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CONTACT INTERACTION OF AXISYMMETRIC INDENTER AND POROELASTIC FOUNDATION

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Abstract. This paper discusses axially symmetric contact problems of rigid punch and poroelastic foundation interaction employing Cowin-Nunziato linear poroelasticity model. It is assumed that the punch base has a flat or parabolic shape, and the contact friction is neglected. Using Hankel's integral transform the considered contact problem is reduced to a solution of a singular integral equation with unknown contact stress. To solve this integral equation, the collocation technique is used. In the case of parabolic punch, the contact pressure and the contact area were calculated. Then it is investigated the relation between the force applied to the punch and its displacement, which is one of the main characteristics in material properties determination by means of instrumented indentation method. A comparative analysis of considered quantities for various porosity values was carried out.

Keywords: contact problem, porous materials, poroelasticity, Cowin-Nunziato model, axisymmetric contact problem, collocation method, instrumented indentation test

1. Introduction

One of the wide-spread classic elasticity theory extension is Cowin-Nunziato's theory of elastic materials with voids. This theory concerns elastic materials with distributed in the volume small pores (voids) that are free of fluid. There are two kinematical variables in this theory – displacement field and micro-dilatation (porosity) field. The theory is of practical use in studying various types of geological, biological, and synthetic porous materials, where classic elasticity theory is not suitable. The theory of elastic materials with voids (micro-dilatation theory) was introduced by Cowin and Nunziato in [1]. In the absence of volume fraction field of empty pores, this theory is reduced to the classical theory of elasticity. The linear micro-dilatation theory was described in [2] by the same authors. Furthermore, the mathematical description of a linear thermoelastic micro-dilatation theory for solids with voids is presented in [3]. The theorems of existence, uniqueness, and reciprocity also have been proven. The general micro-dilatation theory had been well studied in recent years [4-10]. Contact problems formulated in the frame of plane strain state are solved in [5] and [6] for an elastic half-plane and an elastic strip, respectively. In paper [8] a meshless local Petrov-Galerkin model of porous elastic materials based on Cowin-Nunziato theory is developed. In [9] an analysis of the axisymmetric problem for arbitrary loading conditions is presented and a particular example is given for the case of half-space with a point load. To solve the problem analytically, the Hankel's integral transform was applied. The article [10] is about the eigenvalue problems for piezoelectric bodies with voids in contact with massive rigid plane punches and coved by the system of open-circuited and short-circuited electrodes.

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A significant contribution in Cowin-Nunziato theory development had been made in [11,12], where using FEM analysis 2D and 3D static and dynamic problems were considered. The user's type finite elements for Cowin-Nunziato theory were designed.

In the case of the static problem for isotropic materials the micro-dilatation theory operates with five material parameters: Young's modulus, Poisson's ratio, α – voids diffusion parameter, β – coupling modulus and ξ – "void stiffness" modulus (characterizes the material resistance to micro loads). At present, there is no common approach in determining the last three micro-dilatation parameters. Experimental methods for parameter determination are proposed in [13]. The micro-dilatation parameters identification technique was developed on the base of structural and rigidity parameters analysis for lattice structures [14]. In particular, an example of auxetic metamaterials, that obey Cowin-Nunziato's theory in a wide range of parameters' values, is given.

The properties of Cowin-Nunziato composite media with voids, as well as the properties of ordinary porous-elastic materials can be determined, for example, by the method of effective modules in combination with finite element modeling of representative volumes. Such technology for porous composites with a different type of physical-mechanical field coupling was described in [15,16].

It should be noted that there are some other mathematical models to predict the poroelastic bodies' behavior. Biot's model is widely used to simulate geological [17,18] and biological [19] materials. Almost all of these materials contain pores with fluid. Porous medium's deformation influences in pore fluid's movement in a significant degree. Although in general fluid pressure and movement do not affect medium's deformation much.

So, in the context of geomechanics, the problem, considered in the article [20], can be used to simulate the behavior of a consolidated circular base located on an infinitely deep clay layer. Axisymmetric interaction of a rigid circular plain indenter and a poroelastic half-space saturated with a liquid is considered.

