Materials Physics and Mechanics. 2023;51(2): 140-150
Research article
http://dx.doi.org/10.18149/MPM.5122023_13

Submitted: August 10, 2022
Approved: November 14, 2022
Accepted: February 1, 2023

Finite-strain elastic-plastic torsion: comparison of von Mises and Tresca materials

G.M. Sevastyanov ✉

Institute of Machinery and Metallurgy, Khabarovsk Federal Research Center FEB RAS, Komsomolsk-on-Amur, Russia

✉ akela.86@mail.ru

Abstract. Analytical and numerical results for fixed-end torsion of cylindrical specimen are presented. Finite-strain elastoplastic kinematics based on multiplicative split of deformation gradient tensor is adopted. The constitutive relations are a combination of an arbitrary hyperelastic model based on the first invariant of the left Cauchy–Green deformation tensor and the J2–plasticity model with an arbitrary isotropic strain hardening. The integral characteristics of the process, namely, torque and axial force (Swift effect), are compared with the known exact solution for a neo-Hookean hyperelastic material with Tresca yield condition. The axial force predicted by these models can differ markedly, but the torque is almost the same. For the materials with yield stress saturation, we find the limit in torque and axial force.

Keywords: torsion, finite-strain elastoplasticity, hardening, von Mises yield condition, Tresca yield condition, Swift effect, hyperelasticity

***Acknowledgements**. The reported study was carried out within the framework of the state assignment of the KhFRC FEB RAS.*

Citation: Sevastyanov GM. Finite-strain elastic-plastic torsion: comparison of von Mises and Tresca materials. *Materials Physics and Mechanics*. 2023;51(2): 140-150. DOI: 10.18149/MPM.5122023_13.

Introduction

The torsion test is a wide-spread technique to identify the parameters of constitutive relations in elasticity, plasticity or creep. For metallic materials, elastic deformation is usually small, and for an analytical description of experiments, one can utilize the rigid-plastic analysis, the deformation theory of plasticity, or the Prandtl – Reuss theory. For many polymeric materials, the situation is different: these materials are capable to sustain a significant elastic and plastic deformation, and in the plastic flow regime they exhibit nonlinear and sometimes nonmonotonic behavior. For such materials, interpretation of experimental data can be based on FEM modeling [1] or analytical studies [2].

For elastic-plastic torsion of cylindrical specimens, some analytical results are known. In [3], free-end torsion is considered for the Mooney–Rivlin material with Tresca yield condition. For the same material model, a closed-form solution for fixed-end torsion was obtained in [4]; and a solution [5] takes into account the heating caused by plastic dissipation. An approximate analytical solution [6] takes into account the dependence of the yield strength on pressure. All the mentioned results utilize the multiplicative decomposition of the deformation gradient (or decomposition of the total strain tensor, which leads to similar results) into elastic and plastic parts as the elastoplastic kinematics. The hypoelastic formulation was used in [7, 8].

The presented study provides analytical results for a class of isotropic incompressible hyperelastic materials whose elastic potential is a function (possibly nonlinear) of only the first invariant of the left Cauchy – Green deformation tensor, and plastic flow is described by von Mises yield condition with arbitrary isotropic hardening. The mentioned class of elastic materials includes the neo-Hookean solid, as well as Fung [9], Yeoh [10], Arruda–Boyce [11], Gent [12] models.

Finite-strain elastoplastic kinematics

We utilize the multiplicative decomposition of the deformation gradient tensor $\mathbf{F}$ into elastic $\mathbf{F}^e$ and plastic $\mathbf{F}^p$ parts

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p = \left(\mathbf{V}^e\mathbf{R}^e\right)\left(\mathbf{R}^p\mathbf{U}^p\right). \tag{1}$$

Here $\mathbf{R}^e$ and $\mathbf{R}^p$ are proper orthogonal rotation tensors (elastic and plastic, respectively); symmetric $\mathbf{V}^e$ and $\mathbf{U}^p$ are elastic left and plastic right stretch tensors. Following [13], it can be assumed that $\mathbf{R}^p = \mathbf{I}$.

In constitutive equations we utilize the elastic part $\mathbf{c}^e$ of eulerian deformation tensor $\mathbf{c} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} = \mathbf{V}^{-2}$:

$$\mathbf{c}^e = \left(\mathbf{B}^e\right)^{-1} = \left(\mathbf{F}^e\right)^{-T}\left(\mathbf{F}^e\right)^{-1} = \left(\mathbf{V}^e\right)^{-T}\mathbf{R}^e\left(\mathbf{R}^e\right)^{-1}\left(\mathbf{V}^e\right)^{-1} = \left(\mathbf{V}^e\right)^{-T}\left(\mathbf{V}^e\right)^{-1} = \left(\mathbf{V}^e\right)^{-2}.$$

Here $\mathbf{B}$ is the left Cauchy – Green deformation tensor; $\mathbf{I}$ is unit tensor.

