the quantity $\xi(x)$ stands for a symmetric 2-rang spinor, it can be treated as a 2×2-matrix function. Eq. (38) can be presented also with the help of the Ricci rotation coefficients $\gamma_{abc} = +e_{(a)\beta;\alpha} e^\beta_{(b)} e^\alpha_{(c)}$ as follows

$$\left[\sigma^c e^\alpha_{(c)}(x)\partial_\alpha + \sigma^c(\frac{1}{2}\Sigma^{ab}\otimes I + I\otimes\frac{1}{2}\Sigma^{ab})\gamma_{abc}(x)\right]\xi(x) = 0. \tag{39}$$

Let us specify Eq. (38) in the known static de Sitter coordinates:

$$dS^2 = (1 - r^2/\rho^2)c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dr^2}{1 - r^2/\rho^2}, \tag{40}$$

where ρ is the curvature radius. Below we will apply dimensionless coordinates

$ct/\rho \Rightarrow t,\ \ r/\rho \Rightarrow r$. We use the diagonal spherical tetrad [46]:

$$x^\alpha = (t, \theta, \phi,\ r),\ \ \varphi = 1 - r^2,\ \ \varphi' = \frac{d\varphi}{dr},$$

$$e_{(0)\alpha} = (\frac{1}{\sqrt{\varphi}}, 0, 0, 0), \quad e_{(3)\alpha} = (0, 0, 0,\ \sqrt{\varphi}), \qquad (41)$$

$$e_{(1)\alpha} = (0, \frac{1}{r}, 0, 0), \quad e_{(2)\alpha} = (0, 0, \frac{1}{r\sin\theta}, 0).$$

Local Pauli matrices are

$$\sigma^\alpha(x) = (\frac{1}{\sqrt{\varphi}}, -\frac{\sigma^1}{r}, -\frac{\sigma^2}{r\sin\theta}, -\sqrt{\varphi}\sigma^3)\ , \quad \bar{\sigma}^\alpha(x) = (\frac{1}{\sqrt{\varphi}}, \frac{\sigma^1}{r}, \frac{\sigma^2}{r\sin\theta}, \sqrt{\varphi}\sigma^3)\ .$$

Having in mind the general formulas [46]

$$\sigma^{ab} = \frac{1}{4}\begin{vmatrix} \bar{\sigma}^a\sigma^b - \bar{\sigma}^b\sigma^a & 0 \\ 0 & \sigma^a\bar{\sigma}^b - \sigma^b\bar{\sigma}^a \end{vmatrix} = \begin{vmatrix} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{vmatrix},$$

$$\Gamma_\alpha(x) = \frac{1}{2}\begin{vmatrix} \Sigma^{ab} e^\beta_{(a)} \nabla_\alpha(\, e^\alpha_{(b);\beta} & 0 \\ 0 & \bar{\Sigma}^{ab} e^\beta_{(a)} \nabla_\alpha(e^\alpha_{(b);\beta} \end{vmatrix} = \begin{vmatrix} \Sigma_\alpha(x) & 0 \\ 0 & \bar{\Sigma}_\alpha(x) \end{vmatrix}$$

and the known expressions [46] for bispinor connection in de Sitter space, we find the 2-spinor connections:

$$\Gamma_t = \begin{vmatrix} \Sigma_t & 0 \\ 0 & \bar{\Sigma}_t \end{vmatrix} = \frac{\varphi'}{2}\sigma^{03} = \frac{\varphi'}{2}\begin{vmatrix} \sigma^3/2 & 0 \\ 0 & -\sigma^3/2 \end{vmatrix}, \quad \Gamma_r = 0,$$

$$\Gamma_\theta = \begin{vmatrix} \Sigma_\theta & 0 \\ 0 & \bar{\Sigma}_\theta \end{vmatrix} = \sqrt{\varphi}\,\sigma^{31} = \sqrt{\varphi}\begin{vmatrix} -i\sigma^2/2 & 0 \\ 0 & -i\sigma^2/2 \end{vmatrix}, \qquad (42)$$

$$\Gamma_\phi = \begin{vmatrix} \Sigma_\phi & 0 \\ 0 & \bar{\Sigma}_\phi \end{vmatrix} = \sqrt{\varphi}\sin\theta\begin{vmatrix} i\sigma^1/2 & 0 \\ 0 & i\sigma^1/2 \end{vmatrix} + \cos\theta\begin{vmatrix} -i\sigma^3/2 & 0 \\ 0 & -i\sigma^3/2 \end{vmatrix}.$$

Therefore, Eq. (39) takes the form

$$\left\{\frac{1}{\sqrt{\varphi}}\left[\partial_t + \frac{\varphi'}{4}(\sigma^3 \otimes I + I \otimes \sigma^3)\right] + \sigma^3\sqrt{\varphi}\partial_r + \frac{\sigma^1}{r}\left[\partial_\theta - i\frac{\sqrt{\varphi}}{2}(\sigma^2 \otimes I + I \otimes \sigma^2)\right] + \right.$$

$$\left. + \frac{\sigma^2}{r\sin\theta}\left[\partial_\phi + i\frac{\sqrt{\varphi}}{2}\sin\theta(\sigma^1 \otimes I + I \otimes \sigma^1) - i\frac{\cos\theta}{2}(\sigma^3 \otimes I + I \otimes \sigma^3)\right]\right\}\xi = 0.$$

It is convenient to re-group the terms differently, separating the angular operator:

$$\left\{\frac{\partial}{\partial t} + \varphi[\sigma^3\frac{\partial}{\partial r} + \frac{i}{r}(-\sigma^1\frac{\sigma^2 \otimes I + I \otimes \sigma^2}{2} + \sigma^2\frac{\sigma^1 \otimes I + I \otimes \sigma^1}{2}) + \frac{\varphi'}{2\varphi}\frac{\sigma^3 \otimes I + I \otimes \sigma^3}{2}] + \right.$$

$$\left. + \frac{\sqrt{\varphi}}{r}\left[\sigma^1\partial_\theta F - i\sigma^2\frac{i\partial_\phi + \cos\theta(\sigma^3 \otimes I + I \otimes \sigma^3)/2}{\sin\theta}\right]\right\}\xi = 0. \qquad (43)$$

We will construct the solutions with spherical symmetry, by diagonalizing the operators of total angular momentum. In this tetrad basis, it has Schrödinger-like structure [46]:

$$J_1 = l_1 + \frac{\cos\phi}{\sin\theta}S_3, \quad J_2 = l_2 + \frac{\sin\phi}{\sin\theta}S_3, \quad J_3 = l_3 = -i\frac{\partial}{\partial\phi}, \qquad (44)$$

where

$$S_3 = ij^{12} = \frac{1}{2}(\sigma^3 \otimes I + I \otimes \sigma^3), \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}.$$

We are to find the eigenstates of matrix S_3:

$$\frac{1}{2}(\sigma^3 \otimes I + i \otimes \sigma^3)\begin{vmatrix} a & b \\ b & c \end{vmatrix} = \sigma \begin{vmatrix} a & b \\ b & c \end{vmatrix}. \tag{45}$$

Explicitly, this equation reads

$$\begin{vmatrix} a & b \\ -b & -c \end{vmatrix} + \begin{vmatrix} a & -b \\ b & -c \end{vmatrix} = 2\sigma \begin{vmatrix} a & b \\ b & c \end{vmatrix},$$

whence it follows the linear system with three different solutions:

$$\begin{cases} a = \sigma a & \sigma = +1, & a = 1, b = 0, c = 0; \\ 0 = \sigma b \;\Rightarrow & \sigma = 0, & a = 0, b = 1, c = 0; \\ c = -\sigma c & \sigma = -1, & a = 0, b = 0, c = 1. \end{cases} \tag{46}$$

So we have three eigenvalues $\sigma = -1, 0, +1$ and three eigenstates:

$$\sigma = +1, \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \quad \sigma = 0, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma = -1, \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}. \tag{47}$$

We construct the solutions obeying two equations

$$(J_1^2 + J_2^2 + J_3^2)\xi(x) = j(j+1)\xi(x), \quad J_3\,\xi(x) = m\,\xi(x),$$

in accordance with the general theory [46] the spinor $\xi(x)$ should have the form

$$\xi(x) = e^{-i\omega t}\begin{vmatrix} f(r)\,D_{-1} & h(r)\,D_0 \\ h(r)\,D_0 & g(r)\,D_{+1} \end{vmatrix}, \quad D_\sigma = D^j_{-m,-\sigma}(\phi,\theta,0); \tag{48}$$

where f, g, h stand for unknown radial functions; indices j, m at Wigner functions are omitted. The substitution (48) is correct only for the following values of j and m:

$j = 1, 2, 3, \ldots; \quad m = -j, -j+1, \ldots, j-1, j$.