In [21] the effect of a concentrated linear load on a poroelastic half-space with fluid-saturated pores is described. The linear load moves at a constant speed along the surface of the poroelastic half-space. The basic equations for the proposed analysis are based on Biot's theory of dynamics in saturated poroelastic soils.

The present study considers axisymmetric contact problems for a rigid punch and a poroelastic half-space in the frame of Cowin-Nunziato's linear poroelasticity. The analysis is made assuming no friction at the contact area and that the punch is a cylinder with the plain base or a rotated paraboloid.

Using Hankel's integral transform the partial differential equations problem is reduced to integral equations for the unknown contact stress. The solution of the obtained integral equations was derived applying the collocation method. The accuracy of the solution strategy is verified on similar problems for an elastic half-space that has an exact solution [17].

The values of the contact stresses and the contact area in the case of the parabolic punch are determined and the value of the contact area is calculated by the method of successive approximations. The relation between the applied to the punch force and the punch's penetration displacement, which is one of the main relations in material properties determination based on the instrumented indentation testing, is also investigated. The results are presented in the form of graphs for various values of porosity. It is shown that with increasing porosity at the same value of the punch penetration displacement, the contact stresses, and the value of the applied force decrease. At the same time, the sink-in effect became more significant with increasing porosity.

2. Problem formulation

Let us consider the axisymmetric contact problem of the interaction of a rigid punch with a poroelastic half-space in the cylindrical coordinate system (r, φ, z), which deformation is described by the Cowin-Nunziato model [2]. It is assumed that the base of the punch may be flat or in the form of a paraboloid. In Figure 1 the axial section of the problem formulation is illustrated.

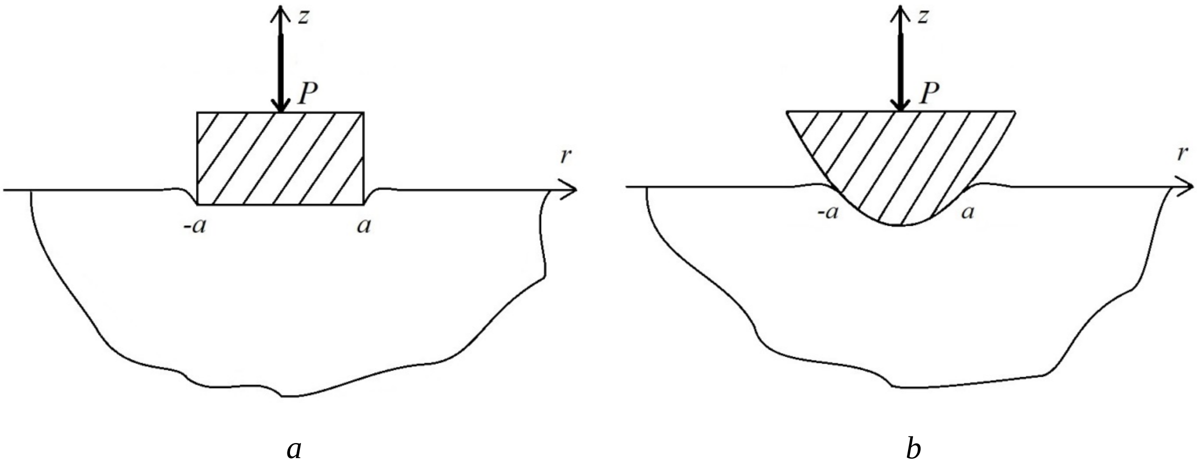

Fig. 1. Axisymmetric contact problem. Axial section. *a*) - Problem 1. *b*) – Problem 2

The function $\phi(r,z)$ describes the change in volume fraction in the reference configuration. The theory of homogeneous isotropic material with voids, according to Cowin-Nunziato theory, is described by the following system of partial differential equations [2].

