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Maxwell electrodynamics, complex rotation group, media

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Abstract. Our treatment will be with definite accents: the main attention is given to classical electrodynamics in material media, focusing on the structure of Minkowski constitutive relations, matrix complex form of Maxwell theory in the form of Riemann–Silberstein–Majorana–Oppenheimer, and the theory of complex rotation group $SO(3.C)$, isomorphic to the Lorentz group. This review includes the topics: introduction; matrix complex form of Maxwell theory in a vacuum; modified Lorentz symmetry in electrodynamics; Minkowski electrodynamics in moving bodies; Minkowski constitutive relations in the complex 3-vector form; symmetry properties of the matrix equation in any linear media; Dirac matrices and electromagnetic field.

Keywords: complex rotation group $SO(3.C)$, Dirac matrices, formalism Majorana–Oppenheimer, Maxwell theory, Minkovski constitutive equations

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1. Introduction

Special Relativity arose from the study of the symmetry properties of the Maxwell equations: Lorentz [1-3], Poincaré [4-10], and Einstein [11-13]. Naturally, an analysis of the Maxwell equations with respect to Lorentz transformations was the first object of relativity theory [14-21]. After discovering the relativistic equation for a particle with spin 1/2 – Dirac [22, 23] – much work was done to study spinors and vectors within the Lorentz group theory: Ivanenko–Landau [24], Neumann [25], van der Waerden [26], Juvet [27]. As was shown, any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason, spinor quantities are considered fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [28], see also Rumer [29]. In 1931, Majorana [30] and Oppenheimer [31] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. Before Majorana and Oppenheimer, the most crucial steps were made by Silberstein [32]; he showed the possibility of formulating the Maxwell equations in the terms of complex 3-vector entities. Silberstein in his second paper [33] wrote that the complex form of Maxwell equations has been known before; he refers to the second

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volume of the lecture notes on the differential equations of mathematical physics by Riemann which were edited and published by H. Weber in 1901 [34]. This not widely used fact is noted by Bialynicki-Birula [35,36].

The Maxwell equations in the matrix Dirac-like form were considered during long time by many authors; the interest to the Majorana–Oppenheimer formulation of electrodynamics has grown in recent years: Luis de Broglie [37-40], Mercier [41], Petiau [42], Proca [43, 44], Kemmer [45-47], Bhabha [48], Belinfante [49,50], Taub [51], Sakata and Taketani [52], Erikson [53], Schrödinger [54-56], Tonnelat [57], Stratton [58], Heitler [59], Einstain-bargnann [60], Harish-Chandra [61-62], Hoffmann [63], Utiyama [64], Schouten [65], Mercier [66], de Broglie and Tonnelat [67], Gupta [68], Bleuler [69], Brulin and Hjalmars [70], Rosen [71], Fujiwara [72], Gürsey [73], Gupta [74], Lichnerowicz [75], Ohmura [76], Borgardt [77,78], Fedorov [79], Kuohsien [80], Bludman [81], Good [82], Moses [83-85], Silveira [86,87], Lomont [88], Post [89], Bogush and Fedorov [90], Sachs and Schwebel [91], Ellis [92], Beckers and Pirotte [93], Casanova [94], Carmeli [95], Weingarten [96], Mignani, Recami and Baldo [97], Newmann [98], Frankel [99], Edmonds [100], Jena et al [101], Venuri [102], Chow [103], Fushchich and Nikitin [104], Cook [105,106], Giannetto [107], Nuez et al [108], Kidd et al [109], Recami [110], Hillion [111], Inagaki [112], Sipe [113], Ghose [114], Gersten [115], Esposito [116], Torres del Castill and Mercado-Perez [117], Dvoeglazov [118,119], Gsponer [120], Ivezic [121-125], Kravchenko [126], Varlamov [127], Khan [128].

In the present paper, the main attention is given to technical aspects of classical electrodynamics based on the theory of rotation complex group SO(3.C) which is isomorphic to the Lorentz group: Macfarlane [129], Fedorov [130].

