Materials Physics and Mechanics 47 (2021) 131-138 Received: March 30, 2020

IMPACTS OF COUPLE STRESS ON S-WAVES IN A PRE-STRESSED ANISOTROPIC SANDY MEDIUM

S. Selvi[1*], L. Anitha[2]

[1]Department of Mathematics, Bannari Amman Institute of Technology, Sathyamangalam, Tamil Nadu, India

[2]Department of Mathematics, Nandha Arts and Science College, Erode, Tamil Nadu, India

*e-mail: selvi@bitsathy.ac.in

Abstract. The effect of initial stress, dry sand with couple stress on shear wave propagation in an inhomogeneous, anisotropic, thinly layered laminated medium is studied. The shear wave velocity for the particular problem has been obtained. The derived velocity expression has been computed numerically for specific rigidity parameter, density parameter, initial stress parameter, anisotropic factor, sand parameter, couple stress parameter as well as wave number, and the outcomes are shown graphically.

Keywords: incompressible, anisotropic, initial stress, couple stress, sand

1. Introduction

Seismology is the elastic wave science that describes the essence of the propagation of waves through the interior of the earth and the execution of origin through earthquakes or explosions. Because of its potential applications in geophysical prospecting, the theory of elastic surface wave inhomogeneous media is of great interest to seismologists. In order to study the internal structure of the earth, seismic waves are used. Explicit numerical results for the Earth's interior are also presented that provide the bridge between modelling results and field applications. In fact, the Earth is very complex and includes numerous forms of rocks. Such as sandiness, inhomogeneity, anisotropy, initial stress, etc., and materials with impressive characteristics. Several researchers have focused on the propagation of elastic (seismic) waves in a finite layer over half-space. In the year 2004, A.M. Abd-Alla, A.H. Hammad, and Abo-Dahab [1] developed the theory of generalized surface waves and investigated the Rayleigh waves under the influence of the gravitational field, the initial stress, and the magnetic field. A.M. Abd-Alla et al. [2] tackled the spread of S-wave in a non-homogeneous anisotropic incompressible pre-stressed medium under the impact of the gravity field in the year 2011. Biot [3] has suggested that studying a laminated medium superimposed by thin adhesive layers that are alternatively hard and soft provides some of the basic properties of anisotropic elastic media. He also suggested that it is possible to use an analogous continuous anisotropic elastic medium to provide useful insight into some of the fundamental characteristics of the static or dynamic elasticity problem. The validity of such an approximation is dependent on the assumption that the layer's rigidity contrast is not too wide and that the layer's thickness is still small enough. The influence of couple stresses on elasticity and viscoelasticity of initially stressed anisotropic solids has been proposed by Biot [4] in order to further refine his theory. He has shown that this principle is intended to provide the dynamics of layered laminated media with an approximate continuous model applicable across a wide range. This theory offers surprisingly clear and useful results when applied to a broad range of geological systems, and can therefore be used to better understand

http://dx.doi.org/10.18149/MPM.4712021_13

the physical characteristics involved. Das et al. [7] investigated the effect of initial stress on the spread of edge waves in homogeneous isotropic plates. In his study, the compound structures of thinly stratified materials were considered and used in the study of twisting and oscillation. The propagation of "Edge Waves" in an anisotropic pre-stressed plate of restricted thickness and the absolute length was explored and the scattering conditions for the edge waves were determined by S. Dey, P.K. De [8] in the year 2009. Pal Roy [9] discussed the propagation nature of the elastic waves in the layered stratified medium under both initial and couple stresses. He also looked at the dynamics of the Maxwell type solid stratified medium and the weakness of the surface of the pre-stressed laminated material. In 1988, Pijush Pal Roy and Lokenath Debnath [10] considered the advancement of edge waves in a pre-stressed laminated medium with stress couples. Predominantly the work based on the similar understanding of a laminated medium by an equivalent pre-stressed anisotropic continuation with couple stresses and concluded that for a particular compression, the couple stress increases the speed of wave propagation with the increment in wave numbers. But in general, the increment in wave numbers diminishes the speed of the wave. Rajneesh Kakar and Shikha Kakar [11] discussed the spreading of magneto shear waves in a non-homogeneous, anisotropic, incompressible, and initially stressed medium. In 2018, F.S. Bayones [12] noted that the velocity of the shear wave was strongly influenced by the initial stress, the field of gravity, and the presence of sand. Santimony Kundu, Manisha Maity [13] studied the spread of plane waves at the edge of an initially stressed dry sandy bounded layer.