$$(\lambda+\mu)\frac{\partial\theta}{\partial r}+\mu(\Delta u-\frac{u}{r^2})+\beta\frac{\partial\phi}{\partial r}=0,$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\Delta w+\beta\frac{\partial\phi}{\partial z}=0, \quad (1)$$

$$\alpha\Delta\phi-\xi\phi-\beta\theta=0,$$

where $\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2}$, $\theta=\frac{u}{r}+\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}$ and μ, λ are the elastic constants, whereas α, β, ξ – are material constants related to the micro-dilatation(porosity), u and w – are the displacements along r and z axis, respectively. Evidently, if $\beta=0$, then the field of micro-dilatation and the displacements field are independent. Thereby, in case $\beta=0$, we have pure elastic deformation of half-space.

If the functions $\phi(r,z)$, $u(r,z)$ and $w(r,z)$ are known, then the components of the stress tensor are defined by the following relations

$$\sigma_z=\lambda\theta+2\mu\frac{\partial w}{\partial z}+\beta\phi,\ \tau_{rz}=\mu\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right). \quad (2)$$

Boundary conditions of the formulated problem over the line z=0 then are

$$\tau_{rz}(r,0)=0,\ \frac{\partial\phi}{\partial z}=0,\ w(r,0)=\delta(r) \quad (r\le a),\ \sigma_z(r,0)=0 \quad (r>a). \quad (3)$$

In case of flat punch $\delta(r)=\delta=const$ (Problem 1), in case of punch, which form is a paraboloid, (Problem 2), $\delta(r)=\delta-r^2/(2R)$, where R is the radius of curvature at the top of the parabola.

3. Main integral equation

Previously, we assume that the normal stresses are known in the contact area: $\sigma_z(r,0) = q(r)$. In the future, to determine it, the integral equation (22) will be constructed. To do this, we find a solution of system (1) with the following boundary conditions

$$\tau_{rz}(r,0) = 0\,, \frac{\partial \phi}{\partial z} = 0\,,\ \sigma_z(r,0) = q(r) \quad (r \le a)\,, \sigma_z(r,0) = 0 \quad (r > a)\,. \tag{4}$$

Using the Hankel integral transform, we have:

$$u(r,z) = \int_0^\infty A(u,z) J_1(ur) u du\,,\ w(r,z) = \int_0^\infty B(u,z) J_0(ur) u du\ , \tag{5}$$

$$\phi(r,z) = \int_0^\infty F(u,z) J_0(ur) u du\ ,$$

where $J_i(u)$ $(i = 0,1)$ are the Bessel functions of zero and first order, respectively. To find the functions $A(u,z)$, $B(u,z)$, $F(u,z)$, we have the following system of ordinary differential equations of second order:

$$\begin{gathered} c^2 A'' - u^2 A - (1-c^2) u B' - uHF = 0\,, \\ (1-c^2) u A' + B'' - c^2 u^2 B + HF' = 0\,, \\ l_1^2 (F'' - u^2 F) - \frac{l_1^2}{l_2^2} F - uA - B' = 0\,. \end{gathered} \tag{6}$$

The boundary conditions (4) in terms of *A*, *B*, and *F* accounting the above relations read:

$$A_z' - uB = 0\,, (1-2c^2) u A + B_z' + HF = Q c^2 \mu^{-1}\,, F_z' = 0\ , \tag{7}$$

$$Q(u) = \int_0^a q(r) J_0(ur) r dr\,. \tag{8}$$

In the above equations we introduced positive value material parameters as follows [5]:

$$c^2 = \frac{\mu}{(\lambda + 2\mu)}\,,\ H = \frac{\beta}{\lambda + 2\mu}\,,\ l_1^2 = \frac{\alpha}{\beta}\,,\ l_2^2 = \frac{\alpha}{\xi}\,. \tag{9}$$

The solution of the system (6) is given in the following form:

$$A = A_1 e^{mz}\,,\ B = A_2 e^{mz}\,,\ F = A_3 e^{mz}\,. \tag{10}$$

Substituting (10) into (7), we obtain a homogeneous system of linear equations for A_i with a determinant.