For the states with $j = 0$, the initial substitution for $\xi(x)$ is different:

$$j = 0, \qquad \xi(x) = e^{-i\omega t}\begin{vmatrix} 0 & h(r) \\ h(r) & 0 \end{vmatrix}. \tag{49}$$

Now we should separate the variables in Eq. (43). First, we find (the factor $e^{-i\omega t}$ is omitted for brevity)

$$\xi(x) = \begin{vmatrix} f(r)\,D_{-1} & h(r)\,D_0 \\ h(r)\,D_0 & g(r)\,D_{+1} \end{vmatrix},$$

$$\frac{\partial}{\partial t}\xi = \begin{vmatrix} -i\omega f\,D_{-1} & -i\omega h\,D_0 \\ -i\omega h\,D_0 & -i\omega g\,D_{+1} \end{vmatrix}, \qquad \sigma^3 \frac{\partial}{\partial r}\xi = \begin{vmatrix} f'D_{-1} & h'D_0 \\ -h'D_0 & -g'D_{+1} \end{vmatrix},$$

$$\frac{i}{r}\left[-\sigma^1 \frac{\sigma^2 \otimes I + I \otimes \sigma^2}{2} + \sigma^2 \frac{\sigma^1 \otimes I + I \otimes \sigma^1}{2})\right]\xi = \frac{1}{r}\begin{vmatrix} f(r)D_{-1} & 2h(r)D_0 \\ -2h(r)D_0 & -g(r)D_{+1} \end{vmatrix},$$

$$\frac{\sigma^3 \otimes I + I \otimes \sigma^3}{2}\xi = \begin{vmatrix} f(r)\,D_{-1} & 0 \\ 0 & -g(r)\,D_{+1} \end{vmatrix}.$$

We find the action of angular operator $\Sigma_{\theta,\phi}$ on spinor $\xi(x)$ (keeping in mind that $i\partial_\phi D^j_{-m,\sigma} = -m\,D^j_{-m,\sigma}$):

$$\Sigma_{\theta,\phi}\xi(x) = \sigma^1\partial_\theta\xi - i\sigma^2\,\frac{-m + \cos\theta(\sigma^3 \otimes I + I \otimes \sigma^3)/2}{\sin\theta}\,\xi =$$

$$=\begin{vmatrix} h\,\partial_\theta D_0 & g\,\partial_\theta D_{+1} \\ f\,\partial_\theta D_{-1} & h\,\partial_\theta D_0 \end{vmatrix} + \frac{1}{\sin\theta}\begin{vmatrix} h\,mD_0 & g\,(m+\cos\theta)D_{+1} \\ f\,(-m+\cos\theta)D_{-1} & -h\,mD_0 \end{vmatrix},$$

or

$$\Sigma_{\theta,\phi}\xi(x)=\begin{vmatrix} h\,(\partial_\theta+m\sin^{-1})\,D_0 & g\,[\partial_\theta+(m+\cos\theta)\sin^{-1}\theta]D_{+1} \\ f\,[\partial_\theta+(-m+\cos\theta)\sin^{-1}\theta]\,D_{-1} & h\,(\partial_\theta-m\sin^{-1})\,D_0 \end{vmatrix}.$$

Using the known recurrent formulas for Wigner functions [46]

$$\partial_\theta D_{-1}=\frac{1}{2}(b\,D_{-2}-a\,D_0),(-m+\cos\theta)\sin^{-1}\theta\,D_{-1}=\frac{1}{2}(-b\,D_{-2}-a\,D_0),$$

$$\partial_\theta D_{+1}=\frac{1}{2}(a\,D_0-b\,D_{+2}),(m+\cos\theta)\sin^{-1}\theta\,D_{+1}=\frac{1}{2}(a\,D_0+b\,D_{+2}),$$

$$\partial_\theta D_0=\frac{1}{2}(a\,D_{-1}-a\,D_{+1}),m\sin^{-1}\theta\,D_0=\frac{1}{2}(a\,D_{-1}+a\,D_{+1}),$$

$$a=\sqrt{j(j+1)},\quad b=\sqrt{(j-1)(j+1)},$$

we derive

$$\Sigma_{\theta,\phi}\xi=a\begin{vmatrix} h\,D_{-1} & g\,D_0 \\ -f\,D_0 & -h\,D_{+1} \end{vmatrix}. \tag{50}$$

Therefore, Eq. (43) takes the form

$$\begin{vmatrix} -i\omega f\,D_{-1} & -i\omega h\,D_0 \\ -i\omega h\,D_0 & -i\omega g\,D_{+1} \end{vmatrix}+\varphi\left\{\begin{vmatrix} f'D_{-1} & h'D_0 \\ -h'D_0 & -g'D_{+1} \end{vmatrix}+\right.$$

$$\left.+\frac{1}{r}\begin{vmatrix} f\,D_{-1} & 2h\,D_0 \\ -2h\,D_0 & -g\,D_{+1} \end{vmatrix}+\frac{\varphi'}{2\varphi}\begin{vmatrix} f\,D_{-1} & 0 \\ 0 & -g\,D_{+1} \end{vmatrix}\right\}+\frac{\sqrt{\varphi}}{r}a\begin{vmatrix} h\,D_{-1} & g\,D_0 \\ -f\,D_0 & -h\,D_{+1} \end{vmatrix}=0,$$

whence it follows the system of four radial equations:

$$-i\omega f+\varphi(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})f+a\frac{\sqrt{\varphi}}{r}h=0,\ +i\omega g+\varphi(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})g+a\frac{\sqrt{\varphi}}{r}h=0,$$

$$-i\omega h+\varphi(\frac{d}{dr}+\frac{2}{r})h+a\frac{\sqrt{\varphi}}{r}g=0,\quad +i\omega h+\varphi(\frac{d}{dr}+\frac{2}{r})h+a\frac{\sqrt{\varphi}}{r}f=0; \tag{51}$$

remembering that $a=\sqrt{j(j+1)}$. The equations for the case $j=0$ follow from (51) by setting $f=0, g=0,\ a=0$:

$$0=0,\quad 0=0,-i\omega h+\varphi(\frac{d}{dr}+\frac{2}{r})h=0,+i\omega h+\varphi(\frac{d}{dr}+\frac{2}{r})h=0. \tag{52}$$

There exists only one and trivial solution, $h(r)=0$, which means that Maxwell equations do not have solutions with $j=0$.

Turning to (51), let us sum and subtract equations 3 and 4; this leads to

$$2\varphi(\frac{d}{dr}+\frac{2}{r})h+a\frac{\sqrt{\varphi}}{r}(f+g)=0,\quad h=\frac{ia}{2\omega}\frac{\sqrt{\varphi}}{r}(f-g). \tag{53}$$

By substituting from equations 3 and 4 the variables $f(r)$ and $g(r)$ expressed through $h(r)$, one can check readily that the first equation in (53) turns out to be an identity $0=0$. This means that we have only three independent equations:

$$h=\frac{ia}{2\omega}\frac{\sqrt{\varphi}}{r}(f-g),$$

$$-i\omega f+\varphi(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})f+a\frac{\sqrt{\varphi}}{r}h=0,\ +i\omega g+\varphi(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})g+a\frac{\sqrt{\varphi}}{r}h=0. \tag{54}$$

Excluding the function $h(r)$, we obtain

$$(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi}-\frac{i\omega}{\varphi})f+\frac{ia^2}{2\omega r^2}(f-g)=0,(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi}+\frac{i\omega}{\varphi})g+\frac{ia^2}{2\omega r^2}(f-g)=0. \tag{55}$$

Let us sum and subtract equations in (55), in the same time introducing new variables, $f+g=F,\ f-g=G$. This results in

$$(\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})F-\frac{i\omega}{\varphi}G+\frac{ia^2}{\omega r^2}G=0,\ (\frac{d}{dr}+\frac{1}{r}+\frac{\varphi'}{2\varphi})G-\frac{i\omega}{\varphi}F=0. \tag{56}$$

System (56) is simplified by introducing sample multipliers: $F=(r\sqrt{\varphi})^{-1}\bar{F}$, $G=(r\sqrt{\varphi})^{-1}\bar{G}$; so we obtain

$$i\omega\frac{d}{dr}\bar{F}+(\frac{\omega^2}{\varphi}-\frac{a^2}{r^2})\bar{G}=0,\quad \varphi\frac{d}{dr}\bar{G}=i\omega\bar{F}, \tag{57}$$

whence it follows the second order equation for the main function $\bar{G}$:

$$\left(\frac{d^2}{dr^2}+\frac{\varphi'}{\varphi}\frac{d}{dr}+\frac{\omega^2}{\varphi^2}-\frac{j(j+1)}{r^2\varphi}\right)\bar{G}=0,\quad \bar{F}(r)=\frac{\varphi(r)}{i\omega}\frac{d}{dr}\bar{G}(r). \tag{58}$$

4. Solutions in Minkowski space

Let us briefly consider the simplest variant of Eq. (58) for Minkowski space (when $\varphi=1$):

$$\left(\frac{d^2}{dr^2}+\omega^2-\frac{j(j+1)}{r^2}\right)\bar{G}=0. \tag{59}$$

Here we have the equation with one regular point $r=0$ and one irregular point $r=\infty$ of the rank 2, so it belongs to a confluent hypergeometric type. The possible asymptotic behavior for solutions is

$$r\to 0,\bar{G}\sim r^{j+1},\ r^{-j};\quad r\to\infty,\bar{G}\sim e^{-i\omega r},\ e^{+i\omega r}. \tag{60}$$