2. Matrix complex form of Maxwell theory in a vacuum

Let us start with Maxwell equations in the homogeneous (ε,μ)-media in presence of external sources:

$$div\, cB = 0, \quad rot\, E = -\frac{\partial cB}{\partial ct}, \quad div\, E = \frac{\rho}{\varepsilon\varepsilon_0}, \quad rot\, cB = \mu\mu_0 cJ + \varepsilon\mu\frac{\partial E}{\partial ct}. \tag{1}$$

With the use of usual notation for current 4-vector

$$j^a = (\rho, J/c), \quad c^2 = 1/\varepsilon_0\mu_0, \tag{2}$$

eqs. (1) read

$$div\, cB = 0, \quad rot\, E = -\frac{\partial cB}{\partial ct}, \quad div\, E = \frac{\rho}{\varepsilon_0}, \quad rot\, cB = \frac{j}{\varepsilon_0} + \frac{\partial E}{\partial ct}, \tag{3}$$

or in an explicit component form (let $x_0 = ct, \partial_0 = c\partial_t$)

$$\partial_1 cB^1 + \partial_2 cB^2 + \partial_3 cB^3 = 0, \quad \partial_2 E^3 - \partial_3 E^2 + \partial_0 cB^1 = 0,$$

$$\partial_3 E^1 - \partial_1 E^3 + \partial_0 cB^2 = 0, \quad \partial_1 E^2 - \partial_2 E^1 + \partial_0 cB^3 = 0,$$

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = j^0/\varepsilon_0, \quad \partial_2 cB^3 - \partial_3 cB^2 - \partial_0 E^1 = j^1/\varepsilon_0,$$

$$\partial_3 cB^1 - \partial_1 cB^3 - \partial_0 E^2 = j^2/\varepsilon_0, \quad \partial_1 cB^2 - \partial_2 cB^1 - \partial_0 E^3 = j^3/\varepsilon_0. \tag{4}$$

Let us introduce 3-dimensional complex vector

$$\psi^k = E^k + icB^k, \tag{5}$$

with the help of which the above equations can be combined into the following ones:

$$\partial_1\Psi^1 + \partial_2\Psi^0 + \partial_3\Psi^3 = j^0/\varepsilon_0, \quad -i\partial_0\psi^1 + (\partial_2\psi^3 - \partial_3\psi^2) = i\, j^1/\varepsilon_0,$$

$$-i\partial_0\psi^2 + (\partial_3\psi^1 - \partial_1\psi^3) = i\,j^2/\varepsilon_0, \quad -i\partial_0\psi^3 + (\partial_1\psi^2 - \partial_2\psi^1) = i\,j^3/\varepsilon_0. \tag{6}$$

These four equations can be re-written in a matrix form; we use a 4-dimensional column ψ with one additional zero-element:

$$\left(-i\partial_0 \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix} + \partial_1 \begin{vmatrix} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{vmatrix} + \partial_2 \begin{vmatrix} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{vmatrix} \right.$$

$$\left. +\partial_3 \begin{vmatrix} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{vmatrix} \right) \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix} = \frac{1}{\varepsilon_0} \begin{vmatrix} j^0 \\ i\,j^1 \\ i\,j^2 \\ i\,j^3 \end{vmatrix}.$$

Here, there arise four ambiguously determined matrices (parameters a_k, b_k, c_k, d_k are arbitrary):

$$(-i\alpha^0\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad \alpha^0 = \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} b_0 & 1 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ b_2 & 0 & 0 & -1 \\ b_3 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} c_0 & 0 & 1 & 0 \\ c_1 & 0 & 0 & 1 \\ c_2 & 0 & 0 & 0 \\ c_3 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} d_0 & 0 & 0 & 1 \\ d_1 & 0 & -1 & 0 \\ d_2 & 1 & 0 & 0 \\ d_3 & 0 & 0 & 0 \end{vmatrix}. \tag{7}$$

