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Hydrogen diffusion in rotating cylindrical elastic bodies

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Abstract. In this paper, we study the distribution of the hydrogen concentration in a rotating cylindrical elastic body compressed by two concentrated forces. This problem is relevant for diagnostics of bearing failure due to the influence of hydrogen on the mechanical properties of structures. During the study, the plane stress state of a loaded cylindrical elastic body is determined by means of the theory of functions of a complex variable. We used the transition to a rotating coordinate system to obtain the static problem of hydrogen diffusion in a body loaded by the known stresses. The solution of the problem includes the methods of asymptotic analysis for the simplification of partial differential equation, the expansion in a Fourier series and the Galerkin approach for finding the expansion coefficients for several harmonics. The results can be useful for calculating the hydrogen concentration distribution in roller bearings.
Keywords: hydrogen diffusion; hydrogen destruction; diffusion equation; stress state; rolling bearings

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Introduction

Hydrogen is one of the factors that have a negative impact on the working capacity of structures. The phenomenon of hydrogen embrittlement can cause the destruction of constructions [1–5]. Therefore, tens of thousands of scientific works are devoted to the study of the nature of hydrogen embrittlement [6,7]. At the moment, there are several main approaches to describing the influence of hydrogen on the mechanical properties of materials: HELP model [8–10], HEDE model [11–13], model with taking into account the internal pressure of hydrogen [14] and multicontinual model [15,16], that takes into account the influence of hydrogen with different bond energies on the mechanical characteristics of the material. Even low hydrogen concentrations lead to degradation of mechanical properties of the material. When the limiting hydrogen concentrations are not exceeded, the diffusion laws formulated by Fick [17,18] are also fulfilled. Fick's second law describes the change in concentration in one dimension with respect to time. Generalizing Fick's equations to the three-dimensional case, we obtain the equation of diffusion of material in elastic body. And the equation of hydrogen diffusion in the field of elastic stresses [6] makes it possible to describe the distribution of hydrogen in a metal due to its diffusion under the action of external loads. The solution of this equation is investigated in our problem of hydrogen diffusion in loaded rotating elastic bodies.

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Bearings are one of the important elements of many machines that use rotary motion in their work. Rolling bearings are used in electric motors, gear reducers, aircraft and vehicles, and in various industrial machine tools. Therefore, the study of the causes of bearing failure has a practical importance. In the literature, there are still no works devoted to the theoretical description of the distribution of hydrogen in rotating rolling elements of bearings. The diffusion of hydrogen in the cylindrical rolling elements of roller bearings will be the subject of our research.

The aim of this work is to investigate the distribution of hydrogen concentration in a rotating cylindrical elastic body under the action of mechanical stresses, as well as to demonstrate how to obtain an analytical solution to the diffusion equation in a rotating loaded body. Thus, it is necessary to solve the problem of the stress state of an elastic body under the action of two compressive concentrated forces, and then calculate the distribution of hydrogen concentrations in a rotating elastic body under the field of found elastic stresses.

Determination of the stress state of a cylindrical elastic body

We considered the cylindrical body under compression in the radial direction by two concentrated forces applied at diametrically opposite points of the outer circle. Cylindrical elastic bodies can be considered as models of rolling elements of roller bearings. The scheme of object of investigation is illustrated in Fig. 1.

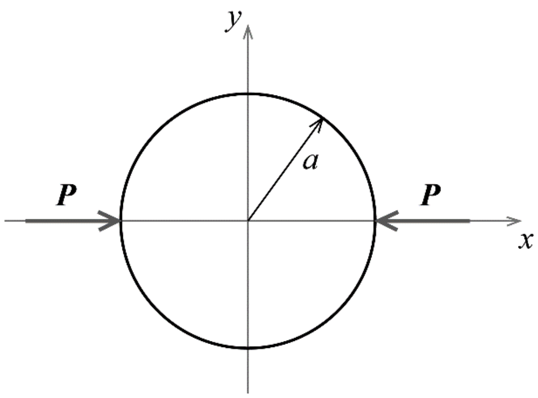

Fig. 1. The scheme of an object of research

It is known that in the plane problem the stress tensor has the form:

$$\boldsymbol{\tau} = \sigma_r \boldsymbol{e_r e_r} + \sigma_\varphi \boldsymbol{e_\varphi e_\varphi} + \tau_{r\varphi}(\boldsymbol{e_r e_\varphi} + \boldsymbol{e_\varphi e_r}). \tag{1}$$