With the use of substitution

$$\bar{G}=r^a r^{br}g(r),\quad a=j+1,-j,\quad b=\pm i\omega, \tag{61}$$

We obtain the following equation (in the variable $x=-2br$)

$$x\frac{d^2g}{dx^2}+(2a-x)\frac{dg}{dx}-a\,g=0, \tag{62}$$

that is identified with the confluent hypergeometric equation

$$x\frac{d^2F}{dx^2}+(c-x)\frac{dF}{dx}-a\,F=0,\quad c=2a. \tag{63}$$

Let us fix parameters a and b: $a=j+1, b=i\omega, x=-2br=-2i\omega r$, then the regular in the point $r=0$ solution has the form (see notations in [47])

$$\bar{G}_1(x)=x^{j+1}e^{-x/2}\Phi(a,c;x),\quad a=j+1,\ c=2a. \tag{64}$$

Taking in mind the known Kummer identity $\Phi(a,c;x)=e^x\Phi(c-a,c;-x)$, we readily prove this solution is a real-valued one in all the points (up to the factor ±1):

$$\bar{G}_1(x)=x^{j+1}e^{-x/2}\Phi(j+1,2j+2;x)=(-1)^{j+1}(x^*)^{j+1}e^{-x^*/2}\Phi(j+1,2j+2;x^*),\quad x^*=(-x).$$

Since the second parameter $c=2(j+1)$ takes on integer values, the singular near the point $r=0$ solution is given by the different function (see in [47]):

$$g(x) = x^a e^{-x/2}\Psi(a,c;x),\ a=-j, c=-2j;\quad g(x\to 0) = x^{-j}\cdot\frac{\Gamma(1+2j)}{\Gamma(j+1)}. \tag{65}$$

The description becomes more symmetric after transforming the main equation (59) to Bessel form:

$$\bar{G}(r) = \sqrt{r}g(r),\quad z=\omega r,\quad \frac{d^2g}{dz^2}+\frac{1}{z}\frac{dg}{dz}+\left(1-\frac{p^2}{z^2}\right)g=0,\ p=(j+1/2). \tag{66}$$

Two independent solutions $J_p(z)$ and $J_{-p}(z)$ refer to the confluent hypergeometric functions according to the formulas

$$J_{\pm p}(z) = (\frac{z}{2})^{\pm p}\frac{e^{iz}}{\Gamma(1\pm p)}\Phi(\pm p+1/2, \pm 2p+1; -2iz); \tag{67}$$

there are known relations

$$J_{\pm p}(z) = (\frac{z}{2})^{\pm p}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1\pm p)}(\frac{iz}{2})^{2n},\ J_{\pm p}(|z|\to\infty) = \sqrt{\frac{2}{\pi z}}\cos[z-(\frac{1}{2}\pm p)\frac{\pi}{2}]. \tag{68}$$

5. Solutions in de Sitter space

Let us study the main radial equation for $\bar{G}(r)$ in de Sitter model:

$$\left(\frac{d^2}{dr^2}+\frac{\varphi'}{\varphi}\frac{d}{dr}+\frac{\omega^2}{\varphi^2}-\frac{j(j+1)}{r^2\varphi}\right)\bar{G}=0, \tag{69}$$

explicitly it reads

$$\left(\frac{d^2}{dr^2}-\frac{2r}{1-r^2}\frac{d}{dr}+\frac{\omega^2}{(1-r^2)^2}-\frac{j(j+1)}{r^2(1-r^2)}\right)\bar{G}=0. \tag{70}$$

In the variable $z=r^2$, we have

$$\left[\frac{d^2}{dz^2}+(\frac{1}{z-1}+\frac{1}{2z})\frac{d}{dz}+\frac{\omega^2}{4(-1+z)^2}-\frac{\omega^2}{4(-1+z)}+\frac{\omega^2}{4z}+\frac{j+j^2}{4(z-1)}-\frac{j+j^2}{4z^2}-\frac{j+j^2}{4z}\right]\bar{G}=0. \tag{71}$$

Here we have an equation of hypergeometric type with three regular points. The behavior of solutions in the vicinity of singular points is described by the formulas:

$$z\to 0,\ \bar{G}=z^a,\ a=\frac{j+1}{2},-\frac{j}{2};\quad z\to 1,\ \bar{G}=(1-z)^b,\quad b=\pm\frac{i\omega}{2}.$$

We search the complete solutions in the form $\bar{G}=z^a(1-z)^b H(z)$, after performing the calculation needed we arrive at

$$4z(1-z)H''+\left[8a(1-z)-8bz+2(1-3z)\right]H'+$$
$$+[[4a(a-1)+2a-j(j+1)]\frac{1}{z}+[4b(b-1)+4b+\omega^2]\frac{1}{1-z}-$$
$$-4a(a-1)-8ab-4b(b-1)-6a-6b]H=0.$$

Equating coefficients at z^{-1} and $(z-1)^{-1}$ to zero, we find the known restrictions on parameters a and b, so obtaining more simple equation

$$z(1-z)H''+[2a+\frac{1}{2}-(2a+2b+3/2)z]H'-(a+b)(a+b+1/2)H=0, \tag{72}$$

that is identified with the hypergeometric equation having parameters

$$\alpha=a+b,\quad \beta=a+b+\frac{1}{2},\quad \gamma=2a+\frac{1}{2}. \tag{73}$$

In order to find the asymptotic behavior of the basic solution $F(z)=u_1(z)$ at $z\to 1$, we

should apply the Kummer relation [47]:

$$u_1(z)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}u_2+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}u_6,$$

$$u_1=H(\alpha,\beta;\gamma;z),\quad u_2=H(\alpha,\beta;\alpha+\beta+1-\gamma;1-z), \tag{74}$$

$$u_6=(1-z)^{\gamma-\alpha-\beta}H(\gamma-\alpha,\gamma-\beta;\gamma+1-\alpha-\beta;1-z).$$

When $z\to 1$, the first formula in(74) gives

$$F_1(z\to 1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}.$$

Therefore, solution $\bar{G}(z)$ at $z\to 1$ behaves as follows (taking in mind $\gamma-\alpha-\beta=-2b$, and $b=\pm i\,\omega/2$)

$$\bar{G}_1(z\to 1)=\Gamma(\gamma)\left[\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}(1-z)^{b}+\frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{-b}\right].$$

Due to identities

$$(\gamma-\alpha-\beta)=-2b,\quad \alpha+\beta-\gamma=+2b=(\gamma-\alpha-\beta)^*,$$

$$(\gamma-\alpha)=a+\frac{1}{2}-b=\beta^*,\quad (\gamma-\beta)=a-b=\alpha^*,$$

we can conclude that the function $\bar{G}(z\to 1)$ is real. It is readily proved that the function $\bar{G}_1(z)$ is real in the whole region of variable z. To this end, we should apply the Kummer identity

$$u_1=F(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z),$$

which provides us with two apparently different representations for one the same function:

$$z^a(1-z)^b F(\alpha,\beta;\gamma;z)=z^a(1-z)^b(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z).$$

This can be rewritten as follows (remembering that a is real)

$$\bar{G}_1(z)=z^a(1-z)^b F(\alpha,\beta;\gamma;z)=z^a(1-z)^{b^*}F(\beta^*,\alpha^*;\gamma;z), \tag{75}$$

so $\bar{G}_1(z)=[\bar{G}_1(z)]^*$.

It is convenient to fix parameters, $a=(j+1)/2, b=+i\omega/2$; this choice corresponds to regular at $z=0$ solution. The singular solution refers to the function $u_5(z)$:

$$u_5(z)=z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z),\ \bar{G}_5(z\to 0)=z^a z^{1-\gamma}=z^{(j+1)/2}z^{-j-1/2}=z^{-j/2}. \tag{76}$$

For function u_5 there are possible two different representations:

$$u_5(z)=z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma;2-\gamma;z)=z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta;2-\gamma;z);$$

so we have two representations for the complete solution:

$$\bar{G}_5(z)=z^{-j/2}(1-z)^{i\omega/2}F(-j/2+i\omega/2,1/2-j/2+i\omega/2;1/2-j;z)=$$

$$=z^{-j/2}(1-z)^{-i\omega/2}F(1/2-j/2-i\omega/2,-j/2-i\omega/2;1/2-j;z),$$

this means that $\bar{G}_5(z)=[\bar{G}_5(z)]^*$.

In order to construct the complex and conjugate to each other solutions with the given behavior at $z\to 1$:

$$u_2\sim(1-z)^b=(1-z)^{+i\omega/2},\quad u_6\sim(1-z)^{-b}=(1-z)^{-i\omega/2},$$

We should apply Kummer solutions $u_2(z)$ and $u_6(z)$.