From (1) by direct time differentiation, we can obtain the following evolution equation for the tensor $\mathbf{c}^e$ [6]:

$$D_{CR}\left(\mathbf{c}^e\right) \equiv \dot{\mathbf{c}}^e + \mathbf{l}^T\mathbf{c}^e + \mathbf{c}^e\mathbf{l} = 2\left(\mathbf{V}^e\right)^{-1}\mathbf{D}^p\left(\mathbf{V}^e\right)^{-1}. \tag{2}$$

Here D_{CR} denotes the Cotter – Rivlin derivative; $\mathbf{l} = \left(\nabla\otimes\mathbf{v}\right)^T = \dot{\mathbf{F}}\mathbf{F}^{-1}$ is the spatial velocity gradient, $\mathbf{F}^{-1} = \mathbf{I} - \left(\nabla\otimes\mathbf{u}\right)^T$ is the spatial deformation gradient; $\mathbf{v} = \dot{\mathbf{u}} = \partial\mathbf{u}/\partial t + \left(\mathbf{v}\nabla\right)\mathbf{u}$ is the velocity vector, $\mathbf{u}$ is the displacement vector in actual basis; $\mathbf{D}^p$ is the plastic strain rate tensor, $2\mathbf{D}^p = \mathbf{R}^e\left[\dot{\mathbf{U}}^p\left(\mathbf{U}^p\right)^{-1} + \left(\mathbf{U}^p\right)^{-1}\dot{\mathbf{U}}^p\right]\left(\mathbf{R}^e\right)^T$. The dot under symbols denotes the material time derivative.

Material model

We consider the class of nonlinear-elastic incompressible materials with free energy represented as a function of the first invariant of the tensor $\mathbf{B}^e$ only. For an incompressible materials the invariants of the tensors $\mathbf{B}^e$ and $\mathbf{c}^e$ are related as $I_1^B = \operatorname{tr}\mathbf{B}^e = I_2^c = (1/2)\left[\operatorname{tr}^2\mathbf{c}^e - \operatorname{tr}\left(\mathbf{c}^e\right)^2\right]$ [14], so the elastic rule in this case can be write out in the form

$$\boldsymbol{\sigma} = -p\mathbf{I} - 2\frac{\partial W}{\partial\mathbf{c}^e}\mathbf{c}^e = -p\mathbf{I} + 2w_2\left(\mathbf{c}^e - I_1^c\mathbf{I}\right)\mathbf{c}^e,\ w_2 = w_2\left(I_2^c\right) = \frac{\partial W}{\partial I_2^c}. \tag{3}$$

Here W is the hyperelastic potential (the volumetric density of free energy), $\boldsymbol{\sigma}$ is the Cauchy (true) stress tensor; the scalar function p is introduced due to incompressibility constraints. For purely elastic deformation, $\mathbf{c}^e$ coincides with the total deformation tensor $\mathbf{c}$. In undeformed state $\mathbf{c}=\mathbf{I}$, $I_1^c=I_2^c=3$, $p=-4w_2(3)$ hold. For the neo-Hookean solid with $W=C_1\left(I_1^B-3\right)=C_1\left(I_2^c-3\right)$, the function w_2 is constant, $w_2=C_1=\mu/2$, where μ has the same meaning as the shear modulus in linear elasticity.

We utilize the von Mises (J_2) yield criterion for isotropic hardening material:

$$\Phi = J_2-\kappa^2=0, \tag{4}$$

where $2J_2=\operatorname{tr}\operatorname{dev}^2\boldsymbol{\sigma}$, $\operatorname{dev}(\ )=(\ )-\mathbf{I}\operatorname{tr}(\ )/3$; $\kappa=\kappa(q)=\kappa_0\left[1+H(q)\right]$ is hardening law for shear yield stress, $H(0)=0$; q is the accumulated plastic strain determined by the equation $\dot{q}=\sqrt{(2/3)\operatorname{tr}\left(\mathbf{D}^p\right)^2}$.

According to the associated flow rule, the plastic strain rate tensor satisfies the equality

$$\mathbf{D}^p=\Lambda\frac{\partial\Phi}{\partial\boldsymbol{\sigma}}=\Lambda\frac{\partial\Phi}{\partial J_2}\frac{\partial J_2}{\partial\boldsymbol{\sigma}}=\Lambda\operatorname{dev}\boldsymbol{\sigma}, \tag{5}$$

where Φ is the plastic potential (4), Λ is the scalar plastic multiplier.
Then $\dot{q}=\Lambda\sqrt{(2/3)\operatorname{tr}\operatorname{dev}^2\boldsymbol{\sigma}}=2\Lambda\sqrt{J_2/3}=\left(2/\sqrt{3}\right)\Lambda\kappa$.

For an isotropic material described by potential relations (3) and (5), the tensors $\boldsymbol{\sigma}$, $\mathbf{D}^c$, $\mathbf{c}^e$ and $\mathbf{V}^e$ are coaxial, commute in product, therefore (2) takes the form

$$\begin{aligned}&\partial\mathbf{c}^e/\partial t=2\Lambda\mathbf{c}^e\operatorname{dev}\boldsymbol{\sigma}-\left[(\mathbf{v}\nabla)\mathbf{c}^e+\mathbf{l}^T\mathbf{c}^e+\mathbf{c}^e\mathbf{l}\right]=\\&=\left(2w_2\sqrt{3}\dot{q}/\kappa\right)\mathbf{c}^e\operatorname{dev}\left\{\left(\mathbf{c}^e-I_1^c\mathbf{I}\right)\mathbf{c}^e\right\}-\left[(\mathbf{v}\nabla)\mathbf{c}^e+\mathbf{l}^T\mathbf{c}^e+\mathbf{c}^e\mathbf{l}\right]\end{aligned} \tag{6}$$

Elastic stage

Let a right-circular cylinder with diameter $2R$ and height H fixed so that one of its ends remains motionless, while the other can only experience the rotation around the axis of symmetry. Thus, the movement of material points along the axis of symmetry is excluded; the height of the cylinder is constant. The angle of rotation of the movable end is a known function of time $\alpha(t)$. The lateral surface of the cylinder is stress-free.