To find a stress state of an elastic body we used means of the theory of functions of a complex variable. According to the theory, the stress field throughout the body can be reconstructed from the known stress vector on the surface of the body. Using Kolosov's formulas [19], we can find the stress tensor components. The detailed description of the stress state solution by means of the theory of functions of a complex variable is written in [20]. Thus, the stress tensor components is determined:

$$\sigma_r = -\frac{P}{\pi a}\left(1 - \frac{r^2}{a^2}\right)\left[\frac{\left(1+\frac{r^2}{a^2}\right)\left(1+\frac{r^4}{a^4}\right)-2cos2\varphi\left(\frac{r^2}{a^2}+2\frac{r^4}{a^4}+1\right)+4\frac{r^2}{a^2}}{\left[1-2\frac{r^2}{a^2}cos2\varphi+\frac{r^4}{a^4}\right]^2}\right], \tag{2}$$

$$\sigma_\varphi = -\frac{P}{\pi a}\left(1 - \frac{r^2}{a^2}\right)\left[\frac{\left(1+\frac{r^2}{a^2}\right)\left(1+\frac{r^4}{a^4}\right)-2cos2\varphi\left(3+\frac{r^2}{a^2}+4\frac{r^4}{a^4}\right)+12\frac{r^2}{a^2}}{\left[1-2\frac{r^2}{a^2}cos2\varphi+\frac{r^4}{a^4}\right]^2}\right], \tag{3}$$

$$\tau_{r\varphi} = \frac{2P}{\pi a}\frac{sin2\varphi(1-\frac{r^4}{a^4})\left(1-\frac{r^2}{a^2}\right)}{\left[(1-\frac{r^2}{a^2}cos2\varphi)^2+\frac{r^4}{a^4}sin^2 2\varphi\right]^2}. \quad (4)$$

Figures 2–4 show the contour plots of the distributions of the stress tensor components Eq. 2 – Eq. 4 in a cylindrical body under the action of two concentrated forces applied at two diametrically opposite points of the outer circle. Figure 5 shows a contour plot of the average normal stress distribution, which is determined through the normal components of the stress tensor as follows:

$$\sigma = \frac{1}{2}\left(\sigma_r + \sigma_\varphi\right) = -\frac{P}{\pi a}\frac{1-\left(\frac{r}{a}\right)^4}{\left(1-2\frac{r^2}{a^2}cos2\varphi+\frac{r^4}{a^4}\right)}. \quad (5)$$

As expected, from the figures we see that the highest values of stresses are achieved near the points of influence of the forces, and as they move away from the line of action of the load, the fields decrease rather quickly to a uniform distribution.

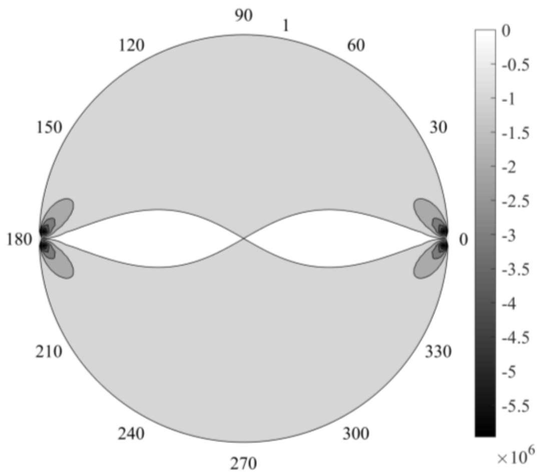

Fig. 2. The σ_r component distribution, [Pa]

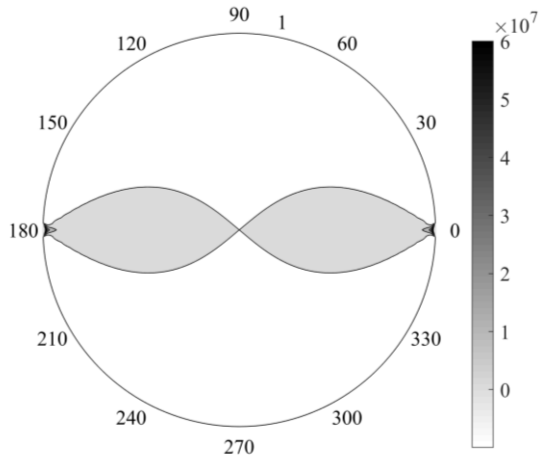

Fig. 3. The σ_φ component distribution, [Pa]