In order to clarify the physical meaning of the arising mathematical task, we turn back to Eq. (69)

$$\varphi\left(\frac{d}{dr}\varphi\frac{d}{dr}+\omega^2-\frac{j(j+1)}{r^2}\varphi\right)\bar{G}=0 \tag{77}$$

and transform it using the new variable r_*:

$$\varphi\frac{d}{dr}=\frac{d}{dr_*} \quad\Rightarrow\quad r_*=\frac{1}{2}\ln\frac{1+r}{1-r};\quad r\to 0, r_*\to 0;\quad r\to 1, r_*\to+\infty. \tag{78}$$

Correspondingly, Eq. (77) reads

$$[\frac{d^2}{dr_*^2}+\omega^2-\frac{j(j+1)}{r^2}(1-r^2)]\bar{G}=0,\quad r\to 0,\quad [\frac{d^2}{dr_*^2}-\frac{j(j+1)}{r^2}]\bar{G}=0; \tag{79}$$

note the behavior near the point $r=1$:

$$[\frac{d^2}{dr_*^2}+\omega^2]\bar{G}=0,\quad \bar{G}=e^{\pm i\omega r_*}=(\cos\omega r_*\pm i\sin\omega r_*);$$

the solutions behave as massless harmonic waves. Eq. (79) may be treated as Schrödinger-like equation with an effective potential

$$[\frac{d^2}{dr_*^2}+\omega^2-U(r_*)]\bar{G}=0,\quad U(r)=\frac{j(j+1)}{r^2}(1-r^2); \tag{80}$$

we should take in mind relations

$$r=\frac{e^{2r_*}-1}{e^{2r_*}+1},\quad U(r_*)=(j+1)(\frac{1}{r^2}-1)=\frac{4j(j+1)e^{2r_*}}{(e^{2r_*}-1)^2}.$$

6. Solutions in anti de Sitter model

The study of the previous section can be extended to anti de Sitter space-time:

$$dS^2=\varphi dt^2-r^2(d\theta^2+\sin^2\theta d\phi^2)-\frac{dr^2}{\varphi},\ \varphi=1+r^2,\ r\in(0,+\infty). \tag{81}$$

We do not need to repeat the most of the above calculation, and can start with Eq. (58)

$$\left(\frac{d^2}{dr^2}+\frac{\varphi'}{\varphi}\frac{d}{dr}+\frac{\omega^2}{\varphi^2}-\frac{j(j+1)}{r^2\varphi}\right)\bar{G}=0, \tag{82}$$

here it reads

$$\left(\frac{d^2}{dr^2}+\frac{2r}{1+r^2}\frac{d}{dr}+\frac{\omega^2}{(1+r^2)^2}-\frac{j(j+1)}{r^2(1+r^2)}\right)\bar{G}=0. \tag{83}$$

Transforming it to the new variable, $r^2=y,\ y\in(0,+\infty)$, we get

$$\left(\frac{d^2}{dy^2}+P^2\right)\bar{G}=0,\ P^2(y)=\frac{[\omega^2-j(j+1)]y-j(j+1)}{4y^2(1+y)^2}, \tag{84}$$

that can be interpreted as the equation of Scrödinger type with an effective linear momentum $P^2(y)$, its behavior at singular points is described by the formulas:

$$y\to 0,\quad P^2\sim\frac{-j(j+1)}{4y^2}\to-\infty;$$

$$y\to\infty,\quad P^2\sim\frac{\omega^2-j(j+1)}{4y^3}=\begin{cases}+0, & \omega^2-j(j+1)>0,\ (A)\\ -0, & \omega^2-j(j+1)<0.\ (B)\end{cases} \tag{85}$$

In the quantum-mechanical context here, we have easily interpretable only the case (A), when $\omega^2>j(j+1)$; the situation (B) is anomalous, for instance, a corresponding classical particle with such parameters cannot exist.

Let us transform Eq. (84) to the new variable, $y=-z,\ z=-r^2,\ z\in(-\infty,0)$, so obtaining

$$\left(\frac{d^2}{dz^2}+\frac{1-3z}{2z(1-z)}\frac{d}{dz}-\frac{\omega^2}{4z(1-z)^2}-\frac{j(j+1)}{4z^2(1-z)}\right)\bar{G}=0. \tag{86}$$

Applying the substitution $\bar{G}=z^a(1-z)^b H(z)$; we derive the equation for $H(z)$ (we can use result (72) with the change ω^2 to $-\omega^2$)

$$4z(1-z)H''+\left[8a(1-z)-8bz+2(1-3z)\right]H'+$$
$$+\{[4a(a-1)+2a-j(j+1)]\frac{1}{z}+[4b(b-1)+4b-\omega^2]\frac{1}{1-z}-$$
$$-4a(a-1)-8ab-4b(b-1)-6a-6b\}H=0. \tag{87}$$

Impose the evident restrictions, $a=(j+1)/2,-j/2$, $b=\pm\omega/2$; then fix the parameters as follows

$$a=\frac{j+1}{2}, b=-\frac{\omega}{2}<0, \quad \bar{G}(z)=z^{(j+1)/2}(1-z)^{-\omega/2}H(z). \tag{88}$$

All the possible functions $H(z)$ must be solutions of the hypergeometric equation

$$z(1-z)F''+[\gamma-(\alpha+\beta+1)zF'-\alpha\beta F=0,$$
$$\alpha=\frac{j+1-\omega}{2}, \beta=\alpha+\frac{1}{2}, \gamma=2a+\frac{1}{2}=j+3/2. \tag{89}$$

Taking $F(z)$ as the Kummer solution $u_1(z)$ (see in [47]):

$$u_1(z)=F(\alpha,\beta,\gamma;z)=F(\frac{j+1-\omega}{2},\frac{j+2-\omega}{2},+\frac{3}{2};z), \tag{90}$$

we come to the situation, where it is possible to obtain the solutions in polynomials, $\alpha=-n, n=0,1,2,...$:

$$\omega=2n+j+1, \quad u_1(z)=F(-n,-n+\frac{1}{2},j+3/2;z). \tag{91}$$

The corresponding complete solution is given by

$$\bar{G}_1(z)=z^{(j+1)/2}(1-z)^{-n-(j+1)/2}\left(1+c_1z+...+c_mz^n\right);$$

at $z\to-\infty$ we have

$$\bar{G}_1(z\to-\infty)=z^{(j+1)/2}(-z)^{-n-(j+1)/2}\left(1+c_1z+...+c_nz^n\right)\to\text{const}.$$

Thus, we have constructed the solutions $\bar{G}_1(z)$ in the quasi-polynomial form, which is finite at two singular points, $r=0$, and $r=\infty$; the parameter ω is quantized, $\omega=2n+j+1, \quad n=0,1,2,...$.

Let us study the case of singular solutions, when $\bar{G}(z\to0)\sim z^{-j/2}$. To this end, we should use another Kummer solution

$$u_5(z)=z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)=z^{-j-1/2}F(\frac{-j-\omega}{2},\frac{-j-\omega+1}{2},-j+\frac{1}{2};z); \tag{92}$$

the corresponding complete solution is

$$\bar{G}_5(z)=z^{-j/2}z^{-\omega/2}F(\frac{-j-\omega}{2},\frac{-j-\omega+1}{2},-j+\frac{1}{2};z). \tag{93}$$

As above, we can apply some quantization condition:

$$\frac{-j+1-\omega}{2}=-n', \quad \omega=2n'-j+1, \bar{G}_5(z)=z^{-j/2}z^{-n+j/2}F(-n-\frac{1}{2},-n,-j+\frac{1}{2};z). \tag{94}$$

The structure of this spectrum is substantially different from the previous one: in particular, at each j there are possible negative values for ω.

To find the behavior of that solution at infinity, we apply the following Kummer

formula

$$u_5(z)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}u_3(z)+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)}e^{i\pi(1-\gamma)}u_4(z);$$

at $z\to-\infty$ it gives

$$u_5(z\to-\infty)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}(-z)^{-\alpha}+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)}e^{i\pi(1-\gamma)}(-z)^{-\beta};$$

so the corresponding complete solution behaves as

$$\bar{G}_5(z\to-\infty)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}z^{-j/2}z^{-n+j/2}(-z)^{+n}+$$

$$+\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(1-\beta)\Gamma(\alpha+1-\gamma)}e^{i\pi(1-\gamma)}z^{-j/2}z^{-n+j/2}(-z)^{+n-1/2},$$

whence, ignoring the second term, we arrive at

$$\bar{G}_5(z\to-\infty)=\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta+1-\gamma)}e^{i\pi(1-\gamma)}. \tag{95}$$

Thus, the function $u_5(z)$ leads to the complete solution $\bar{G}_5(z)$ with a quasi-polynomial structure, which is singular at $z=0$ and regular at infinity $z=-\infty$; the corresponding quantization rule is $\omega=2n'-j$. This type of solution is of questionable physical interest.

In order to clarify the physical meaning of the arising problem, let us turn back to Eq. (82), written in the form

$$\varphi\left(\frac{d}{dr}\varphi\frac{d}{dr}+\omega^2-\frac{j(j+1)}{r^2}\varphi\right)\bar{G}=0; \tag{96}$$

and transform it to the new variable:

$$\varphi\frac{d}{dr}=\frac{d}{dr_*}\quad\Rightarrow\quad r_*=\arctan r,\quad \tan r_*=r;\, r\to 0, r_*\to 0;\, r\to+\infty, r_*\to+\frac{\pi}{2}. \tag{97}$$

Then Eq. (96) in this variable reads

$$\left[\frac{d^2}{dr_*^2}+\omega^2-j(j+1)(1+\frac{1}{\tan^2 r_*})\right]\bar{G}=0,\quad r_*\in(0,\frac{\pi}{2}). \tag{98}$$

Here we have the Schrödinger type equation with an effective potential

$$\left[\frac{d^2}{dr_*^2}+\omega^2-U(r_*)\right]\bar{G}=0,\quad U=j(j+1)(1+\frac{1}{\tan^2 r_*}). \tag{99}$$

This problem is easily interpretable in quantum mechanics if the following inequality is valid

$$\omega^2>j(j+1)\iff\Omega^2>\frac{c^2}{\rho^2}j(j+1). \tag{100}$$

In this point we should recall that the related to solutions $\bar{G}_1(z)$ spectrum for ω satisfies the needed requirement

$$\omega^2-j(j+1)=4n^2+(4n+1)(j+1)>0. \tag{101}$$

Let us consider from this point of view the spectrum related to solutions $\bar{G}_5(z)$:

$$\omega=k-j,\quad \text{where}\quad k=(2n'+1)\in\{1,3,5,...\}. \tag{102}$$

From (102) it follows $\omega^2-j(j+1)=k^2-2kj-j$. Taking in mind the roots of this polynomial

$$k_1=j-j\sqrt{1+1/j},\, -1<k_1<0;\qquad k_2=j+j\sqrt{1+1/j},\, k_2>2j,$$

we conclude that

$$\omega^2 - j(j+1) < 0, \text{ when } \quad (2n'+1) < j + j\sqrt{1+1/j}, \qquad (103)$$
$$\omega^2 - j(j+1) > 0, \text{ when } \quad (2n'+1) > j + j\sqrt{1+1/j}.$$

Therefore, the solutions of type $\bar{G}_5(z)$ are relevant just to the situation badly interpretable from the physical point of view.