$$\begin{vmatrix} -u^2 + c^2 m^2 & -um(1-c^2) & -uH \\ um(1-c^2) & m^2 - c^2 u^2 & mH \\ -u & -m & l_1^2 (m^2 - u^2 - l_2^{-2}) \end{vmatrix}. \tag{11}$$

Equating the determinant value to zero, we obtain the following equation to find the constant *m*:

$$m^6 l_2^2 + m^4(-3u^2 l_2^2 + N - 1) + m^2 u^2 (2(1-N) + 3u^2 l_2^2) - u^4 (1 - N + u^2 l_2^2) = 0\,, \tag{12}$$

where ($N = H \cdot l_2^2 / l_1^2$). The roots of (12) are:

$$m_1 = u\,,\ m_2 = u\,,\ m_3 = -u\,,\ m_4 = -u\,, m_5 = \sqrt{1 - N + u^2 l_2^2} \,/ l_2\,,\ m_6 = -\sqrt{1 - N + u^2 l_2^2} \,/ l_2\,. \tag{13}$$

We have to mention, that $N < 1 - c^2$ [5].

Taking into account that the stresses with $z \to -\infty$ in the half-space decay, we consider only the single root m_5 and the double root m_1.

In the case of the root m_5, according to the ordinary differential equations theory, we can substitute the solution in the following form in the first two equations of the system (6):

$$A(z)=a_5e^{m_5z},\, B(z)=b_5e^{m_5z},\; F(z)=c_5e^{m_5z}. \tag{14}$$

This leads us to

$$b_5=-a_5\frac{\sqrt{u^2l_2^2-N+1}}{ul^2},\; c_5=a_5\frac{1-N}{uNl_1^2} \tag{15}$$

In the case we employ the double root m_1, according to the ODE theory, we substitute into (7) the corresponding solution, given below:

$$A(u,z)=(a_1+a_2z)e^{uz},\; B(u,z)=(b_1+b_2z)e^{uz},\; F(u,z)=(c_1+c_2z)e^{uz} \tag{16}$$

As a result, we obtain six equations, relating the coefficients b_1, b_2, c_1, c_2, a_1, a_2

$$\begin{aligned}
&1)\; (c^2-1)ua_1+2c^2a_2+(c^2-1)ub_1+(c^2-1)b_2-N\,c_1l_1^2/l_2^2=0,\\
&2)\; (1-c^2)ua_2+(1-c^2)ub_2+N\,c_2l_1^2/l_2^2=0,\\
&3)\; (1-c^2)u^2a_1+(1-c^2)ua_2+(1-c^2)u^2b_1+2ub_2+N(uc_1+c_2)l_1^2/l_2^2=0,\\
&4)\; (1-c^2)ua_2+(1-c^2)ub_2+N\,c_2l_1^2/l_2^2=0\,(3.14),\\
&5)\; ua_1+ub_1+b_2+c_1l_1^2/l_2^2-2l_1^2uc_2=0,\\
&6)\; ua_2+ub_2+c_2l_1^2/l_2^2=0.
\end{aligned} \tag{17}$$

Here, it is worth to mention that the second and the fourth equations match up.

To express the coefficients b_1, b_2, c_1, c_2 through a_1, a_2, we take four equations from (17), namely 1), 4), 5) and 6), and as a result we get:

$$b_1=-a_1-a_2\frac{c^2+1-N}{u(c^2-1+N)},\; b_2=-a_2,\; c_1=a_2\frac{2c^2l_2^2}{l_1^2(c^2-1+N)},\; c_2=0\;. \tag{18}$$

This solution scheme was tested when solving a similar system of differential equations for the problem, considered in [5], where a different approach was suggested for deriving the solution. The result is found to be identical when constructing the integral equation. Finally, taking into account relations (14), (16), we obtain a solution of the system of equations (6) in the form:

$$\begin{aligned}
A(u,z)&=a_5e^{m_5z}+(a_1+a_2z)e^{uz},\\
B(u,z)&=b_5e^{m_5z}+(b_1+b_2z)e^{uz},\\
F(u,z)&=c_5e^{m_5z}+(c_1+c_2z)e^{uz},
\end{aligned} \tag{19}$$

where the coefficients b_5, b_1, b_2, c_5, c_1, c_2 can be expressed through a_5, a_1 and a_2 according to the relations (15) and (18).