Consider the products of these matrices. Taking into account the identity

$$(\alpha^0)^2 = \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} a_0 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & 0 & 1 & 0 \\ a_3 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_0a_0 & 0 & 0 & 0 \\ a_1a_0 + a_1 & 1 & 0 & 0 \\ a_2a_0 + a_2 & 0 & 1 & 0 \\ a_3a_0 + a_3 & 0 & 0 & 1 \end{vmatrix},$$

and requiring that $(\alpha^0)^2 = +I$, we get equations

$$a_0a_0 = 1, \quad a_1a_0 + a_1 = 0, \quad a_2a_0 + a_2 = 0, \quad a_3a_0 + a_3 = 0;$$

their simple solution is

$$a_0 = \pm 1, \quad a_j = 0, \quad \alpha^0 = \begin{vmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad (\alpha^0)^2 = I. \tag{8}$$

In the same manner, we obtain

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad (\alpha^1)^2 = -I; \tag{9}$$

$$\alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad (\alpha^2)^2 = -I; \tag{10}$$

$$\alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad (\alpha^3)^2 = -I. \tag{11}$$

For other products we have

$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3, \tag{12}$$

$$\alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1, \tag{13}$$

$$\alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2. \tag{14}$$

Let us turn to $\alpha^0\alpha^i$:

$$\delta = \pm 1, \quad \alpha^0\alpha^1 = \begin{vmatrix} 0 & \delta & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^1\alpha^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

It should be noted that only at $\delta = +1$ we have a simple commutation rule:

$$\alpha^0 = I, \quad \alpha^i\alpha^0 = \alpha^0\alpha^i = \alpha^i. \tag{15}$$

Thus, the eight Maxwell equations are presented in the following matrix form:

$$(-i\partial_0 + \alpha^j\partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad J = \frac{1}{\varepsilon_0}\begin{vmatrix} j^0 \\ i\,j^1 \\ i\,j^2 \\ i\,j^3 \end{vmatrix},$$

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

$$(\alpha^1)^2 = -I, \quad (\alpha^2)^2 = -I, \quad (\alpha^2)^2 = -I,$$

$$\alpha^1\alpha^2 = -\alpha^2\alpha^1 = \alpha^3, \quad \alpha^2\alpha^3 = -\alpha^3\alpha^2 = \alpha^1, \quad \alpha^3\alpha^1 = -\alpha^1\alpha^3 = \alpha^2. \tag{16}$$

Such a complex matrix form can be transformed to a real 8-dimensional presentation (see in [93]). Indeed, we have two conjugated equations:

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = J, \quad (+i\partial_0 + \alpha^j \partial_j)\Psi^* = J^*, \tag{17}$$

$$E = (\psi + \psi^*)/2, \quad B = (\psi - \psi^*)/2i. \tag{18}$$

Summing and subtracting them we get

$$\partial_0 \frac{\Psi - \Psi^*}{2i} + \alpha^j \partial_j \frac{\Psi + \Psi^*}{2} = \frac{J + J^*}{2}, \quad -\partial_0 \frac{\Psi + \Psi^*}{2} + \alpha^j \partial_j \frac{\Psi - \Psi^*}{2i} = \frac{J - J^*}{2i};$$

that is

$$\partial_0 B + \alpha^j \partial_j E = Re(J), \quad -\partial_0 E + \alpha^j \partial_j B = Im(J).$$

These equations can be presented in a matrix 8-dimensional form

$$\begin{vmatrix} \alpha^j \partial_j & \partial_0 \\ -\partial_0 & \alpha^j \partial_j \end{vmatrix} \begin{vmatrix} E \\ B \end{vmatrix} = \begin{vmatrix} Re(J) \\ Im(J) \end{vmatrix}, \quad (\Gamma^0 \partial_0 + \Gamma^i \partial_i)\Psi = J, \tag{19}$$

where

$$\Gamma^0 = \begin{vmatrix} 0 & I \\ -I & 0 \end{vmatrix}, \quad \Gamma^i = \begin{vmatrix} \alpha^i & 0 \\ 0 & \alpha^i \end{vmatrix}, \tag{20}$$