In the present work, the attempt has been made to discover the phase velocity of S-waves in a non-homogeneous, anisotropic, incompressible and initially stressed medium under the presence of dry sand with couple stress. Likewise, the dispersion equations for this specific issue have been acquired to determine the S-wave velocity using linear inhomogeneities and results are shown graphically using MATLAB.

2. Formulation of the problem

The propagation of the shear wave in a pre-stressed inhomogeneous thinly layered laminated dry sandy medium of n layers and of total thickness H on the x-direction is considered. If the layers are sufficiently thin and the rigidity contrast of the layers is not too large, such a medium behaves like an elastic continuum with anisotropic properties, although the individual layers may be isotropic. Suppose that the z-axis is perpendicular to the plane of the laminations, the stress-strain relationship in the composite medium is given by [3]

$$s_{11} = \frac{2N}{\eta} e_{xx} + s , \tag{1}$$

$$s_{33} = \frac{2N}{\eta} e_{zz} + s , \tag{2}$$

$$s_{13} = \frac{2Q}{\eta} e_{xz} , \tag{3}$$

where s_{11} and s_{33} are the components of principal stress along x and z directions respectively, s_{13} is the shear stress component in the xz-plane and $s = \frac{1}{2}(s_{11} + s_{33})$, e_{xx} and e_{zz} are the principal strain components and e_{xz} is the shear strain component, N and Q are the rigidities of the medium and $\eta > 1$ is a measure of sandiness and $\eta = 1$ corresponds to an isotropic elastic medium.

The components of incremental strain are given by [1]

$$e_{xx} = \frac{\partial u_1}{\partial x},\ e_{zz} = \frac{\partial u_3}{\partial z},\ e_{xz} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}\right) \tag{4}$$

with the composite elastic coefficients N and Q where

$$N = \sum_{i=1}^{n} N_i\, \delta \ and\ \ \frac{1}{Q} = \sum_{i=1}^{n} \frac{\delta_i}{Q_i}.$$

The i^{th} layer occupies the fraction δ_i of the total thickness and if the laminations are made of isotropic materials, then $N_i = Q_i = \lambda_i \quad (i = 1,2,3,\ldots.n)$. [4]

The bending moment of the i^{th} layer of thickness h_i given by [4]

$$D_i = \frac{1}{3}(Q_i - Q)\frac{N_i}{Q_i} h_i^3 \frac{\partial^2 u_3}{\partial x^2},\ (i = 1, 2, 3, \ldots.n).$$

It follows that the average bending moment D in each layer is

$$D = \frac{1}{H}\sum_{i=1}^{n} D_i = r_s \frac{\partial^2 u_3}{\partial x^2},$$

where $r_s = \frac{1}{3} H^2 \sum_{i=1}^{n}\left(1 - \frac{Q}{Q_i}\right) N_i\, .\delta_i^3 = \sum_{i=1}^{n} r_i$ is the average couple stress coefficient.

We consider the influence of initial stresses in the composite structure. If $s_{11}^{(i)}$ and $s_{11}^{(i+1)}$ are the principle initial stresses along the $x-$direction is given by

$$s_{11} = \sum_{i=1}^{n} \delta_i\, s_{11}^{(i)}\ .$$

The z components of the initial stresses $s_{33} = 0$, the quantities $p^{(i)} = -s_{11}^{(i)} \quad (i = 1, 2, 3, \ldots.n)$ represent compressive stresses in a direction parallel to the layers. The average initial compression is $P = \sum_{i=1}^{n} \delta_i\, p^{(i)}$.