Let introduce a cylindrical coordinate system with basis vectors $\mathbf{e}_r$, $\mathbf{e}_\varphi$, $\mathbf{e}_z$; in the actual configuration, the displacement vector has the form [5,14]

$$\mathbf{u}=r\left[1-\cos\left(\alpha zH^{-1}\right)\right]\mathbf{e}_r+r\sin\left(\alpha zH^{-1}\right)\mathbf{e}_\varphi \tag{7}$$

Under purely elastic deformation, the tensor $\mathbf{c}^e$ coincides with the total deformation tensor $\mathbf{c}=\mathbf{F}^{-T}\mathbf{F}^{-1}$ and has the following coordinate representation:

$$\mathbf{c}^e=\begin{pmatrix}c_{rr}^e & 0 & 0\\ 0 & c_{\varphi\varphi}^e & c_{\varphi z}^e\\ 0 & c_{\varphi z}^e & c_{zz}^e\end{pmatrix}, \tag{8}$$

$$c_{rr}^e=1,\ c_{\varphi\varphi}^e=1,\ c_{\varphi z}^e=-\gamma\tilde{r},\ c_{zz}^e=1+\left(\gamma\tilde{r}\right)^2. \tag{9}$$

We introduced here two dimensionless parameters: the radial coordinate $\tilde{r}=r/R$ and the surface shear strain $\gamma=\alpha R/H$. The latter is conveniently used as a time-like loading parameter.

The stress state of the cylinder is described by the well-known universal (i.e., valid for any elastic potential) Rivlin solution [15]. Taking into account the incompressibility condition $\det \mathbf{c}^e = 1$, we have $c^e_{\varphi z} = -\sqrt{c^e_{\varphi\varphi} c^e_{zz} - 1/c^e_{rr}}$. Then according to (3)

$$\boldsymbol{\sigma} = -p\mathbf{I} - 2w_2 \begin{pmatrix} c^e_{rr}\left(c^e_{\varphi\varphi} + c^e_{zz}\right) & 0 & 0 \\ 0 & c^e_{rr} c^e_{\varphi\varphi} + 1/c^e_{rr} & c^e_{\varphi z} c^e_{rr} \\ 0 & c^e_{\varphi z} c^e_{rr} & c^e_{rr} c^e_{zz} + 1/c^e_{rr} \end{pmatrix} \qquad (10)$$

and taking into account (9):

$$\boldsymbol{\sigma} = -p\mathbf{I} - 2w_2 \begin{pmatrix} 2 + (\gamma\tilde{r})^2 & 0 & 0 \\ 0 & 2 & -\gamma\tilde{r} \\ 0 & -\gamma\tilde{r} & 2 + (\gamma\tilde{r})^2 \end{pmatrix}, \; w_2 = w_2\left(I^c_2\right), \; I^c_2 = 3 + (\gamma\tilde{r})^2 .$$

The non-vanished component of the equilibrium equation $\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$ is

$$\frac{\partial \sigma_{rr}}{\partial \tilde{r}} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\tilde{r}} = 0 ;$$

it serves to calculate the function p:

$$p = -2\left(I^c_2 - 1\right) w_2 - W + W\big|_{r=R}, \quad W\big|_{r=R} = W\left(I^c_2\big|_{r=R}\right), \quad I^c_2\big|_{r=R} = 3 + \gamma^2$$

Here we used the boundary condition $\sigma_{rr}\big|_{r=R} = 0$ in determining the integration constant. For example, for neo-Hookean solid with $W\left(I^c_2\right) = (\mu/2)\left(I^c_2 - 3\right)$, $w_2 = \mu/2 = const$, one can find $p = -(\mu/2)\left[4 + \gamma^2\left(3\tilde{r}^2 - 1\right)\right]$.

Plastic flow stage

At a certain angle of rotation, the yield condition is satisfied for the first time on the lateral surface of the cylinder $r = R$. We assume that after the plastic flow starts, the expression (7) for the displacement vector remains valid. From this point on, two regions will be present in the cylinder: the elastic deformation region $r \in \left[0, r_{ep}\right]$ and the plastic deformation region $r \in \left[r_{ep}, R\right]$. These regions are separated by a moving elastic-plastic boundary $r = r_{ep}(t)$; on this interface the accumulated plastic strain q is equal to zero and the yield condition is fulfilled in the form $J_2 = \kappa_0^2$. In the elastic region, the elastic deformation tensor (8) satisfies the equalities (9). In the plastic region, $\mathbf{c}^e$ does not satisfy (9); the evolution of its components is determined by equation (6), while the initial condition for $\mathbf{c}^e$ components in the plastic region is the elastic solution (9). Equation (10) for stress tensor holds in the plastic flow region too; the function p must be continuous on the elastic-plastic boundary.