with the properties

$$(\Gamma^0)^2 = -I, \quad (\Gamma^1)^2 = -I, \quad (\Gamma^1)^2 = -I, \quad (\Gamma^1)^2 = -I,$$

$$\Gamma^1\Gamma^2 = -\Gamma^2\Gamma^1 = \Gamma^3, \quad \Gamma^2\Gamma^3 = -\Gamma^3\Gamma^2 = \Gamma^1, \quad \Gamma^3\Gamma^1 = -\Gamma^1\Gamma^3 = \Gamma^2,$$

$$\Gamma^0\Gamma^i = \Gamma^i\Gamma^0 = \begin{vmatrix} 0 & \alpha^i \\ -\alpha^i & 0 \end{vmatrix} \neq \Gamma^i. \tag{21}$$

Now let us turn again to the 4-dimensional complex form as a more compact one and consider the problem of relativistic invariance of this equation. The lack of explicit invariance of 3-vector complex form of Maxwell theory has been intensively discussed in various aspects by Ivezic [114-118]. Let us start with the relations

$$(-i\partial_0 + \alpha^j \partial_j)\Psi = J, \quad \Psi = \begin{vmatrix} 0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{vmatrix}, \quad J = \frac{1}{\varepsilon_0} \begin{vmatrix} j^0 \\ i\,j^1 \\ i\,j^2 \\ i\,j^3 \end{vmatrix}.$$

Arbitrary Lorentz transformation over the function Ψ is given by (take notice that one may introduce four undefined parameters $s_0, ..., s_3$, but we will take $s_0 = 1, s_j = 0$)

$$S = \begin{vmatrix} s_0 & 0 & 0 & 0 \\ s_1 & . & . & . \\ s_2 & . & O(k) & . \\ s_3 & . & . & . \end{vmatrix}, \quad \Psi' = S\Psi, \quad \Psi = S^{-1}\Psi', \tag{22}$$

where $O(k)$ stands for a (3×3)-rotation complex matrix from the group $SO(3,C)$, isomorphic to the Lorentz one, more detail see in the book by Fedorov [130]. Equation for a primed function Ψ' reads

$$(-i\partial_0 + S\alpha^j S^{-1} \partial_j)\Psi' = SJ. \tag{23}$$

When working with matrices α^j we will use vectors e_i and (3×3)-matrices τ_i; then the structure $S\alpha^j S^{-1}$ is presented as follows

$$S\alpha^j S^{-1} = \begin{vmatrix} 1 & 0 \\ 0 & O(k) \end{vmatrix} \begin{vmatrix} 0 & e_j \\ -e_j^t & \tau_j \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & O^{-1}(k) \end{vmatrix} = \begin{vmatrix} 0 & e_j O^{-1}(k) \\ -O(k)e_j^t & O(k)\tau_j O^{-1}(k) \end{vmatrix}.$$

This relationship can be rewritten with the help of indices

$$S\alpha^j S^{-1} = \begin{vmatrix} 0 & \delta_{ji} O^{-1}(k)_{in} \\ -O(k)_{ni}\delta_{ij} & \tau_n O(k)_{nj} \end{vmatrix} = \begin{vmatrix} 0 & O(k)_{nj} \\ -O(k)_{nj} & \tau_n O(k)_{nj} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (e_m)_n O(k)_{mj} \\ -(e_m)_n O(k)_{mj} & \tau_m O(k)_{mj} \end{vmatrix} = \alpha^m O_{mj}(k). \qquad (24)$$

Therefore, after the above transformation, the matrix equation is brought to

$$(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\Psi' = SJ, \quad (-i\partial_0 + \alpha^m O_{mj}\partial_j)\Psi' = SJ, \quad O_{mj}\partial_j = \partial_m',$$

that is

$$(-i\partial_0 + \alpha^m \partial'_m)\Psi' = SJ. \qquad (25)$$

Let us specify this result for simple cases when S looks as follows (to a real value a corresponds Euclidean rotation, to an imaginary value a corresponds a Lorentzian rotation):