7. Maxwell equation in Schwarzschild metric

We can start with the main radial equation, specifying to Schwarzschild space-time, when $\varphi = 1 - \frac{1}{r}$:

$$\left(\frac{d^2}{dr^2} + \frac{\varphi'}{\varphi}\frac{d}{dr} + \frac{\omega^2}{\varphi^2} - \frac{j(j+1)}{r^2\varphi} \right)\bar{G} = 0, \qquad (104)$$

or explicitly

$$\left(\frac{d^2}{dr^2} + \frac{1}{r(r-1)}\frac{d}{dr} + \frac{\omega^2 r^2}{(r-1)^2} - \frac{j(j+1)}{r^2}\frac{r}{r-1} \right)\bar{G} = 0. \qquad (105)$$

Here we have the equation with three singular points, the points $r = 0.1$ are regular, the point $r = \infty$ is irregular of rang 2; this is the class of confluent Heun functions.

Equation (104) becomes more understandable after transforming it to other variable:

$$\left(\varphi\frac{d}{dr}\varphi\frac{d}{dr} + \omega^2 - \frac{j(j+1)}{r^2}\varphi \right)\bar{G} = 0, \; dr_* = \frac{dr}{\varphi} = dr(1 + \frac{1}{r-1}),$$
$$r_* = r + \ln(r-1), \; r \to \infty, \; r_* \to +\infty; \quad r \to 1+0, \; r_* \to -\infty, \qquad (106)$$
$$\left(\frac{d^2}{dr_*^2} + \omega^2 - U(r_*) \right)\bar{G} = 0, \quad U(r_*) = \omega^2 - \frac{j(j+1)}{r^2}\varphi.$$

Let us note the behavior of the effective potential at two infinities:

$$U(r_* \to +\infty) = \frac{j(j+1)}{r^2}\frac{r-1}{r} = 0, \; U(r_* \to -\infty) = \frac{j(j+1)}{r^2}\frac{r-1}{r} = 0; \qquad (107)$$

this means that here we have the effective potential of barrier type, tending to zero both at $r \to 1, r_* \to -\infty$ and $r \to \infty, r_* \to +\infty$.

Now we are to construct solutions of Eq. (105):

$$\left[\frac{d^2}{dr^2} + (\frac{1}{r-1} - \frac{1}{r})\frac{d}{dr} + \omega^2(1 + \frac{2}{r-1} + \frac{1}{(r-1)^2}) + j(j+1)(\frac{1}{r} - \frac{1}{r-1}) \right]\bar{G} = 0. \qquad (108)$$

In (nonphysical) singular point $r = 0$, the solutions behave as

$$G'' - \frac{1}{r}G' + \frac{j(j+1)}{r}G = 0, \quad G \sim r^c, \quad c = 0, c = 2. \qquad (109)$$

Near the point $r = 1$ we have

$$\left[\frac{d^2}{dr^2} + \frac{1}{r-1}\frac{d}{dr} + \frac{\omega^2}{(r-1)^2} \right]\bar{G} = 0, \quad G \sim (r-1)^a, \quad a = \pm i\omega. \qquad (110)$$

To find the asymptotic at infinity, we transform the equation to the variable $x = 1/r$:

$$\left[\frac{d^2}{dx^2} + (\frac{2}{x} - \frac{1}{1-x})\frac{d}{dx} + \frac{\omega^2}{x^4(1-x)^2} - \frac{j(j+1)}{x^2(1-x)} \right]G = 0.$$

It becomes simpler near the point $x = 0$:

$$\left[\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{\omega^2}{x^4} - \frac{j(j+1)}{x^2} \right]G = 0, \quad G \sim x^A e^{B/x},$$

further we derive

$$-\frac{AB}{x^3}-\frac{B(A-2)}{x^3}-\frac{2B}{x^3}+\frac{B^2}{x^4}+\frac{\omega^2}{x^4}=0,$$

whence it follows

$$B^2+\omega^2 \Rightarrow B=\pm i\omega, \quad AB+B(A-2)+2B=0, \quad 2AB=0, \quad A=0. \tag{111}$$

Therefore, the general solution of equation (105) can be searched in the form $\bar{G}=r^c(r-1)^a e^{br}\ g(r)$. Performing the necessary calculation we arrive at

$$g''+\left(\frac{2c}{r}+\frac{2a}{r-1}+2b+\frac{1}{r(r-1)}\right)g'+$$
$$+\left[\frac{c(c-1)}{r^2}+\frac{a(a-1)}{(r-1)^2}+\frac{2ca}{r-1}-\frac{2ca}{r}+\frac{2cb}{r}+\frac{2ab}{r-1}+b^2+\right.$$
$$+\frac{c}{r-1}-\frac{c}{r^2}-\frac{c}{r}+\frac{a}{r}+\frac{a}{(r-1)^2}-\frac{a}{r-1}+\frac{b}{r-1}-\frac{b}{r}+$$
$$\left.+\omega^2+\frac{2\omega^2}{r-1}+\frac{\omega^2}{(r-1)^2}+\frac{j(j+1)}{r}-\frac{j(j+1)}{r-1}\right]g=0.$$

Imposing the evident restrictions, we obtain eight variants of the parameters:
$a=\pm i\omega, \quad c=0,\ 2, \quad b=\pm i\omega.$

So the resulting equation becomes simpler

$$g''+\left(\frac{2a+1}{r-1}+\frac{2c-1}{r}+2b\right)g'+$$
$$+\left[\frac{2ca+2ab+c-a+b-j(j+1)+2\omega^2}{r-1}+\frac{-2ca+2cb-c+a-b+j(j+1)}{r}\right]g=0. \tag{112}$$

Because the physical region for the radial variable is the interval $r\in(1,+\infty)$, the most interesting would be a series in the variable $x=r-1$. Transformed to this variable x the equation reads (the prime designates the derivative d/dx):

$$g''+(p+\frac{p_1}{x}+\frac{p_2}{x+1})g'+(\frac{q_1}{x}+\frac{q_2}{x+1})g=0, \quad x\in(0,+\infty). \tag{113}$$

Its solutions can be constructed as the power series, $g(x)=\sum_{k=0}^{\infty}c_k x^k$. After performing the needed calculation we derive the recurrent formulas

$$n=0, \qquad q_1c_0+p_1c_1=0;$$
$$n=1, \qquad (q_1+q_2)c_0+(p+p_1+p_2+q_1)c_1+(2+2p_1)c_2=0;$$
$$n=2,3,\ldots, \qquad [p(n-1)+q_1+q_2]c_{n-1}+$$
$$+[n(n-1)+(p+p_1+p_2)n+q_1]c_n+[(n+1)n+p_1(n+1)]c_{n+1}=0.$$

The possible convergence radii can be found by Poincaré – Perrone method. To this end, dividing the last relation by n^2c_{n-1}

$$\frac{1}{n^2}[p(n-1)+q_1+q_2]+$$
$$+\frac{1}{n^2}[n(n-1)+(p+p_1+p_2)n+q_1]\frac{c_n}{c_{n-1}}+\frac{1}{n^2}[(n+1)n+p_1(n+1)]\frac{c_{n+1}}{c_n}\frac{c_n}{c_{n-1}}=0$$

and tending $n\to\infty$, we obtain the algebraic equation which determines the possible convergence radii:

$$R=\lim_{n\to\infty}\frac{c_n}{c_{n-1}}=R,\quad R+R^2=0,\quad R_{conv}=\frac{1}{|R|}=1,\infty.$$

Recall that complete solutions have the structure

$$\bar{G}(r)=r^c(r-1)^a e^{br}g(r)\quad\Rightarrow\quad \bar{G}(x)=(1+x)^c x^a e^{b(1+x)}g(x), \tag{114}$$

where

$$c=0,2;\quad a=-i\omega,+i\omega;\quad b=-i\omega,+i\omega;\quad x\in(0,+\infty).$$