An elastic-plastic boundary is determined by the yield condition (4) $J_2\big|_{r=r_{ep}} = \kappa_0^2$. Taking into account (9) and the following expression

$$J_2 = \frac{1}{2}\operatorname{tr}\operatorname{dev}^2 \boldsymbol{\sigma} = \frac{1}{2}(2w_2)^2 \operatorname{tr}\operatorname{dev}^2\left\{\left(\mathbf{c}^e - I^c_1 \mathbf{I}\right)\mathbf{c}^e\right\} = \frac{(2w_2)^2}{3}\left[\left(c^{e\,2}_{\varphi z} + c^{e\,2}_{rr}\right)^2 + \frac{c^{e\,2}_{\varphi z}}{c^e_{rr}} - 2c^e_{rr} + \frac{1}{c^{e\,2}_{rr}}\right],$$

we have the following:

$$\left(\gamma\tilde{r}_{ep}\right)^2=\frac{3}{2}\left[\sqrt{1+\frac{4}{3}\left(\frac{\kappa_0}{2\,w_2\big|_{r=r_{ep}}}\right)^2}-1\right],\ w_2\big|_{r=r_{ep}}=w_2\left(I_2^c\big|_{r=r_{ep}}\right),\ I_2^c\big|_{r=r_{ep}}=3+\left(\gamma\tilde{r}_{ep}\right)^2. \quad (11)$$

This equation gives $r_{ep}\left(\gamma\right)$ explicitly only for the neo-Hookean material.

Let's make some *preliminary remarks*. The Hamiltonian operator in a cylindrical coordinate system is

$$\nabla=\frac{\partial}{\partial r}\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi+\frac{\partial}{\partial z}\mathbf{e}_z .$$

Then we have $\mathbf{v}\nabla=\left(z/R\right)\left(d\gamma/dt\right)\left(\partial/\partial\varphi\right)$, and taking into account the derivatives of basis vectors ($\partial\mathbf{e}_r/\partial\varphi=\mathbf{e}_\varphi$, $\partial\mathbf{e}_\varphi/\partial\varphi=-\mathbf{e}_r$, all other vanish), we obtain

$$\mathbf{v}=z\tilde{r}\frac{d\gamma}{dt}\mathbf{e}_\varphi,\ \mathbf{l}=\left(\nabla\otimes\mathbf{v}\right)^T=\frac{d\gamma}{dt}\begin{pmatrix}0 & -z/R & 0\\ z/R & 0 & \tilde{r}\\ 0 & 0 & 0\end{pmatrix}$$

Now we can write the coordinate representation of the tensor $\left(\mathbf{v}\nabla\right)\mathbf{c}^e+\mathbf{l}^T\mathbf{c}^e+\mathbf{c}^e\mathbf{l}$ which appears on the right-hand side of equation (6):

$$\left(\mathbf{v}\nabla\right)\mathbf{c}^e+\mathbf{l}^T\mathbf{c}^e+\mathbf{c}^e\mathbf{l}=\tilde{r}\frac{d\gamma}{dt}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & c_{\varphi\varphi}^e\\ 0 & c_{\varphi\varphi}^e & 2c_{\varphi z}^e\end{pmatrix} \quad (12)$$

Further we have

$$\mathbf{c}^e\,\mathrm{dev}\left\{\left(\mathbf{c}^e-I_1^c\mathbf{I}\right)\mathbf{c}^e\right\}=$$

$$=\frac{1}{3}\begin{pmatrix}2-c_{rr}^{e\,2}\left(c_{\varphi\varphi}^e+c_{zz}^e\right) & 0 & 0\\ 0 & 3-\dfrac{c_{\varphi\varphi}^e}{c_{rr}^e}-c_{rr}^e c_{\varphi\varphi}^e\left(c_{\varphi\varphi}^e+c_{zz}^e\right) & -\dfrac{c_{\varphi z}^e}{c_{rr}^e}\left[1+c_{rr}^{e\,2}\left(c_{\varphi\varphi}^e+c_{zz}^e\right)\right]\\ 0 & -\dfrac{c_{\varphi z}^e}{c_{rr}^e}\left[1+c_{rr}^{e\,2}\left(c_{\varphi\varphi}^e+c_{zz}^e\right)\right] & 3-\dfrac{c_{zz}^e}{c_{rr}^e}-c_{rr}^e c_{zz}^e\left(c_{\varphi\varphi}^e+c_{zz}^e\right)\end{pmatrix}. \quad (13)$$

Here it is taken into account that the incompressibility condition $\det\mathbf{c}^e=1$ is also fulfilled in the plastic region [6], therefore, $c_{\varphi z}^{e\,2}=c_{\varphi\varphi}^e c_{zz}^e-1/c_{rr}^e$. It is easy to see that if $c_{rr}^e=c_{\varphi\varphi}^e$ then the components of tensor (13) satisfy the equality $\left(\mathbf{c}^e\,\mathrm{dev}\left\{\left(\mathbf{c}^e-I_1^c\mathbf{I}\right)\mathbf{c}^e\right\}\right)_{rr}=\left(\mathbf{c}^e\,\mathrm{dev}\left\{\left(\mathbf{c}^e-I_1^c\mathbf{I}\right)\mathbf{c}^e\right\}\right)_{\varphi\varphi}$ and the corresponding components of tensor (12) are zero. This allows us to conclude that in this case $\partial c_{rr}^e/\partial t=\partial c_{\varphi\varphi}^e/\partial t$. But from (9) it follows that $c_{rr}^e=c_{\varphi\varphi}^e=1$ on the elastic-plastic boundary. Consequently, in the plastic region, the components c_{rr}^e and $c_{\varphi\varphi}^e$ also coincide, although, in contrast to the elastic region, they change during loading. This interesting property, which is typical only for materials with I_1^B–based hyperelastic potentials, makes it possible (as for a viscoelastic material [16]) to obtain a simple analytical solution of the problem.

Thus, the system of differential equations (6) contains unknown functions c^e_{rr}, $c^e_{\varphi\varphi}$, c^e_{zz}, $c^e_{\varphi z}$, q, which are also related by algebraic equations $c^e_{\varphi\varphi} = c^e_{rr}$ and $c^{e\,2}_{\varphi z} = c^e_{\varphi\varphi} c^e_{zz} - 1/c^e_{rr}$, as well as the yield condition (4).