The modified dynamical equations of equilibrium in $x - z$ plane of the composite anisotropic medium under the impact of couple stress for the present problem are given by [5] and [9]

$$\frac{\partial s_{11}}{\partial x} + \frac{\partial s_{13}}{\partial z} - P\frac{\partial \omega}{\partial z} = \rho \frac{\partial^2 u_1}{\partial t^2}, \tag{5}$$

$$\frac{\partial s_{13}}{\partial x} + \frac{\partial s_{33}}{\partial z} - P\frac{\partial \omega}{\partial x} = \rho \frac{\partial^2 u_3}{\partial t^2} + r_s \frac{\partial^4 u_3}{\partial x^4}, \tag{6}$$

where ρ is the density and u_1, u_3 are the displacement components along x and z directions respectively, ω is the component of the local rotation perpendicular to $x - z$ plane

$$\omega = \frac{1}{2}\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right). \tag{7}$$

The condition of incompressibility $e_{xx} + e_{zz} = 0$ is satisfied by

$$u_1 = -\frac{\partial \varsigma}{\partial z}\ \ \&\ \ u_3 = \frac{\partial \varsigma}{\partial x}, \tag{8}$$

where $\varsigma = \varsigma(x, z, t)$.

3. Solution of the problem

Applying the equations from (1) to (3) and (7) to (8) in (5) and (6), we obtain

$$\frac{\partial s}{\partial x}-\frac{2N}{\eta}\frac{\partial^3\varsigma}{\partial x^2\partial z}+\frac{\partial}{\partial z}\left[\frac{Q}{\eta}\left(\frac{\partial^2\varsigma}{\partial x^2}-\frac{\partial^2\varsigma}{\partial z^2}\right)\right]-\frac{P}{2}\left(\frac{\partial^3\varsigma}{\partial x^2\partial z}+\frac{\partial^3\varsigma}{\partial z^3}\right)=-\rho\frac{\partial^3\varsigma}{\partial t^2\partial z}, \tag{9}$$

$$\frac{\partial s}{\partial z}+\frac{2N}{\eta}\frac{\partial^3\varsigma}{\partial x\partial z^2}+\frac{Q}{\eta}\left(\frac{\partial^3\varsigma}{\partial x^3}-\frac{\partial^3\varsigma}{\partial x\partial z^2}\right)-\frac{P}{2}\left(\frac{\partial^3\varsigma}{\partial z^2\partial x}+\frac{\partial^3\varsigma}{\partial x^3}\right)=\rho\frac{\partial^3\varsigma}{\partial t^2\partial x}+r_s\frac{\partial^5\varsigma}{\partial x^5}. \tag{10}$$

Assuming the inhomogeneities

$$Q=Q_{11}(1+az),\ N=N_{11}(1+bz),\ \rho=\rho_{11}(1+cz), \tag{11}$$

where Q_{11} and N_{11} are rigidities and ρ_{11} is the density in the homogeneous isotropic generalized medium.

Applying the equation (11) in (9) and (10)

$$\left\{\begin{aligned}&\left[\frac{Q_{11}(1+az)}{\eta}-\frac{P}{2}\right]\frac{\partial^4\varsigma}{\partial x^4}-\left[\frac{Q_{11}(1+az)}{\eta}+\frac{P}{2}\right]\frac{\partial^4\varsigma}{\partial z^4}+\\&\left[\frac{4N_{11}(1+bz)}{\eta}-\frac{2Q_{11}(1+az)}{\eta}\right]\frac{\partial^4\varsigma}{\partial x^2\partial z^2}+\left[\frac{2bN_{11}}{\eta}-\frac{aQ_{11}}{\eta}\right]\frac{\partial^3\varsigma}{\partial x^2\partial z}\\&+\frac{aQ_{11}}{\eta}\frac{\partial^3\varsigma}{\partial z^3}-\rho_{11}(1+cz)\left[\frac{\partial^4\varsigma}{\partial t^2\partial x^2}+\frac{\partial^4\varsigma}{\partial t^2\partial z^2}\right]+-\frac{\partial^3\varsigma}{\partial t^2\partial z}\rho_{11}c-r_s\frac{\partial^6\varsigma}{\partial x^6}\end{aligned}\right\}=0. \tag{12}$$