Below we list the eight variants of solutions (they are collected in pairs of conjugate ones)

$$\begin{aligned}
&c=0,\quad a=+i\omega,\quad b=+i\omega,\quad \bar{G}_1=x^{+i\omega}e^{+i\omega(1+x)}g_1(x),\\
&c=0,\quad a=-i\omega,\quad b=-i\omega,\quad \bar{G}_1^*=x^{-i\omega}e^{-i\omega(1+x)}g_1^*(x);\\
&c=0,\quad a=+i\omega,\quad b=-i\omega,\quad \bar{G}_2=x^{+i\omega}e^{-i\omega(1+x)}g_2(x),\\
&c=0,\quad a=-i\omega,\quad b=+i\omega,\quad \bar{G}_2^*=x^{-i\omega}e^{+i\omega(1+x)}g_2^*(x);
\end{aligned} \tag{115}$$

$$\begin{aligned}
&c=2,\quad a=+i\omega,\quad b=+i\omega,\quad \bar{G}_3=(1+x)^2x^{+i\omega}e^{+i\omega(1+x)}g_3(x),\\
&c=2,\quad a=+i\omega,\quad b=-i\omega,\quad \bar{G}_4=(1+x)^2x^{+i\omega}e^{-i\omega(1+x)}g_4(x);\\
&c=2,\quad a=-i\omega,\quad b=+i\omega,\quad \bar{G}_4^*=(1+x)^2x^{-i\omega}e^{+i\omega(1+x)}g_4^*(x),\\
&c=2,\quad a=-i\omega,\quad b=-i\omega.\quad \bar{G}_3^*=(1+x)^2x^{-i\omega}e^{-i\omega(1+x)}g_x^3(x).
\end{aligned} \tag{116}$$

8. Solutions in spherical Riemann space

Now we consider Maxwell equations in spherical Riemann model (let $x^\alpha=(t,r,\theta,\phi)$):

$$dS^2=dt^2-dr^2-\sin^2 r d\theta^2-\sin^2\theta d\phi^2,\quad g_{\alpha\beta}=\begin{vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-\sin^2 r&0\\0&0&0&-\sin^2 r\sin^2\theta\end{vmatrix}, \tag{117}$$

$$e^\alpha_{(0)}=(1,0,0,0),\quad e^\alpha_{(3)}=(0,1,0,0),\quad e^\alpha_{(1)}=(0,0,\frac{1}{\sin r},0),\quad e^\alpha_{(2)}=(1,0,0,\frac{1}{\sin r\sin\theta}); \tag{118}$$

by changing the numeration for coordinates $(t,r,\theta,\phi)\Rightarrow(t,\theta,\phi,r)$ the tetrad (118) becomes diagonal, Ricci rotations coefficient being $\gamma_{ab0}=0,\quad \gamma_{ab3}=0$, and

$$\gamma_{ab1}=\begin{vmatrix}0&0&0&0\\0&0&0&-\frac{1}{\tan r}\\0&0&0&0\\0&+\frac{1}{\tan r}&0&0\end{vmatrix},\quad \gamma_{ab2}=\begin{vmatrix}0&0&0&0\\0&0&+\frac{1}{\tan\theta\sin r}&0\\0&-\frac{1}{\tan\theta\sin r}&0&-\frac{1}{\tan r}\\0&0&+\frac{1}{\tan r}&0\end{vmatrix}.$$

Starting with the general spinor equation

$$\left[\sigma^c e^\alpha_{(c)}(x)\partial_\alpha+\sigma^c(\frac{1}{2}\Sigma^{ab}\otimes I+I\otimes\frac{1}{2}\Sigma^{ab})\gamma_{abc}(x)\right]\xi(x)=0, \tag{119}$$

we arrive at

$$\left[\partial_t+\left\{\sigma^3\partial_r+\frac{i}{\tan r}\left(-\sigma^1\frac{\sigma^2\otimes I+I\otimes\sigma^2}{2}+\sigma^2\frac{\sigma^1\otimes I+I\otimes\sigma^1}{2}\right)\right\}+\right.$$

$$\left.+\frac{1}{\sin r}\left\{\sigma^1\partial_\theta-i\sigma^2\frac{i\partial_\varphi+\cos\theta(\sigma^3\otimes I+I\otimes\sigma^3)/2}{\sin\theta}\right\}\right]\xi=0.$$

Comparing this equation with Eq. (43), we can write down the resulting radial equations just by formal changes in system (51):

$$\begin{aligned}&-i\omega f+(\frac{d}{dr}+\frac{1}{\tan r})f+\frac{a}{\sin r}h=0,\ i\omega g+(\frac{d}{dr}+\frac{1}{\tan r})g+\frac{a}{\sin r}h=0,\\&-i\omega h+(\frac{d}{dr}+\frac{2}{\tan r})h+\frac{a}{\sin r}g=0,\ i\omega h+(\frac{d}{dr}+\frac{2}{\tan r})h+\frac{a}{\sin r}f=0.\end{aligned}\tag{120}$$

Summing and subtracting the third and fourth equations, we derive

$$2(\frac{d}{dr}+\frac{2}{\tan r})h+\frac{a}{\sin r}(f+g)=0,\quad 2i\omega h+\frac{a}{\sin r}(f-g)=0.\tag{121}$$

It is readily checked that the first equation in (120) is a result of combining three remaining ones. Therefore, we have only the three independent equations

$$\begin{aligned}&h=-\frac{a}{2i\omega\sin r}(f-g),\\&-i\omega f+(\frac{d}{dr}+\frac{1}{\tan r})f+\frac{a}{\sin r}h=0,\ i\omega g+(\frac{d}{dr}+\frac{1}{\tan r})g+\frac{a}{\sin r}h=0.\end{aligned}\tag{122}$$

Excluding the variable $h(r)$ we obtain:

$$\begin{aligned}&(\frac{d}{dr}+\frac{1}{\tan r}-i\omega)f+\frac{ia^2}{2\omega\sin^2 r}(f-g)=0,\\&(\frac{d}{dr}+\frac{1}{\tan r}+i\omega)g+\frac{ia^2}{2\omega\sin^2 r}(f-g)=0.\end{aligned}\tag{123}$$

Summing and subtracting these two equations, and using the new variables, $f+g=F,\ f-g=G$, we arrive at the system

$$(\frac{d}{dr}+\frac{1}{\tan r})F-i\omega G+\frac{ia^2}{\omega\sin^2 r}G=0,\quad (\frac{d}{dr}+\frac{1}{\tan r})G-i\omega F=0.\tag{124}$$

System (124) can be simplified by separating multipliers, $F=\sin^{-1}r\,\bar F$, $G=\sin^{-1}r\,\bar G$, in this way we obtain

$$\frac{d}{dr}i\omega\bar F+(\omega^2-\frac{a^2}{\sin^2 r})\bar G=0,\quad \frac{d}{dr}\bar G=i\omega\bar F,\tag{125}$$

whence it follows the equation for the main function $\bar G$:

$$\left(\frac{d^2}{dr^2}+\omega^2-\frac{a^2}{\sin^2 r}\right)\bar G=0.\tag{126}$$

In the new variable $y=\dfrac{1-\cos r}{2}$, the last equation reads

$$\left[y(1-y)\frac{d^2}{dy^2}+(\frac{1}{2}-y)\frac{d}{dy}+\omega^2-\frac{a^2}{4y(1-y)}\right]\bar G=0.\tag{127}$$

Its solutions are searched in the form $\bar G=y^A(1-y)^B g(y)$ that results in

$$y(1-y)g''+[2A+1/2-(2A+2B+1)y]g'+$$

$$+\frac{1}{y}[A(A-1)+\frac{1}{2}A-\frac{a^2}{4}]g+\frac{1}{1-y}[B(B-1)+\frac{1}{2}B-\frac{a^2}{4}]g+$$
$$+[\,\omega^2-2AB-A(A-1)-B(B-1)-A-B\,]\,g=0.$$

Equating the coefficients at y^{-1} and $(1-y)^{-1}$ to zero, we obtain $A=(j+1)/2,-j/2$; $B=(j+1)/2,-j/2$. The above equation for $g(y)$ is simplified and recognized as the hypergeometric equation with parameters

$$\gamma=2A+1/2,\quad \alpha=A+B-\omega,\quad \beta=A+B+\omega. \tag{128}$$

Let us fix parameters A and B: $A=(j+1)/2$, $B=(j+1)/2$, so obtaining

$$\gamma=j+3/2,\quad \alpha=j+1-\omega,\quad \beta=j+1+\omega. \tag{129}$$

We obtain the polynomials imposing the standard restriction

$$\alpha=-n,\quad n=1,2,3,\ldots,\quad \omega=n+j+1. \tag{130}$$

The corresponding complete solution has the structure

$$\bar{G}(y)=y^{(j+1)/2}(1-y)^{(j+1)/2}F(-n,n+2j+2,j+3/2;y), \tag{131}$$

this function tends to zero at the points $y\to 0,\ y\to 1$ $(r\to 0,\ r\to\pi)$.