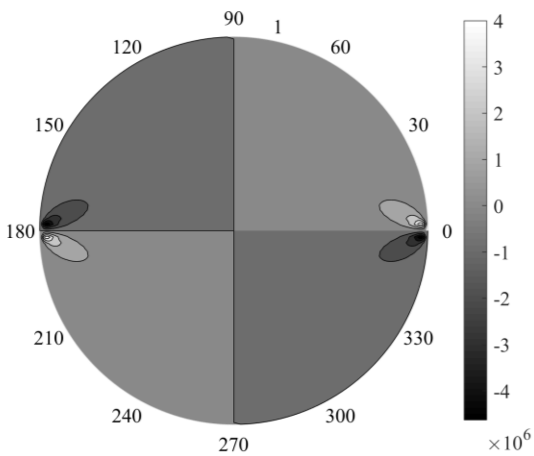

Fig. 4. The $\tau_{r\varphi}$ component distribution, [Pa]

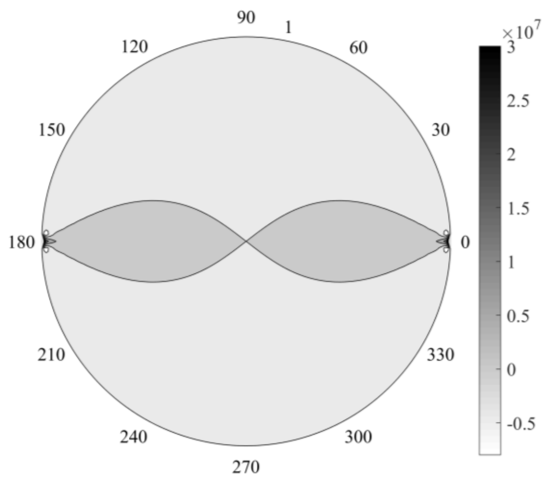

Fig. 5. The distribution of the average normal stress, [Pa]

Solution of the equation of hydrogen diffusion in a rotating cylindrical elastic body

The problem of hydrogen diffusion in rotating cylindrical elastic bodies is considered. According to [6], hydrogen diffusion in the field of elastic stresses is described by the following equation:

$$\dot{C} = D\Delta C - \frac{DV_H}{RT}\nabla C \cdot \nabla\sigma - \frac{DV_H}{RT} C\Delta\sigma, \tag{6}$$

where C − hydrogen concentration, D − hydrogen diffusion coefficient, V_H − partial molar volume of hydrogen, σ − average normal stress, R − gas constant, T − absolute temperature.

The differential equation is considered in a polar coordinate system, so the hydrogen concentration $C(r,\varphi,t)$ is a function of three independent variables: radius r, angle φ and time t. We assumed that the body rotates at a constant speed. In order to reduce the number of independent variables, the transition to rotating coordinate system is made and now the material derivative has the form:

$$\dot{C} = \frac{dC}{dt} = \frac{\partial C}{\partial t} + \frac{\partial C}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial C}{\partial \varphi}\frac{\partial \varphi}{\partial t} = \omega\frac{\partial C}{\partial \varphi}. \tag{7}$$

This expression is substituted in the Eq. 6 and the hydrogen diffusion equation in rotating coordinate system is obtained. In that equation as a stress field acting on a body the hydrostatic component of the found stress field Eq. 5 is operated and the following equation is obtained:

$$\Delta C + \frac{V_H}{RT}\frac{4P}{\pi a}\left[\frac{\frac{r}{a^2}\left(cos2\varphi - 2\frac{r^2}{a^2} + \frac{r^4}{a^4}cos2\varphi\right)}{\left(1-2\frac{r^2}{a^2}cos2\varphi+\frac{r^4}{a^4}\right)^2}\frac{\partial C}{\partial r} - \frac{1}{a^2}\frac{(1-\frac{r^4}{a^4})sin2\varphi}{\left(1-2\frac{r^2}{a^2}cos2\varphi+\frac{r^4}{a^4}\right)^2}\frac{\partial C}{\partial \varphi}\right] - \frac{\omega}{D}\frac{\partial C}{\partial \varphi} = 0. \tag{8}$$