The solution of equation (12) for the propagation of sinusoidal waves at any direction is

$$\varsigma(x,z,t)=M_1e^{ik(x\cos\phi+z\sin\phi-v_1t)}, \tag{13}$$

where ϕ is the angle of propagation with the $x-$axis, v_1 is the S-wave velocity, and k is the wave number.

Applying the equation (13) in (12) and equating the real part

$$\left(\frac{v_1}{\beta}\right)^2=\frac{1}{\eta(1+cz)}\left\{\begin{aligned}&\left[(1+az)+\frac{P\eta}{2Q_{11}}\right]\sin^4\phi+\left[(1+az)-\frac{P\eta}{2Q_{11}}\right]\cos^4\phi\\&+\left[\frac{4N_{11}}{Q_{11}}(1+bz)-2(1+az)\right]\sin^2\phi\cos^2\phi+\frac{r_sk^2\eta}{Q_{11}}\cos^6\phi\end{aligned}\right\}, \tag{14}$$

where $\beta=\left(\frac{Q_{11}}{\rho_{11}}\right)^{\frac{1}{2}}$ is the S-wave velocity in a generalized homogeneous isotropic medium.

The equation (14) represents the square of the phase velocity V_R of the shear waves.

4. Inspection of a particular problem in general homogeneous medium: Investigation of equation (14):

Case A: When $a\to 0$, Q is homogeneous

i.e., Constant rigidity along the vertical direction.

$$\left(\frac{v_1}{\beta}\right)^2=\frac{1}{\eta(1+cz)}\left\{\begin{aligned}&\left[1+\frac{P\eta}{2Q_{11}}\right]\sin^4\phi+\left[1-\frac{P\eta}{2Q_{11}}\right]\cos^4\phi\\&+\left[\frac{4N_{11}}{Q_{11}}(1+bz)-2\right]\sin^2\phi\cos^2\phi+\frac{r_sk^2\eta}{Q_{11}}\cos^6\phi\end{aligned}\right\}. \tag{15}$$

When $\cos\phi=1,\sin\phi=0$ the velocity along (horizontal) $x-$ direction is

$$v_{11}^2=\frac{\beta^2}{\eta(1+cz)}\left[1-\frac{P\eta}{2Q_{11}}+\frac{r_s k^2\eta}{Q_{11}}\right]. \tag{16}$$

Equation (16) depends on the couple stress coefficient.

When $\cos\phi=0,\sin\phi=1$ the velocity along (vertical) $z-$direction is

$$v_{22}^2=\frac{\beta^2}{\eta(1+cz)}\left[1+\frac{P\eta}{2Q_{11}}\right]. \tag{17}$$

Case B: When $b\to 0$ N is homogeneous
i.e., Constant rigidity along the horizontal direction.

$$\left(\frac{v_1}{\beta}\right)^2=\frac{1}{\eta(1+cz)}\left\{\begin{array}{l}\left[(1+az)+\frac{P\eta}{2Q_{11}}\right]\sin^4\phi+\left[(1+az)-\frac{P\eta}{2Q_{11}}\right]\cos^4\phi\\ +\left[\frac{4N_{11}}{Q_{11}}-2(1+ay)\right]\sin^2\phi\cos^2\phi+\frac{r_s k^2\eta}{Q_{11}}\cos^6\phi\end{array}\right\}. \tag{18}$$

When $\cos\phi=1,\sin\phi=0$ the velocity along (horizontal) $x-$ direction is

$$v_{11}^2=\frac{\beta^2}{\eta(1+cz)}\left[(1+az)-\frac{P\eta}{2Q_{11}}+\frac{r_s k^2\eta}{Q_{11}}\right]. \tag{19}$$

Equation (19) depends on the average couple stress coefficient.