Euclidean rotation $(1-2)$,

$$S\alpha^1 S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & -\sin a & 0 \\ 0 & \sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & +\sin a & 0 \\ 0 & -\sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \cos a\,\alpha^1 + \sin a\,\alpha^2 = \alpha^j O_{j1}. \qquad (26)$$

$$S\alpha^2 S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & -\sin a & 0 \\ 0 & \sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & \sin a & 0 \\ 0 & -\sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\sin a\,\alpha^1 + \cos a\,\alpha^2 = \alpha^j O_{j2}. \qquad (27)$$

$$S\alpha^3 S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & -\sin a & 0 \\ 0 & \sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos a & \sin a & 0 \\ 0 & -\sin a & \cos a & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \alpha^3 = \alpha^j O_{j3}. \qquad (28)$$

Now, one should give special attention to the fact that the symmetry properties given by (25) look satisfactory only at real values of parameter a; in this case, it describes the symmetry of the Maxwell equations under Euclidean rotations. However, if the values of a are imaginary, the above transformation S gives a Lorentzian boost:

$$a = ib, b^* = b, \sin a = i\,sinh b, \cos a = cosh b, S(a = ib) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & cosh b & -i\,sinh b & 0 \\ 0 & i\,sinh b & cosh b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \qquad (29)$$

and formulas (26) will take the form

$$S\alpha^1 S^{-1} = cosh b\alpha^1 + i\,sinh b\alpha^2, \quad S\alpha^2 S^{-1} = -i\,sinh b\alpha^1 + cosh b\alpha^2, \quad S\alpha^3 S^{-1} = \alpha^3. \qquad (30)$$

So the Maxwell matrix equation after transformations (29), (30) will look asymmetric

$$\left[(-i\partial_0 + \alpha^3\partial_3) + (cosh b\alpha^1 + i\,sinh b\alpha^2)\partial_2 + (-i\,sinh b\alpha^1 + cosh b\alpha^2)\partial_3\right]\Psi' = SJ. \qquad (31)$$

One can note the identity

$$(cosh b - i\,sinh b\alpha^3)(-i\partial_0 + \alpha^3\partial_3)$$

$$= -i(cosh b\partial_0 - sinh b\partial_3) + \alpha^3(-sinh b\partial_0 + cosh b\partial_3) = -i\partial_0' + \alpha^3\partial_3', \qquad (32)$$

where derivatives are changed in accordance with the Lorentzian symmetry. It remains to determine the action of the operator

$$\Delta = cosh b - i\,insh b\alpha^3 \qquad (33)$$

on two other terms in eq. (31); one might expect two relations:

$$(cosh b - i\,sinh b\alpha^3)(cosh b\alpha^1 + i\,sinh b\alpha^2) = \alpha^2,$$

$$(cosh b - i\,sinh b\alpha^3)(-i\,sinh b\alpha^1 + cosh b\alpha^2) = \alpha^3. \qquad (34)$$

As easily verified they hold indeed. We should determine the term $\Delta S J$. First, we have

$$S J = \varepsilon_0^{-1} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & cosh b & -i\,sinh b & 0 \\ 0 & i\,sinh b & cosh b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} j^0 \\ ij^1 \\ ij^2 \\ ij^3 \end{vmatrix} = \begin{vmatrix} j^0 \\ i\,cosh b\, j^1 + sinh b\, j^2 \\ -sinh b\, j^1 + i\,cosh b\, j^2 \\ ij^3 \end{vmatrix},$$

and then

$$\Delta S J = \begin{vmatrix} cosh b\, j^0 + sinh b\, j^3 \\ i\, j^1 \\ i\, j^2 \\ i(sinh b\, j^0 + cosh b\, j^3) \end{vmatrix}; \qquad (35)$$

the right-hand side of (35) is what we need. Thus, the symmetry of the matrix Maxwell equation under the Lorentzian boost in the plane $(0-3)$ is described by the relations.