Instead of the equation for component c^e_{zz}, it is more convenient to use the evolution equation for the second invariant of the elastic deformation tensor $I^c_2 = (1/2)\left[\mathrm{tr}^2\,\mathbf{c}^e - \mathrm{tr}\left(\mathbf{c}^e\right)^2\right] = c^e_{rr}c^e_{\varphi\varphi} + c^e_{\varphi\varphi}c^e_{zz} + c^e_{rr}c^e_{zz} - c^{e\,2}_{\varphi z} = c^{e\,2}_{rr} + c^e_{rr}c^e_{zz} + 1/c^e_{rr}$, which can be obtained from (6):

$$\frac{\partial I^c_2}{\partial t} = \left(2c^e_{rr} + c^e_{zz} - \frac{1}{c^{e\,2}_{rr}}\right)\frac{\partial c^e_{rr}}{\partial t} + c^e_{rr}\frac{\partial c^e_{zz}}{\partial t} =$$

$$= -2\frac{2w_2\sqrt{3}}{\kappa}\frac{\partial q}{\partial t}\left(\frac{1}{3}I^{c2}_2 - \frac{I^c_2}{c^e_{rr}} - c^e_{rr} + \frac{1}{c^{e\,2}_{rr}}\right) + 2\tilde{r}\frac{d\gamma}{dt}c^e_{rr}\sqrt{I^c_2 - c^{e\,2}_{rr} - 2/c^e_{rr}}.$$

Here it is taken into account that for the considered deformation pattern $\dot{q} = \partial q/\partial t$ holds.

Just as in [5,17], we assume that the components of the elastic deformation tensor are functions of the accumulated plastic strain q, which in turn is a function of the self-similar variable $\gamma\tilde{r}$. In what follows, it is convenient to consider as a parameter the elastic strain component c^e_{rr}. In this case, the system of governing differential equations (6) has the form

$$\frac{dq}{dc^e_{rr}} = -\frac{1}{\sqrt{3}}\frac{\kappa}{2w_2}\left(\frac{1}{3}I^c_2 c^e_{rr} - 1\right)^{-1} \tag{14a}$$

$$\frac{d\left(\gamma\tilde{r}\right)}{dc^e_{rr}} = -\frac{\dfrac{I^c_2}{c^e_{rr}} - \dfrac{1}{2}\dfrac{dI^c_2}{dc^e_{rr}} + \dfrac{1}{c^{e\,2}_{rr}}\dfrac{1-c^{e\,3}_{rr}}{c^e_{rr}I^c_2/3-1}}{c^e_{rr}\sqrt{I^c_2 - c^{e\,2}_{rr} - 2/c^e_{rr}}} \tag{14b}$$

In the last of these equations, dI^c_2/dc^e_{rr} is calculated according to the chain rule as the derivative of the composite function $I^c_2 = I^c_2\left(c^e_{rr}, q\right)$:

$$\frac{dI^c_2}{dc^e_{rr}} = \frac{\partial I^c_2}{\partial c^e_{rr}} + \frac{\partial I^c_2}{\partial q}\frac{dq}{dc^e_{rr}} = \frac{\partial I^c_2}{\partial c^e_{rr}} - \frac{\partial I^c_2}{\partial q}\frac{1}{\sqrt{3}}\frac{\kappa}{2w_2}\left(\frac{1}{3}I^c_2 c^e_{rr} - 1\right)^{-1}$$

Equation (14a) contains the known functions $w_2 = \partial W/\partial I^c_2 = w_2\left(I^c_2\right)$ and $\kappa(q)$ describing the elastic and plastic properties of the material. The invariant I^c_2 can be expressed in terms of c^e_{rr} and q with yield condition (4)

$$J_2 = \left(2w_2\right)^2\left(\frac{1}{3}I^{c2}_2 - \frac{I^c_2}{c^e_{rr}} - c^e_{rr} + \frac{1}{c^{e\,2}_{rr}}\right) = \kappa^2. \tag{15}$$

Then integration (14a) with the initial condition ($c^e_{rr} = 1$ when $q = 0$) implicitly determines the function $c^e_{rr}(q)$. It is worth noting that expression $I^c_2 = I^c_2\left(c^e_{rr}, q\right)$ from (15) can be quite complex.

Given that the function $q = q\left(c_{rr}^{e}\right)$ is known from solution (14a), the right-hand side of equation (14b) is a function of c_{rr}^{e}. This allows us to integrate (14b) directly, taking into account the initial condition (11) ($\gamma\tilde{r} = \gamma\tilde{r}_{ep}$ when $c_{rr}^{e} = 1$), by finding $\gamma\tilde{r}$ as a function of c_{rr}^{e}.