When $\cos\phi=0,\sin\phi=1$ the velocity of propagation along (vertical) $z-$direction is

$$v_{22}^2=\frac{\beta^2}{\eta(1+cz)}\left[(1+az)+\frac{P\eta}{2Q_{11}}\right]. \tag{20}$$

The velocity along vertical direction increases for $P>0$ and the waves are dispersive.

Case C: When N($b\to 0$), Q($a\to 0$), and ρ ($c\to 0$) are homogeneous

$$\left(\frac{v_1}{\beta}\right)^2=\frac{1}{\eta}\left\{\begin{array}{l}\left[1+\frac{P\eta}{2Q_{11}}\right]\sin^4\phi+\left[1-\frac{P\eta}{2Q_{11}}\right]\cos^4\phi\\ +\left[\frac{4N_{11}}{Q_{11}}-2\right]\sin^2\phi\cos^2\phi+\frac{r_s k^2\eta}{Q_{11}}\cos^6\phi\end{array}\right\}. \tag{21}$$

The velocity equation when the initial stress is absent becomes

$$\left(\frac{v_{11}}{\beta}\right)^2=\frac{1}{\eta}\left\{\begin{array}{l}\sin^4\phi+\cos^4\phi\\ +\left[\frac{4N_{11}}{Q_{11}}-2\right]\sin^2\phi\cos^2\phi+\frac{r_s k^2\eta}{Q_{11}}\cos^6\phi\end{array}\right\}. \tag{22}$$

When $\cos\phi=1,\sin\phi=0$ the velocity along (horizontal) $x-$ direction is

$$v_{11}^2=\frac{\beta^2}{\eta}\left[1+\frac{r_s k^2\eta}{Q_{11}}\right]. \tag{23}$$

When $\cos\phi=0,\sin\phi=1$ the velocity along (vertical) $z-$direction is

$$v_{22}^2=\frac{\beta^2}{\eta}.$$

In this case, the anisotropy does not affect the velocity.

5. Mathematical Analysis and Discussion. Consider the non-dimensional parameters to get numerical data on the phase velocity V_R of shear waves in the non-homogeneous pre-stressed medium:

$A=\frac{a}{b}$ (rigidity parameter), $C=\frac{c}{b}$ (density parameter), $\overline{N}=\frac{N_{11}}{Q_{11}}$ (anisotropy factor), $B=bz$ (depth), $V_R=\frac{v_1}{\beta}$ (Shear wave velocity), $\overline{P}=\frac{P\eta}{2Q_{11}}$ (initial stress factor), $\Re=\frac{r_s\eta}{Q_{11}}$ (couple stress parameter) and $\eta'=\frac{1}{\eta}$ (sandiness parameter).

Using these parameters in the Equation (14)

$$V_R^2=\frac{\eta'}{1+BC}\left\{\begin{matrix}\left[(1+AB)+\overline{P}\right]\sin^4\phi+\left[1+AB-\overline{P}\right]\cos^4\phi \\ +\left[4\overline{N}(1+B)-2(1+AB)\right]\cos^2\phi\sin^2\phi+\Re k^2\cos^6\phi\end{matrix}\right\}. \tag{24}$$

The outcomes are plotted for the parameters A=4; B=0:0.1:0.7; C=0.8; $\overline{P}$ =0.5; $\overline{N}$ =2.5; $\Re$ =0.08; k=1; $\eta'=1.5$ given by [9] and [11].