9. Solutions in hyperbolic Lobachevsky space

The main radial equation reads

$$\left(\frac{d^2}{dr^2}+\omega^2-\frac{a^2}{\sinh^2 r}\right)\bar{G}=0,\quad r\in(0,\infty). \tag{132}$$

In the new variable $y=\dfrac{1-\cosh r}{2}$, it takes the form

$$\left[y(y-1)\frac{d^2}{dy^2}+(y-1/2)\frac{d}{dy}+\omega^2-\frac{a^2}{4y(y-1)}\right]\bar{G}=0. \tag{133}$$

Formally, this equation differs from that of used in previous section only in the sign at ω^2. Substitution for $\bar{G}(y)$ is the same

$$\bar{G}=y^A(1-y)^B g(y),\quad A=\frac{j+1}{2},-\frac{j}{2};\quad B=\frac{j+1}{2},-\frac{j}{2}; \tag{134}$$

for function $g(y)$ we obtain the equation of hypergeometric type

$$y(1-y)g''+\left[\,2A+\frac{1}{2}-(2A+2B+1)y\,\right]g'-\left[(A+B)^2+\omega^2\right]g=0$$

with parameters

$$\gamma=2A+1/2,\quad \alpha=A+B-i\omega,\quad \beta=A+B+i\omega. \tag{135}$$

Let us fix parameters as follows (negative B ensures the term $(1-y)^B$ tending to zero when $y\to\infty$):

$$A=\frac{j+1}{2},\ B=-\frac{j}{2};\quad \gamma=j+3/2,\ \alpha=1/2-i\omega,\ \beta=1/2+i\omega. \tag{136}$$

Thus, we have constructed the needed solution (see notations in [45])

$$\bar{G}_1(y)=y^{(j+1)/2}(1-y)^{-j/2}u_1(y),\qquad u_1(y)=F(\alpha,\beta,\gamma;y) \tag{137}$$

it tends to zero at the points $y=0\,(r=0)$.

The singular point $y=1$ does not belong to the physical region. To find the behavior of this solution in infinity, we should apply the Kummer formula [47]

$$u_1(y)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}u_3(y)+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}u_4(y), \tag{138}$$

where

$$u_3(y)=(-y)^{-\alpha}F(\alpha,\alpha+1-\gamma,\alpha+1-\beta;\frac{1}{y})=(-y)^{(-1/2+i\omega)}F(1/2-i\omega,-i\omega-j,1-2i\omega;\frac{1}{y}),$$

$$u_4(y)=(-y)^{-\beta}F(\beta+1-\gamma,\beta,\beta+1-\alpha;\frac{1}{y})=(-y)^{(-1/2-i\omega)}F(i\omega-j,(1/2+i\omega),1+2i\omega;\frac{1}{y}).$$

As $y\to+\infty$, the last formula gives

$$u_1(y)=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-y)^{(-1/2+i\omega)}+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}(-y)^{(-1/2-i\omega)}.$$

Therefore, the complete solution behaves as follows

$$\bar{G}_1(y\to\infty)=(-1)^{-(j+1)/2}\left\{\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(-y)^{i\omega}+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\Gamma(\alpha)}(-y)^{-i\omega}\right\}. \tag{139}$$

Taking in mind the identities

$\beta-\alpha=2i\omega,\ \alpha-\beta=-2i\omega,\ \gamma-\alpha=j+i\omega+1,\ \gamma-\beta=j-i\omega+1,$

we conclude that $\bar{G}_1(y\to\infty)$ is real up to a simple phase factor. In the initial variable r, asymptotic (139) is described by the formulas

$$(-y)^{i\omega}\approx(\frac{1}{4})^{i\omega}e^{i\omega r},\ \ \bar{G}_1(r\to\infty)=Me^{i\omega r}+M^*e^{-i\omega r},\ \ M=\frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\beta)}(\frac{1}{4})^{i\omega}. \tag{140}$$

It is readily can be proved that, using Kummer solutions u_3 and u_4, the corresponding complete solutions $\bar{G}_3$ and $\bar{G}_4$ are conjugate to each other and have the asymptotic $u_3\sim\text{const}\ e^{i\omega r}$, $u_4\sim\text{const}\ e^{-i\omega r}$.

10. Solutions with cylindric symmetry

Let us consider spinor Maxwell equations in the cylindric coordinates of the spherical Riemann model, which are specified by the formulas

$u_1=\sin r\cos\phi,\ u_2=\sin r\sin\phi,\ u_3=\cos r\sin z,$

$u_0=\cos r\cos z,\quad u_0^2+u_1^2+u_2^2+u_3^2=1,$

$$dS^2=dt^2-dr^2-\sin^2 r\,d\phi^2-\cos^2 r\,dz^2,\ x^\alpha=(t,r,\phi,z), \tag{141}$$

$$e^\beta_{(a)}(x)=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&\sin^{-1}r&0\\0&0&0&\cos^{-1}r\end{vmatrix},\ r\in[0,+\pi/2],\phi\in[-\pi,+\pi],\ z\in[-\pi,+\pi].$$

Ricci rotation coefficients are (we write down only non-vanishing ones)

$$\gamma_{ab0}=0,\quad \gamma_{ab1}=0,\quad \gamma_{122}=+\frac{\cos r}{\sin r},\quad \gamma_{313}=+\frac{\sin r}{\cos r}. \tag{142}$$

Starting with the general spinor form of Maxwell equations

$$\left[\sigma^c e^\alpha_{(c)}(x)\partial_\alpha+\sigma^c(\frac{1}{2}\Sigma^{ab}\otimes I+I\otimes\frac{1}{2}\Sigma^{ab})\gamma_{abc}(x)\right]\xi(x)=0, \tag{143}$$

we obtain

$$\left[\partial_t+\sigma^1\partial_r-\frac{i\sigma^2}{2}(\sigma^3\otimes I+I\otimes\sigma^3)\frac{\cos r}{\sin r}-\right.$$
$$\left.-\frac{i\sigma^3}{2}(\sigma^2\otimes I+I\otimes\sigma^2)\frac{\sin r}{\cos r}+\frac{\sigma^2}{\sin r}\partial_\phi+\frac{\sigma^3}{\cos r}\partial_z\right]\xi=0. \tag{144}$$

The structure of this equation allows the following substitution for an electromagnetic spinor

$$\xi(t,r,\phi,z)=e^{-i\omega t}\,e^{im\phi}\,e^{ikz}\begin{vmatrix} f(r) & h(r) \\ h(r) & g(r)\end{vmatrix}, \tag{145}$$

so we derive

$$\left\{-i\omega I+\sigma^1\partial_r-\frac{i\cos r}{2\sin r}\sigma^2(\sigma^3\otimes I+I\otimes\sigma^3)-\frac{i\sin r}{2\cos r}\sigma^3(\sigma^2\otimes I+I\otimes\sigma^2)+\right.$$
$$\left.+\frac{im}{\sin r}\sigma^2+\frac{ik}{\cos r}\sigma^3\right\}\begin{vmatrix} f(r) & h(r) \\ h(r) & g(r)\end{vmatrix}=0, \tag{146}$$

further we find the system of four equations:

$$\begin{aligned}
&(\frac{d}{dr}+\frac{m}{\sin r}-\frac{\sin r}{\cos r})h+(-i\omega+\frac{ik}{\cos r})f=0,\\
&(\frac{d}{dr}-\frac{m}{\sin r}-\frac{\sin r}{\cos r})h+(-i\omega-\frac{ik}{\cos r})g=0;\\
&(\frac{d}{dr}+\frac{m}{\sin r}+\frac{\cos r}{\sin r}-\frac{1}{2}\frac{\sin r}{\cos r})g+\frac{1}{2}\frac{\sin r}{\cos r}f+(-i\omega+\frac{ik}{\cos r})h=0,\\
&(\frac{d}{dr}-\frac{m}{\sin r}+\frac{\cos r}{\sin r}-\frac{1}{2}\frac{\sin r}{\cos r})f+\frac{1}{2}\frac{\sin r}{\cos r}g+(-i\omega-\frac{ik}{\cos r})h=0.
\end{aligned} \tag{147}$$

Summing and subtracting equations in each pair, we obtain (let $F=f+g, G=f-g$)

$$\begin{aligned}
&\frac{ik}{\cos r}F-i\omega G+\frac{2m}{\sin r}h=0, \qquad \frac{ik}{\cos r}G-i\omega F+2(\frac{d}{dr}-\frac{\sin r}{\cos r})h=0,\\
&-\frac{2ik}{\cos r}h-\frac{m}{\sin r}F+(\frac{d}{dr}+\frac{\cos r}{\sin r}-\frac{\sin r}{\cos r})G=0,\\
&-2i\omega h-\frac{m}{\sin r}G+(\frac{d}{dr}+\frac{\cos r}{\sin r})F=0.
\end{aligned} \tag{148}$$

Expressing from 1-st, 2-nd, and 4-th equations the variables $\omega G, \omega F, 2i\omega h$, and substituting them into the third equation, we arrive at the identity $0\equiv 0$. Therefore, only three equations in (147) are independent:

$$\begin{aligned}
&\frac{ik}{\cos r}F-i\omega G+\frac{2m}{\sin r}h=0,\\
&\frac{ik}{\cos r}G-i\omega F+2(\frac{d}{dr}-\frac{\sin r}{\cos r})h=0,\ -2i\omega h-\frac{m}{\sin r}G+(\frac{d}{dr}+\frac{\cos r}{\sin r})F=0.
\end{aligned} \tag{149}$$

Taking into account the formulas

$$F=\frac{1}{\sin r}\bar{F},\ (\frac{d}{dr}+\frac{\cos r}{\sin r})F=\frac{1}{\sin r}\frac{d\bar{F}}{dr};\quad h=\frac{1}{\cos r}\bar{h},\ (\frac{d}{dr}-\frac{\sin r}{\cos r})h=\frac{1}{\cos r}\frac{d\bar{h}}{dr},$$

we can simplify equations (149):

$$\begin{aligned}
&\frac{ik}{\cos r}\frac{1}{\sin r}\bar{F}-i\omega G+\frac{2m}{\sin r}\frac{1}{\cos r}\bar{h}=0,\\
&\frac{ik}{\cos r}G-i\omega\frac{1}{\sin r}\bar{F}+\frac{2}{\cos r}\frac{d\bar{h}}{dr}=0,\ -2i\omega\frac{1}{\cos r}\bar{h}-\frac{m}{\sin r}G+\frac{1}{\sin r}\frac{d\bar{F}}{dr}=0.
\end{aligned} \tag{150}$$