This two-dimensional partial differential equation appeared to be very complex. We had to simplify it by the methods of asymptotic analysis. We considered the concentration only near the line of action of the force, thus, the equation is simplified by assuming that the angles φ are small. So the following partial differential equation is obtained:

$$\Delta C + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{r}{\left(1-\frac{r^2}{a^2}\right)^2}\frac{\partial C}{\partial r} - \left(\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}sin2\varphi + \frac{\omega}{D}\right)\frac{\partial C}{\partial \varphi} = 0. \tag{9}$$

Since the equation does not have coefficients that depend in a complex way on the variable φ, the required unknown function of hydrogen concentration can be represented in the form of a Fourier series:

$$C(r,\varphi) = \sum_{n=0}^{\infty}(A_n(r)sinn\varphi + B_n(r)cosn\varphi). \tag{10}$$

The expansion coefficients $A_n(r)$ and $B_n(r)$ need to be found. For this, we substituted Eq. 10 into the Eq. 9, multiplied the resulting expression by the basis functions $sinn\varphi$ and $cosn\varphi$ (where $n = 0, 1, 2 \ldots$) and integrated it over the variable φ on the interval $[0;\ 2\pi]$. And we got an infinite system of ordinary differential equations for finding an infinite number of coefficients $A_n(r)$ and $B_n(r)$.

After completing this procedure, we see that A_0 can be taken equal to zero, and ordinary differential equations for finding $B_0(r),\ A_1(r),\ B_1(r)$ and other coefficients are determined. The equations for finding $B_0(r),\ A_1(r),\ B_1(r)$ have the following structure:

$$B_0''(r) + \left(\frac{1}{r} + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{r}{\left(1-\frac{r^2}{a^2}\right)^2}\right)B_0'(r) + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}B_2(r) = 0, \tag{11}$$

$$A_1'' + \left(\frac{1}{r} + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{r}{\left(1-\frac{r^2}{a^2}\right)^2}\right)A_1' - \left(\frac{1}{r^2} + \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}\right)A_1 +$$

$$+\frac{3}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}A_3 + \frac{\omega}{D}B_1 = 0. \tag{12}$$

$$B_1'' + \left(\frac{1}{r} + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{r}{\left(1-\frac{r^2}{a^2}\right)^2}\right)B_1' - \left(\frac{1}{r^2} - \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}\right)B_1 +$$
$$+\frac{3}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}B_3 - \frac{\omega}{D}A_1 = 0. \quad (13)$$

In order to find expressions for the unknown coefficients, it is necessary to restrict the infinite system to a finite number of equations and the required functions. Let B_2 be equal to zero, then the Eq. 11 can be calculated for finding B_0. Using the Mathematica symbolic computation package, the solution is obtained:

$$B_0(r) = C_2 + \frac{1}{2}C_1\left[-Ei\left(\frac{K}{2\left(-1+\frac{r^2}{a^2}\right)}\right) + exp\left(-\frac{K}{2}\right)Ei\left(\frac{\frac{r^2}{a^2}K}{2\left(-1+\frac{r^2}{a^2}\right)}\right)\right], \quad (14)$$

where $K = \frac{V_H}{RT}\frac{4P}{\pi a}$, $Ei(x) -$ the exponential integral.

The coefficients of integration C_1 and C_2 are found from boundary conditions. Due to linearity of solution, we supposed that at the outer boundary the hydrogen concentration is set equal to one. Using this boundary condition and the condition of boundedness of the solution at zero, the expansion coefficient B_0 has been calculated:

$$B_0(r) = 1. \quad (15)$$

Next, we considered the solution for $B_1(r)$. In order to simplify its calculation, we assumed that the hydrogen concentration at the outer boundary is set according to the following law:

$$C(a,\varphi) = B_0(a) + B_1(a)cos\varphi, \quad (16)$$

where $B_0(a)$ и $B_1(a)$ are known. Due to symmetry, only an even function of the angle is considered. And the following simplified equation is obtained:

$$B_1'' + \left(\frac{1}{r} + \frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{r}{\left(1-\frac{r^2}{a^2}\right)^2}\right)B_1' - \left(\frac{1}{r^2} - \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\frac{1}{a^2}\frac{\left(1+\frac{r^2}{a^2}\right)}{\left(1-\frac{r^2}{a^2}\right)^3}\right)B_1 = 0. \quad (17)$$