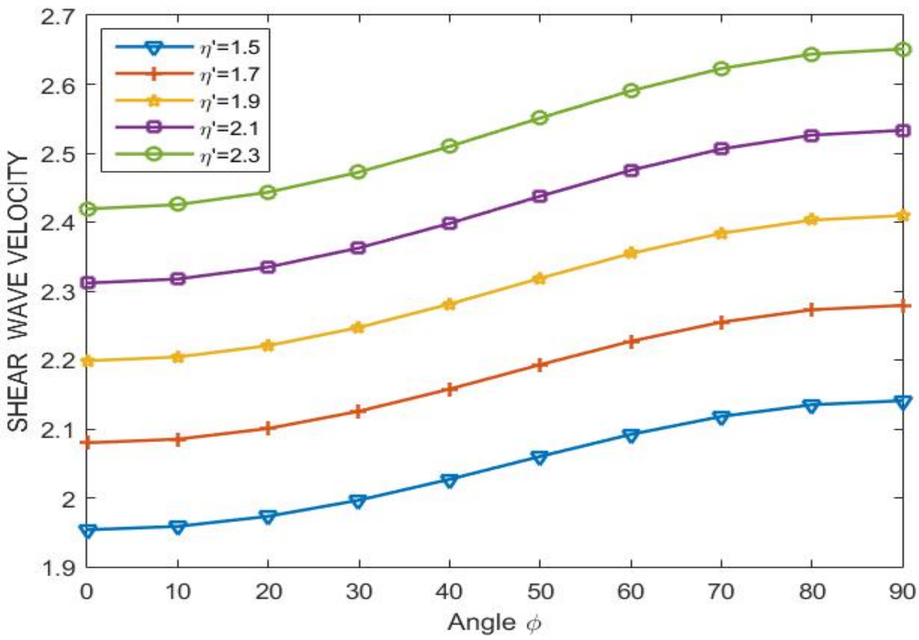

Fig. 1. Changes in wave velocity with various angle ϕ and distinct sandiness parameter $\eta'=1.5, 1.7, 1.9, 2.1, 2.3$

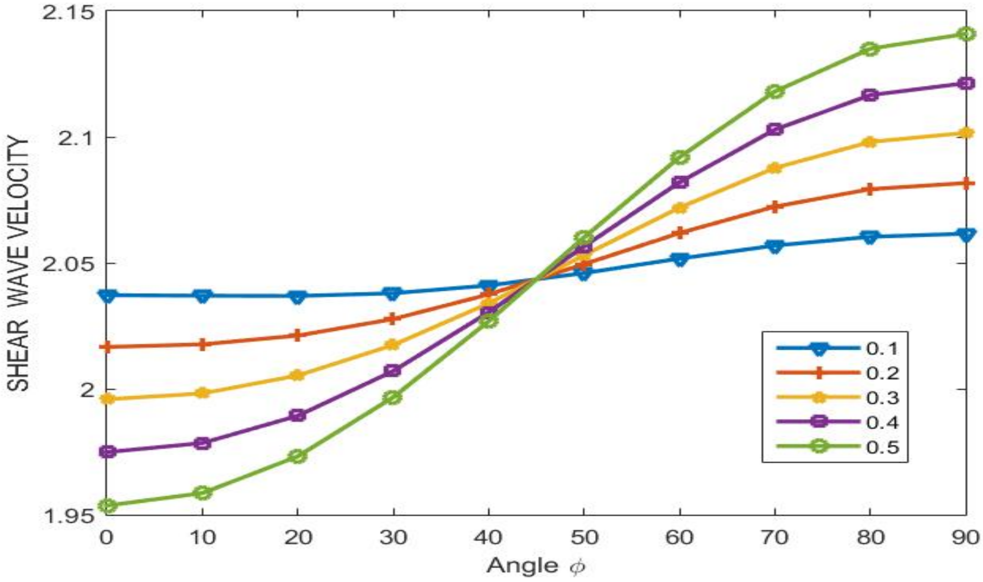

Fig. 2. Changes in wave velocity with various angle ϕ and distinct initial stress parameter $\overline{P} = 0.1, 0.2, 0.3, 0.4, 0.5$