Satisfying the boundary conditions (7), we obtain equations for determining the unknown coefficients a_1, a_2, a_5. In order not to clutter the presentation, the expressions for these coefficients are not explicitly given here. The expression for $B(u,0)$, which we need for constructing the integral equation to determine the contact stress, $q(r)$, is given:

$$B(u,0)=\frac{Q(u)}{2\mu(1-c^2)}L(u),\; L(u)=\frac{L_1(u)}{L_2(u)}\; L_i(u)=\tilde{L}_i(s)\;(i=1,2)\; s=ul_2, \tag{20}$$

$$\tilde{L}_1(s)=-2(1-c^2)l_2\left((N-1-s^2)(N(2s^2+1)-1)-2Ns^3\sqrt{s^2-N+1}\right),$$

$$\tilde{L}_2(s)=2s\left(-2Ns^3\sqrt{s^2-N+1}+N^2(2s^2+1)++N(c^2(1+2s^2)^2-2s^4-3s^2-2)+(1+s^2)(1-c^2)\right)$$

Satisfying the boundary condition $w(r,0)=\delta(r)$ at $r \le a$ and $z=0$, we have:

$$w(r,0)=\frac{1}{2\mu(1-c^2)}\int_0^\infty Q(u)L(u)uJ_0(ur)du=\delta(r)\ (r\le a). \quad (21)$$

Substituting expression (8) for *Q(u)* in (21), we obtain, after simple transformations, the desired integral equation

$$\int_0^a q(\rho)\rho\, k(\rho,r)d\rho=\frac{\mu}{1-\nu}\delta(r) \quad (r\le a), \quad (22)$$

$$k(\rho,r)=\int_0^\infty L(u)uJ_0(ur)J_0(u\rho)du. \quad (23)$$

Here, we used that $2(1-c^2)=(1-\nu)^{-1}$, where ν is the Poisson's ratio.

If N=0, then $L(u)=1/u$, that matches the case of the contact problem for elastic half-space [17].

4. Solution of the integral equation

To solve the integral equation (22), (23) with the kernel $L(u)$ (20) let us apply the direct collocation method [22]. Let the segment [0, a] be divided into n subsegments $[b_{j-1},b_j]$ according to the rule $b_j=\varepsilon\, j$ ($\varepsilon=a/n$, $j=0,1,...,n$), assuming contact stress q_j is a constant value in each subsegment. With $r_k=(b_k+b_{k-1})/2$ being the collocation points, then the integral equation (23) can be discretized according to.

$$\sum_{j=1}^n q_j\int_{b_{j-1}}^{b_j} k(\rho,r_k)\rho d\rho=\frac{\mu}{1-\nu}\delta(r_k),\ (k=\overline{1,n}). \quad (24)$$

Thus we arrive to a system of linear algebraic equations:

$$\sum_{j=1}^n q_j a_{kj}=\frac{\mu}{1-\nu}\delta(r_k)\ ,\quad (k=\overline{1,n}), \quad (25)$$

with $a_{kj}=\int_0^\infty L(u)uJ_0(ur_k)du\int_{b_{j-1}}^{b_j}\rho J_0(u\rho)d\rho$.