With the notation $2i\bar{h}=\bar{H}$, the last system is presented as follows

$$\omega G=\frac{k\bar{F}-m\bar{H}}{\cos r\sin r},$$

$$\frac{k}{\cos r}\omega G-\frac{\omega^2}{\sin r}\bar{F}-\frac{\omega}{\cos r}\frac{d\bar{H}}{dr}=0,\quad -\omega^2\frac{1}{\cos r}\bar{H}-\frac{m}{\sin r}\omega G+\frac{\omega}{\sin r}\frac{d\bar{F}}{dr}=0.$$

Excluding the function G, we derive

$$\frac{1}{\cos r}t(\omega\frac{d}{dr}+\frac{km}{\cos r\sin r}t)\bar{H}+\frac{1}{\sin r}(\omega^2-\frac{k^2}{\cos^2 r})\bar{F}=0,$$

$$\frac{1}{\sin r}(\omega\frac{d}{dr}-\frac{km}{\sin r\cos r})\bar{F}+\frac{1}{\cos r}(-\omega^2+\frac{m^2}{\sin^2 r})\bar{H}=0. \tag{151}$$

Let us transform this system to the new variable, $\sin r=\sqrt{z}$, $z\in[0,1]$. Then we arrive at

$$\left(2\omega\frac{d}{dz}+\frac{km}{z(1-z)}\right)\bar{H}+\frac{\omega^2-k^2-\omega^2 z}{z(1-z)}\bar{F}=0,\ \left(2\omega\frac{d}{dz}-\frac{km}{z(1-z)}\right)\bar{F}+\frac{m^2-\omega^2 z}{z(1-z)}\bar{H}=0. \tag{152}$$

Note that from Eq. (152) straightforwardly it follows two 2-nd order equations with four singular points:

$$\bar{H},\quad z=0,1,\infty,(1-\frac{k^2}{\omega^2});\qquad \bar{F},\quad z=0,1,\infty,\frac{m^2}{\omega^2}.$$

There exists possibility to reduce the problem to equations with three singular points. Indeed, let us define new variables, $\bar{H}=V+W,\ \bar{F}=V-W$, then system (152) reads

$$\left[2\omega\frac{d}{dz}+\frac{km}{z(1-z)}\right](V+W)+\frac{\omega^2-k^2-\omega^2 z}{z(1-z)}(V-W)=0,$$

$$\left[2\omega\frac{d}{dz}-\frac{km}{z(1-z)}\right](V-W)+\frac{m^2-\omega^2 z}{z(1-z)}(V+W)=0.$$

Summing and subtracting these equations we obtain

$$\left[4\omega\frac{d}{dz}+\frac{\omega^2-k^2+m^2-2\omega^2 z}{z(1-z)}\right]V-\frac{\omega^2-(k+m)^2}{z(1-z)}W=0,$$

$$\left[4\omega\frac{d}{dz}W-\frac{\omega^2-k^2+m^2-2\omega^2 z}{z(1-z)}\right]W+\frac{\omega^2-(k-m)^2}{z(1-z)}V=0. \tag{153}$$

Further we readily derive the 2-nd order equation for function $W(z)$:

$$z(z-1)W''+(2z-1)W'+\left\{-\frac{\omega(\omega+2)}{4}-\frac{k^2}{4(z-1)}+\frac{m^2}{4z}\right\}W=0. \tag{154}$$

Near the points $z=0,1$ the solutions behave as

$$z\to 0,\ W=z^A,\ A=\pm\frac{|m|}{2};\quad z\to 1,\ W=(z-1)^B,\ B=\pm\frac{|k|}{2}. \tag{155}$$

In the whole region of the variable z, solutions are searched in the form $W(z)=z^A(z-1)^B\bar{W}(z)$. After the needed calculation we arrive at

$$(z-1)z\bar{W}''+\left[2A(z-1)+2Bz+(2z-1)\right]\bar{W}'+$$

$$+\left((A+B)(A+B+1)-\frac{\omega(\omega+2)}{4}-\frac{k^2}{4(z-1)}+\frac{B^2}{z-1}+\frac{m^2}{4z}-\frac{A^2}{z}\right)\bar{W}=0.$$

Imposing the known restrictions (155), we obtain the equation

$$z(1-z)\bar{W}''+[2A+1-(2A+2B+2)z)]\bar{W}'-[(A+B)(A+B+1)-\frac{1}{4}\omega(\omega+2)]\bar{W}=0, \tag{156}$$

that is identified with the equation of a hypergeometric type

$$z(1-z)\frac{d^2F}{dz^2}+[\gamma-(\alpha+\beta+1)\gamma]\frac{dF}{dz}-\alpha\beta F=0.$$

Let us fix parameters A and B, so that solutions be finite at the points $z=0.1$:

$$A=+\frac{|m|}{2},\ B=+\frac{|k|}{2},\ \gamma=|m|+1,\ \alpha=\frac{|k|+|m|-\omega}{2},\ \beta=\frac{|k|+m|+\omega}{2}+1, \tag{157}$$

and impose the standard requirement for the polynomials:

$$\alpha=-n,\ \omega=2n+|k|+|m|,\ \beta=n+1+|m|+|k|,$$
$$n=0,1,2,...,\quad W(z)=z^{|m|/2}(z-1)^{|k|/2}F(\alpha,\beta,\gamma;z). \tag{158}$$

Now, let us turn to the equation for the second function $V(z)$. There exists symmetry between two equations in (153): the system is invariant under the formal changes

$$V\Longleftrightarrow W,\quad \omega\Longleftrightarrow-\omega,\quad m\Longleftrightarrow-m. \tag{159}$$

Therefore, from the 2-nd order equation (154) for $W(z)$, without any calculation we obtain the similar equation for $V(z)$:

$$z(z-1)V''+(2z-1)W'+[-\frac{\omega(\omega-2)}{4}-\frac{k^2}{4(z-1)}+\frac{m^2}{4z}]V=0. \tag{160}$$

We are to apply the same substitution $V(z)=z^A(z-1)^B\bar{V}(z)$. Performing the needed calculation we find the equation for $\bar{V}(z)$:

$$(z-1)z\bar{V}''+\left[2A(z-1)+2Bz+(2z-1)\right]\bar{V}'+$$
$$+[(A+B)^2+A+B-\frac{1}{4}\omega(\omega-2)-\frac{k^2}{4(z-1)}+\frac{B^2}{z-1}+\frac{m^2}{4z}-\frac{A^2}{z}]\bar{V}=0.$$

Imposing the same restrictions on A and B, we arrive at the equation of hypergeometric type

$$z(1-z)\bar{V}''+\left[2A+1-(2A+2B+2)z)\right]\bar{V}'-[(A+B)(A+B+1)-\frac{1}{4}\omega(\omega-2)]\bar{V}=0, \tag{161}$$

with parameters

$$A=+\frac{|m|}{2},\ B=+\frac{|k|}{2},\ \gamma'=|m|+1,\ \alpha'=\frac{|k|+|m|+\omega}{2},\ \beta'=\frac{|k|+m|-\omega}{2}+1. \tag{162}$$

Further, applying the polynomial condition, we find the needed solutions

$$\beta'=-n',\quad \omega=2(n'+1)+|k|+|m|,\quad \alpha'=n'+1+|m|+|k|,$$
$$n'=0,1,2,...,\quad V(z)=z^{|m|/2}(z-1)^{|k|/2}F(\alpha',\beta',\gamma';z). \tag{163}$$

In a similar way, we could study the spinor Maxwell equations also in hyperbolic Lobachevsky space, being parameterized by cylindric coordinates according to the formulas:

$$dS^2=dt^2-dr^2-\sinh^2 r\,d\phi^2-\cosh^2 r\,dz^2,\quad x^\alpha=(t,r,\phi,z), \tag{164}$$

$$e^{\beta}_{(a)}(x)=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&\sinh^{-1}r&0\\0&0&0&\cosh^{-1}r\end{vmatrix},\quad r\in[0,+\infty),\ \phi\in[-\pi,+\pi],\ z\in(-\infty,+\infty). \tag{165}$$

11. Conclusions

We have studied the possibility of interpretation for vacuum Maxwell equations on the background of any pseudo-Riemannian space-time as Maxwell equations in Minkowski space but specified in some effective medium, which constitutive relations are determined by metric of the curved space-time model. In that context, we have considered de Sitter, anti de Sitter,

and Schwarzschild spaces. Also we have studied hyperbolic Lobachevsky and spherical Riemann models, parameterized by coordinates with spherical and cylindric symmetry. We have proved that in all the examined cases, effective tensors of electric permittivity $\varepsilon_{ij}(x)$ and magnetic permeability $\mu_{ij}(x)$ obey one the same condition: $\varepsilon_{ij}(x)\mu_{jk}(x)=\delta_{ik}$. The simplicity of expressions for the explicitly found tensors $\varepsilon_{ij}(x)$ and $\mu_{jk}(x)$ is misleading, because for the each curved space-time models we are to solve Maxwell equations separately and anew. We have constructed such solutions, applying Maxwell equations in spinor form.

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