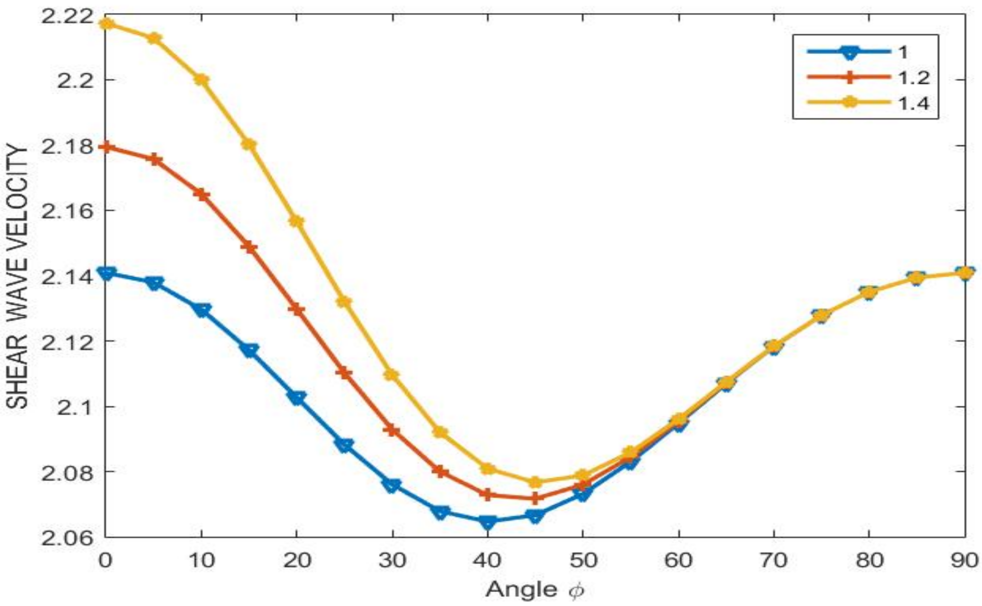

Fig. 3. Changes in wave velocity with various angle ϕ and distinct couple stress parameter $\Re = 1, 1.2, 1.4$

Figure 1 gives the variation of Shear wave velocity at different directions ϕ with x-axis at different values of sandiness parameter $\eta' = 1.5, 1.7, 1.9, 2.1, 2.3$. The velocity plots in Fig. 1 show that the shear wave velocity increases with the increase of sandiness.

Figure 2 gives the variation of shear wave velocity at different directions ϕ with x-axis at different values of the initial stress parameter $\overline{P} = 0.1, 0.2, 0.3, 0.4, 0.5$. The velocity plots in Fig. 2 show that the shear wave velocity decreases at $\varphi \in (0°, 44°)$ and increases at

$\varphi \in (46°, 90°)$ with the increasing values of the initial stress parameter but the velocity is equal at the direction $\varphi = 45°$.

Figure 3 gives the variation of shear wave velocity at different directions ϕ with x-axis at different values of couple stress parameters $\Re = 1, 1.2, 1.4$. The velocity plots in Fig. 3 show that the shear wave velocity increases at $\varphi \in (0°, 69°)$ but the velocity is equal within the range $(70°, 90°)$.

Conclusion

The S-wave velocity increases when the sandiness parameter increases. The increasing values of the initial stress parameter decrease the S-wave velocity within the range $(0°, 44°)$ and increase the velocity within the range $(46°, 90°)$ but the phase velocity is equal at $\varphi = 45°$. Also, the wave velocity increases when the couple stress parameter increases within the range $(0°, 69°)$ but the velocity is equal within the range $(70°, 90°)$. In various branches of science and technology, especially in earthquake science, acoustics, geophysics, and plasma physics, the problem of shear waves in an anisotropic elastic medium is very important.

Acknowledgements. *No external funding was received for this study.*

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