Formulas for torque and axial force

Using the found solution the torque can be expressed as

$$M = 2\pi\int_0^R r^2\sigma_{\varphi z}dr = 4\pi R^3\gamma\int_0^{\tilde{r}_{ep}}\tilde{r}^3 w_2 d\tilde{r} - 4\pi R^3\int_{\tilde{r}_{ep}}^1 \tilde{r}^2 c_{\varphi z}^{e} c_{rr}^{e} w_2 d\tilde{r} =$$
$$= 4\pi R^3\gamma\int_0^{\tilde{r}_{ep}}\tilde{r}^3 w_2 d\tilde{r} + \frac{4\pi R^3}{\gamma^3}\int_{c_{rr}^e|_{r=R}}^{1}\left(\frac{I_2^c}{c_{rr}^e} - \frac{1}{2}\frac{dI_2^c}{dc_{rr}^e} + \frac{1}{c_{rr}^{e\,2}}\frac{1-c_{rr}^{e\,3}}{c_{rr}^e I_2^c/3-1}\right)G^2 w_2 dc_{rr}^e \tag{16}$$

Here, the first term reflects the contribution of the elastic region, the second corresponds to the plastic one. In the first term $w_2 = w_2\left(3+\left(\gamma\tilde{r}\right)^2\right)$; in the second term $w_2 = w_2\left(I_2^c\right)$; $I_2^c = I_2^c\left(c_{rr}^e, q\right)$ is the function known from (15); $q = q\left(c_{rr}^e\right)$ is the solution of (14a); the function $G = \gamma\tilde{r}$ is given by equation (14b):

$$G\left(c_{rr}^e\right) = \gamma\tilde{r} = \gamma\tilde{r}_{ep} + \int_{c_{rr}^e}^{1}\frac{\dfrac{I_2^c}{c_{rr}^e} - \dfrac{1}{2}\dfrac{dI_2^c}{dc_{rr}^e} + \dfrac{1}{c_{rr}^{e\,2}}\dfrac{1-c_{rr}^{e\,3}}{c_{rr}^e I_2^c/3-1}}{c_{rr}^e\sqrt{I_2^c - c_{rr}^{e\,2} - 2/c_{rr}^e}}dc_{rr}^e \tag{17}$$

The integration limit $c_{rr}^e\big|_{r=R}$ in (16) can be considered as a solution parameter, while the value of the loading parameter γ corresponding to it can be expressed from (17) as $\gamma = G\left(c_{rr}^e\big|_{r=R}\right)$.

To calculate the axial force $Q = -2\pi R^2\int_0^1\sigma_{zz}\tilde{r}d\tilde{r}$, we represent the integral $\int\sigma_{zz}\tilde{r}d\tilde{r}$ in the following form using expressions (10):

$\int\sigma_{zz}\tilde{r}d\tilde{r} = \int\sigma_{rr}\tilde{r}d\tilde{r} + 2\int\left(c_{rr}^{e\,2} - 1/c_{rr}^e\right)\tilde{r}w_2 d\tilde{r}$.

Next, we integrate by parts the first integral on the right-hand side of this expression and use the equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial\tilde{r}} = -\frac{\sigma_{rr}-\sigma_{\varphi\varphi}}{\tilde{r}} = -2w_2\frac{c_{rr}^e c_{zz}^e + 1/c_{rr}^e}{\tilde{r}} = -2w_2\frac{I_2^c - c_{rr}^{e\,2}}{\tilde{r}}.$$

We have:

$$\int\sigma_{zz}\tilde{r}d\tilde{r} = \frac{1}{2}\sigma_{rr}\tilde{r}^2 - \frac{1}{2}\int\tilde{r}^2\frac{d\sigma_{rr}}{d\tilde{r}}d\tilde{r} + 2\int\left(c_{rr}^{e\,2} - 1/c_{rr}^e\right)\tilde{r}w_2 d\tilde{r} = \frac{1}{2}\sigma_{rr}\tilde{r}^2 + \int\left(3c_{rr}^{e\,2} - I_2^c\right)\tilde{r}w_2 d\tilde{r}.$$

This expression holds for both the elastic and plastic regions. Then, taking into account the continuity of the radial stress on the elastic-plastic boundary, as well as the fact that the lateral surface of the cylinder is stress-free, $\sigma_{rr}\big|_{\tilde{r}=1} = 0$, we have:

$$Q = -2\pi R^2\left\{\left[\frac{1}{2}\sigma_{rr}\tilde{r}^2\right]_{\tilde{r}=0}^{\tilde{r}=1} + \int_0^1\left(3c_{rr}^{e\,2} - I_2^c\right)\tilde{r}w_2 d\tilde{r}\right\} = 2\pi R^2\int_0^1\left(I_2^c - 3c_{rr}^{e\,2}\right)\tilde{r}w_2 d\tilde{r} =$$
$$= 2\pi R^2\gamma^2\int_0^{\tilde{r}_{ep}}\tilde{r}^3 w_2 d\tilde{r} + \frac{2\pi R^2}{\gamma^2}\int_{c_{rr}^e|_{r=R}}^{1}\left(\frac{I_2^c}{c_{rr}^e} - \frac{1}{2}\frac{dI_2^c}{dc_{rr}^e} + \frac{1}{c_{rr}^{e\,2}}\frac{1-c_{rr}^{e\,3}}{c_{rr}^e I_2^c/3-1}\right)\frac{\left(I_2^c - 3c_{rr}^{e\,2}\right)G w_2 dc_{rr}^e}{c_{rr}^e\sqrt{I_2^c - c_{rr}^{e\,2} - 2/c_{rr}^e}} \tag{18}$$