Given the value of the integral $\int xJ_0(x)dx=xJ_1(x)$ for a_{kj} we have:

$$a_{kj}=\int_0^\infty L(u)J_0(ur_k)\left[b_jJ_1(ub_j)-b_{j-1}J_1(ub_{j-1})\right]du. \quad (26)$$

Thereafter, the force applied to the punch can be calculated:

$$P=2\pi\int_0^a q(r)rdr=2\pi\varepsilon\sum_{k=1}^n q_k r_k. \quad (27)$$

5. Numerical results

To verify the solution strategy and the numerical scheme described above, we consider the case when N=0 (pure elastic half-space without voids). In this case, the problem has an exact solution [23]. For Problem 1 (flat punch) the solution:

$$q(r)=\frac{2\mu\delta}{\pi(1-\nu)\sqrt{a^2-r^2}},\ P=\frac{4\mu\,\delta\, a}{1-\nu}. \quad (28)$$

And for Problem 2 (parabolic punch) the solution is given as:

$$q(r) = \frac{4\mu\sqrt{a^2 - r^2}}{\pi R(1-\nu)}, \; P = \frac{8\mu\delta a}{3(1-\nu)}, \; a = \sqrt{\delta R} \;. \quad (29)$$

In Table 1 the values of $q_1^* = q_1(1-\nu)/\mu$ and $P^* = P(1-\nu)/\mu$ for $\delta = 1$ and a = 1 (Problem 1) and the values of q_1^*, P^* and a for $\delta = 1$ and R = 2 (Problem 2) for different values of the number of equations n in the system (25) are given. The last row in Table 1 shows the corresponding values of the quantities calculated by formulas (28), (29). We note that here the contact stresses are calculated for r = 0. There is a good agreement of the results even at n = 50. It should also be noted that the collocation method gives a slightly larger error in the calculation of the contact stresses in the vicinity of r = 0 and r = a, which does not affect the value of the force applied to the punch.

Table 1. Numerical scheme and exact solution comparison (no voids case)

	Problem 1		Problem 2		
n	q_1^*	P^*	q_1^*	P^*	a
50	0.640	3.98	0.900	3.77	1.44
100	0.633	3.99	0.900	3.77	1.43
[17]	0.637	4.00	0.891	3.77	1.41

The contact stresses and the displacements of the free surface apart of the contact with the punch were calculated for both Problem1 and Problem 2. In the case of parabolic punch (Problem 2) the size of the contact area a has been also determined. For Problem 1 $\delta = 1$ and a = 1, while for Problem 2, $\delta = 1$, R = 2, and the radius of the contact region was calculated by the method of successive approximations based on the initial value $a = \sqrt{\delta R}$ corresponding to contact area size for elastic half-space without voids. The calculation results for Problem 1 are shown in Figs. 2-3, while Figs. 4 and 5 present the results for Problem 2. The results reveal that with the increase in porosity at the same value of the punch penetration displacement, the contact stresses, and the magnitude of the applied force decrease, while the sink-in effect increases. Another outcome of this analysis specifically for Problem 2, is that the size of the contact area decreases with increasing of porosity.

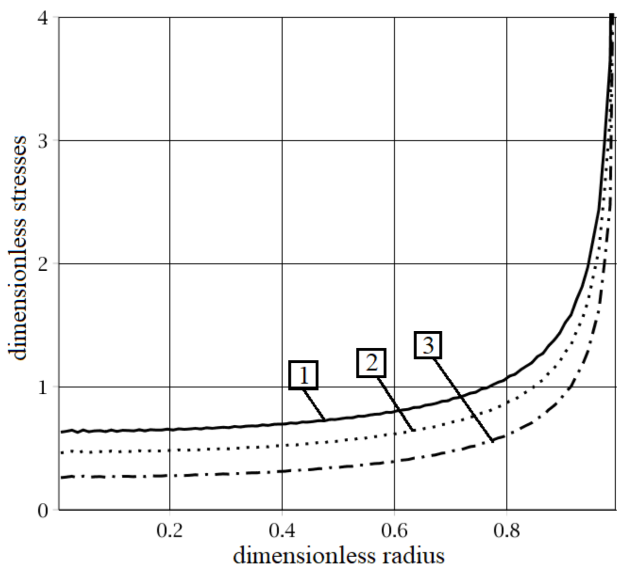

Fig. 2. Contact stresses for flat punch. c^2=0.35; l_2=1/3; 1 – N=0; 2 – N=0.3; 3 – N=0.5