This equation has a complex structure, since the coefficients depend on the variable r in a complex way. Thus, we found the approximate solution of this equation using the Galerkin approach. Since the method assumes the zero boundary condition, the solution is considered in the form:

$$B_1(r) = B_1(a) + C_1 f(r), \quad (18)$$

where the basis function vanishes at the boundary:

$$f(r) = 1 - \frac{r}{a}. \quad (19)$$

The constant C_1 should be calculated. To do this, the Eq. 18 is substituted into the Eq. 17, it is multiplied by the basis function and integrated over the variable r on the interval $[0;\ a]$. The constant C_1 has this view: $C_1 = -B_1(a)\left(1 + \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\right)$. Then the coefficient $B_1(r)$ is equal to:

$$B_1(r) = -B_1(a)\frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a} + B_1(a)\left(1 + \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\right)\frac{r}{a}. \quad (20)$$

As a result, the linear approximate solution of the hydrogen concentration distribution is obtained:

$$C(r,\varphi) = B_0(a) - B_1(a)\frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}cos\varphi + B_1(a)\left(1 + \frac{1}{2}\frac{V_H}{RT}\frac{4P}{\pi a}\right)\frac{r}{a}cos\varphi. \quad (21)$$

Therefore, the diffusion equation in a rotating body under the action of elastic stresses with taking into account the smallness of the angles and considering the region near the point of load application can be solved by representing the function of the hydrogen concentration in the form of a Fourier series. The coefficients of this expansion can be found using numerical or approximate analytical methods. Using the Galerkin approach, in the first approximation, the coefficient standing in front of the cosine in the expansion of the function in the Fourier series is obtained, which we only considered due to the symmetry of this problem.

Discussion

We investigated the problem of the distribution of the hydrogen concentration in a rotating cylindrical elastic body compressed by two concentrated forces. The stress state of an elastic body was determined by means of the theory of functions of complex variable. As a result, we constructed the contour graphs of the stress distribution in the cylindrical body. The stress field has an areas of concentration of the maximum values of compressive stresses, located near the points of application of concentrated forces, however, the stress values rather quickly decrease with distance from these points. The stress field is distributed evenly in the remaining area of the cross-section of cylinder.

Then we wrote down the hydrogen diffusion equation in an elastic body under the action of concentrated forces. The obtained equation turned out to be very complex, it does admit a closed form solution. This equation was reduced to ordinary differential equations by expansion the sought-for function in a Fourier series. It was discovered to be possible to solve them only asymptotically for some harmonics. However, the asymptotic simplification is justified, since in this case the search of the hydrogen concentration is performed near the zone of action of external forces, where the stresses reach their maximum values.

In this research, we demonstrated the possibility of analytical determination of the hydrogen concentration distribution in a rotating cylindrical elastic body near the points of application of external forces. This problem has a practical importance, since the object of research can be interpreted as a model of many structural elements, for example, as rolling elements of roller bearings. Determination of the distribution of hydrogen concentration in bodies in a stressed state can help in studies of diffusion processes of hydrogen in metals, which affect their mechanical characteristics.

Conclusion

In this paper, the diffusion of hydrogen in a rotating cylindrical elastic body under the action of external concentrated forces was investigated. As a result, a stress field was found. The method of calculating a solution of hydrogen diffusion equation in a rotating body was demonstrated. Method includes a transition to a rotating coordinate system, expanding the function in a Fourier series and applying numerical and approximate methods for solving ordinary differential equations to find the expansion coefficients. The solution of ordinary differential equations was obtained asymptotically for several harmonics. The result confirms the possibility of applying the presented methods to solve the hydrogen diffusion equation in loaded rotating elastic bodies. This problem requires further research in order to obtain the hydrogen distribution throughout the cross-section of cylindrical body. The determination of hydrogen distribution in rolling elements is important in diagnostics of the residual life of roller bearings during their operation in environments with a high hydrogen content.

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