Limit values for torque and axial force. If the strain hardening of the material is limited (meaning the saturation of the yield stress), then there are limit values of the torque and axial force, to which these quantities tend with an increase in the angle of rotation. There is also a limiting stress state, which corresponds to a certain value of the elastic deformation tensor. It follows from (14a) that $I_2^c c_{rr}^e \to 3$ with $\gamma \to \infty$. Then (15) becomes $c_{rr}^{e\,-2} - c_{rr}^e = \tilde{\kappa}_\infty^{\,2}$, where $\tilde{\kappa}_\infty = \lim\limits_{q\to\infty} \kappa/(2w_2)$. The root of this equation, up to quadratic terms, is $c_{rr}^e = 1 - \tilde{\kappa}_\infty^{\,2}/3$. Applying L'Hopital's rule to formulas (16) and (18) taking into account (14b) we have:

$$\lim_{\gamma\to\infty} M \approx \frac{2\pi R^3}{3}\kappa_\infty\left(1-\frac{\tilde{\kappa}_\infty^{\,2}}{3}\right) \approx \frac{2\pi R^3}{3}\kappa_\infty,\ \lim_{\gamma\to\infty} Q \approx \frac{3}{2}\pi R^2 \kappa_\infty \tilde{\kappa}_\infty\left(1-\frac{\tilde{\kappa}_\infty^{\,2}}{3}\right) \approx \frac{3}{2}\pi R^2 \kappa_\infty \tilde{\kappa}_\infty \qquad (19)$$

Plastically non-hardening neo-Hookean solid. Let's illustrate the solution with a simple example of the neo-Hookean solid with $W = (\mu/2)(I_2^c - 3)$, $w_2 = \mu/2 = \text{const}$. If the material is non-hardening in the plastic range, then from (15) it follows that

$$I_2^c = \frac{3}{2}\frac{S+1}{c_{rr}^e},\ \tilde{\kappa}_0 = \frac{\kappa_0}{\mu},\quad S\left(c_{rr}^e\right) = \sqrt{\frac{4}{3}c_{rr}^{e\,2}\left(c_{rr}^e + \tilde{\kappa}_0^{\,2}\right) - \frac{1}{3}} \qquad (20)$$

The integral of equation (14a) is

$$q = \frac{2}{\sqrt{3}}\tilde{\kappa}_0 \int_{c_{rr}^e}^{1} \frac{d\xi}{S(\xi)-1}$$

From (20) $\dfrac{dI_2^c}{dc_{rr}^e} = \dfrac{\partial I_2^c}{\partial c_{rr}^e} = \dfrac{1}{2}\dfrac{2c_{rr}^{e\,3}+1-3S}{Sc_{rr}^{e\,2}}$, then

$$G = \gamma\tilde{r} = \gamma\tilde{r}_{ep} + \frac{\sqrt{2}}{4}\int_{c_{rr}^e}^{1} \frac{(S+1)\left[3S(2S-1)+1\right] - 2(5S-1)\xi^3}{S(S-1)\xi^{5/2}\left(3S-2\xi^3-1\right)^{1/2}} d\xi \qquad (21)$$

where from (11) we state

$$\left(\gamma\tilde{r}_{ep}\right)^2 = \frac{3}{2}\left[\sqrt{1+\frac{4}{3}\tilde{\kappa}_0^{\,2}} - 1\right]. \qquad (22)$$

The torque is

$$M = \frac{\pi\mu R^3}{2\gamma^3}\left\{\left(\gamma\tilde{r}_{ep}\right)^4 + \int_{c_{rr}^e|_{r=R}}^{1} \frac{(S+1)\left[3S(2S-1)+1\right] - 2\xi^3(5S-1)}{S(S-1)\xi^2} G^2 d\xi\right\}$$

The axial force is

$$Q = \frac{\pi\mu R^2}{4\gamma^2}\left\{\left(\gamma\tilde{r}_{ep}\right)^4 + \frac{3}{\sqrt{2}}\int_{c_{rr}^e|_{r=R}}^{1} \frac{(S+1)\left[3S(2S-1)+1\right] - 2(5S-1)\xi^3}{S(S-1)\xi^{5/2}\left(3S-1-2\xi^3\right)^{1/2}} \frac{S+1-2\xi^3}{\xi} G d\xi\right\}$$

The solution parameter $c_{rr}^e\big|_{r=R}$ defines by (21) the surface shear strain $\gamma = G\left(c_{rr}^e\big|_{r=R}\right)$.

Comparison with the exact solution for the Tresca yield condition

The torque and axial force in the fixed-end elastic-plastic torsion for a neo-Hookean material with the Tresca yield condition have the following expressions [4,6]:

$$M = 2\pi\int_0^R \sigma_{\varphi z} r^2 dr = \left(a^2-4\right)^2 \frac{\pi\mu R^3}{2\gamma^3}\left[1+2a^2\int_0^\delta \frac{f^2-1}{f^2+1}\left(1+a\varsigma\right)^2 d\varsigma\right]$$

$$Q = -2\pi\int_0^R \sigma_{zz} r dr = \left(a^2-4\right)^2 \frac{\pi\mu R^2}{4\gamma^2}\left[1+2a\int_0^\delta \left(\frac{6a}{\sqrt{a^2-4}}\frac{f}{f^2+1}-1\right)\left(1+a\varsigma\right)d\varsigma\right]$$

$$f\left(\varsigma\right)=\sqrt{\frac{a+2}{a-2}\frac{1+\left(a-1\right)\tanh\varsigma}{1+\left(a+1\right)\tanh\varsigma}},\ a=\min\left\{\sqrt{4+\gamma^2},\sqrt{2\left(\sqrt{1+\tilde{\kappa}_0^{\,2}}+1\right)}\right\},\ \delta=\left(\frac{\gamma}{\sqrt{a^2-4}}-1\right)\frac{1}{a}$$

Under purely elastic deformation ($a=\sqrt{4+\gamma^2}$, $\delta=0$), the formulas above take the well-known form $M=\left(1/2\right)\pi\mu\gamma R^3$, $Q=\left(1/4\right)\pi\mu\gamma^2R^2$.