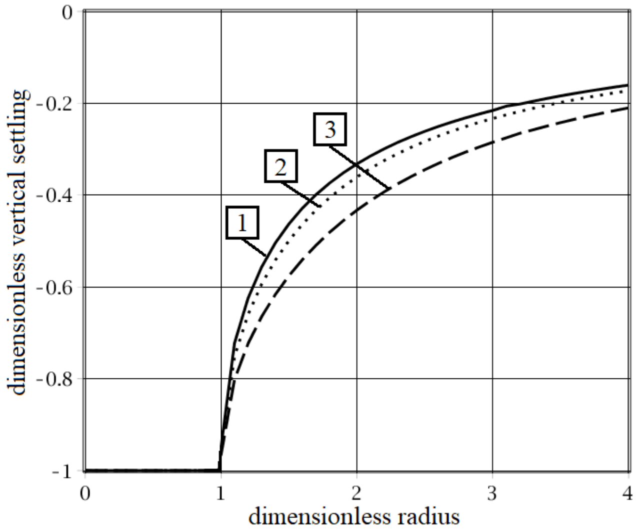

Fig. 3. Vertical settling (sink-in effect) at z=0 for a flat punch. c^2=0.35; l_2=1/3; 1 – N=0; 2 – N=0.3; 3 – N=0.5

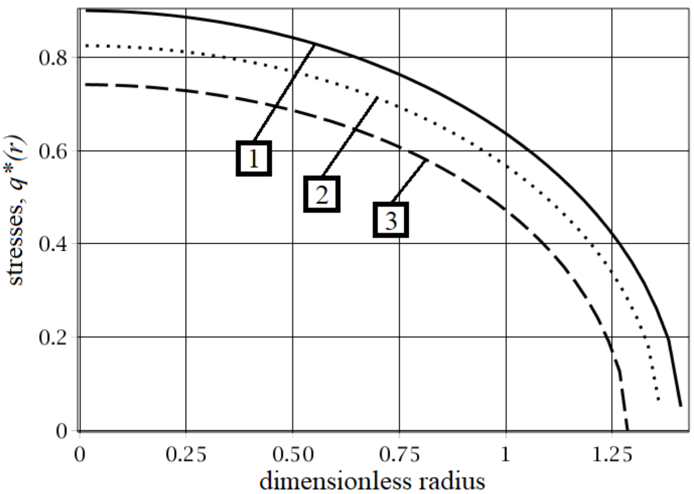

Fig. 4. Contact stresses for a parabolic punch. c^2=0.35; l_2=1/3; 1 – N=0; 2 – N=0.3; 3 – N=0.5

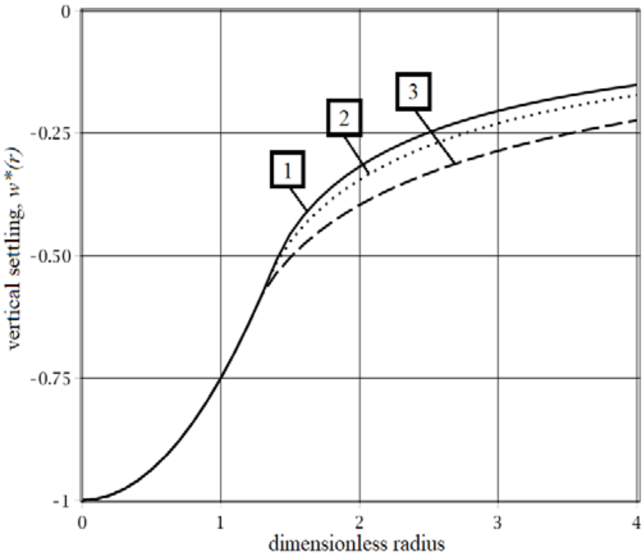

Fig. 5. Vertical settling (sink-in effect) at z=0 for a parabolic punch. c^2=0.35; l_2=1/3; 1 – N=0; 2 – N=0.3; 3 – N=0.5

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