Figure 1 shows the plots of the dimensionless torque $\tilde{M}=2M/\left(\pi\mu R^3\right)$ and axial force $\tilde{Q}=4Q/\left(\pi\mu R^2\right)$ for the Tresca and von Mises plasticity models combined with the hyperelastic neo-Hookean model. The expression for surface shear strain, at which plastic flow starts in the sample, in the Tresca model is $\gamma_y=\sqrt{2\left(\sqrt{1+\tilde{\kappa}_0^{\,2}}-1\right)}$ [4], for von Mises model according to (22) $\gamma_y=\sqrt{\left(3/2\right)\left[\sqrt{1+\left(4/3\right)\tilde{\kappa}_0^{\,2}}-1\right]}$. Both of these quantities at $\tilde{\kappa}_0 << 1$ are practically indistinguishable from the small-strain theory approximation $\gamma_y=\tilde{\kappa}_0$.

The torque is almost the same for von Mises model as for Tresca model, if the models are calibrated such that the pure shear yield strength is the same for both models. The Swift effect (the appearance of an axial force in fixed-end torsion) is more pronounced for the von Mises model, and the difference increases with increasing deformation of the sample. For an ideally plastic (non-hardening) neo-Hokean solid, both models predict that both the torque and the axial force reach their limiting stationary values at large angles of rotation. For the Tresca material

$$\lim_{\gamma\to\infty} M=\frac{2}{3}\frac{\tilde{\kappa}_0}{\sqrt{1+\tilde{\kappa}_0^{\,2}}}\pi\mu R^3,\ \lim_{\gamma\to\infty} Q=\frac{\pi\mu R^2}{2}\frac{\sqrt{1+\tilde{\kappa}_0^{\,2}}+2\tilde{\kappa}_0^{\,2}-1}{\sqrt{1+\tilde{\kappa}_0^{\,2}}}.$$

If $\tilde{\kappa}_0 << 1$ then the first of these quantities differs little from the value $M=\left(2/3\right)\pi R^3\kappa_0$, which can be obtained by completely neglecting elastic effects and assuming that $\sigma_{\varphi z}=\kappa_0$. The second represents a purely non-linear effect and it's approximately equal to $\left(5/4\right)\pi R^2\kappa_0\left(\kappa_0/\mu\right)$ at $\tilde{\kappa}_0 << 1$.

According to (19), for the von Mises material

$$\lim_{\gamma\to\infty} M\approx\frac{2\pi R^3}{3}\kappa_0,\ \lim_{\gamma\to\infty} Q\approx\frac{3\pi R^2\kappa_0}{2}\frac{\kappa_0}{\mu}$$

The limit value of axial force for the von Mises model is 20% higher than for the Tresca model.

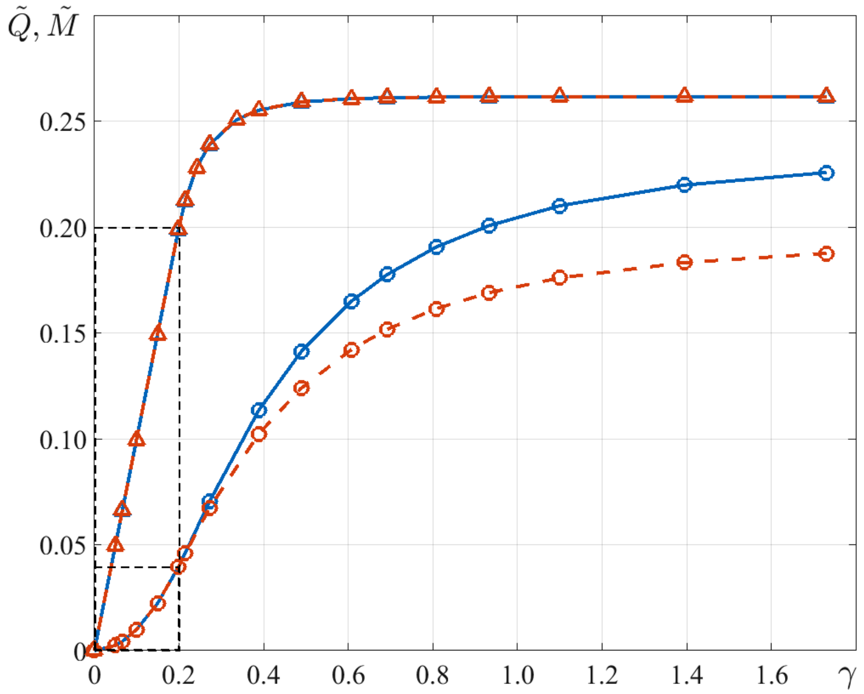

Fig. 1. Dimensionless axial force $\tilde{Q} = 4Q/(\pi\mu R^2)$ (circles) and torque $\tilde{M} = 2M/(\pi\mu R^3)$ (triangles). The solid lines correspond to the presented solution for the von Mises yield condition, the dashed lines correspond to the solution [4] for the Tresca yield condition. The dashed rectangles mark the range of purely elastic torsion. The abscissa axis displays the loading parameter (surface shear strain $\gamma = \alpha R/H$). Neo-Hookean solid with $\tilde{\kappa}_0 = \kappa_0/\mu = 0.2$

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THE AUTHORS

Sevastyanov G.M.
e-mail: akela.86@mail.ru