Pseudo-Euclidean rotation $(0-3)$

$$S(b) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & cosh b & -i\,sinh b & 0 \\ 0 & i\,sinh b & cosh b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad \Delta(b) = cosh b - i\,sinh b\alpha^3,$$

$$\Delta(b)(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\Psi' = \Delta S J \equiv J' \quad \Rightarrow \quad (-i\partial_0' + \alpha^j\partial_j')\Psi' = J',$$

$$cosh b\partial_0 - sinh b\partial_3 = \partial_0', \quad -sinh b\partial_0 + cosh b\partial_3 = \partial_3'. \qquad (36)$$

The symmetry properties of this equation under two other boosts are similar.

Pseudo-Euclidean rotation $(0-1)$

$$S(a=b)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh b & i\sinh b \\ 0 & 0 & -i\sinh b & \cosh b \end{vmatrix}, \quad \Delta(b)=\cosh b + i\sinh b\alpha^1,$$

$$\Delta(b)(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\Psi' = \Delta S J \equiv J' \quad \Rightarrow \quad (-i\partial_0' + \alpha^j\partial_j')\Psi' = J',$$

$$\cosh b\partial_0 + \sinh b\partial_1 = \partial_0', \quad \sinh b\partial_0 + \cosh b\partial_1 = \partial_3'. \tag{38a}$$

Pseudo-Euclidean rotation $(0-2)$

$$S(ib)=\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh b & 0 & i\sinh b \\ 0 & 0 & 1 & 0 \\ 0 & -i\sinh b & 0 & \cosh b \end{vmatrix}, \quad \Delta(b)=\cosh b - i\sinh b\alpha^2,$$

$$\Delta(b)(-i\partial_0 + S\alpha^j S^{-1}\partial_j)\Psi' = \Delta S J \equiv J' \quad \Rightarrow \quad (-i\partial_0' + \alpha^j\partial_j')\Psi' = J',$$

$$\cosh b\partial_0 - \sinh b\partial_2 = \partial_0', \quad -\sinh b\partial_0 + \cosh b\partial_2 = \partial_3'. \tag{38b}$$

For the general case, one can think that for an arbitrarily oriented boost the operator Δ should be of the form:

$$\Delta = \Delta_\alpha = \cosh b - i\sinh b n_j \alpha^j.$$

This can be verified by direct calculation, all details are omitted. Therefore, under general Lorentzian boost (take note that $x^a=(x^0,x)$ is a contra-variant 4-vector, whereas ∂_a is a covariant one)

$$t' = \cosh\beta t + \sinh\beta n x, \quad x' = +n\sinh\beta t + x + (\cosh\beta - 1)n(nx); \tag{39}$$

the matrix Maxwell equation transforms according to

$$(A_0 + \alpha^1 A_1 + \alpha^2 A_2 + \alpha^3 A_3)\Psi' = \Delta S J, \tag{40}$$

where

$$\partial_0' \equiv A_0 = \cosh b\partial_0 - \sinh b(n_1\partial_1 + n_2\partial_2 + n_3\partial_3),$$

$$\partial_1' \equiv A_1 = -\sinh b n_1\partial_0 + [\partial_1 + (\cosh b - 1)n_1(n_1\partial_1 + n_2\partial_2 + n_3\partial_3)],$$

$$\partial_2' \equiv A_2 = -\sinh b n_2\partial_0 + [\partial_2 + (\cosh b - 1)n_2(n_1\partial_1 + n_2\partial_2 + n_3\partial_3)],$$

$$\partial_3' \equiv A_3 = -\sinh b n_3\partial_0 + [\partial_3 + (\cosh b - 1)n_2(n_1\partial_1 + n_2\partial_2 + n_3\partial_3)];$$

so that the transformed equation reads

$$(\partial_0' + \alpha^1\partial_1' + \alpha^2\partial_2' + \alpha^3\partial_3')\Psi' = \Delta S J. \tag{41}$$

It remains to examine the term ΔSJ in the right-hand side. Using

$$\Delta = \cosh b - i\sinh b n_1\alpha^1 - i\sinh b n_2\alpha^2 - i\sinh b n_3\alpha^3$$

$$= \begin{vmatrix} \cosh b & -i\sinh b n_1 & -i\sinh b n_2 & -i\sinh b n_3 \\ i\sinh b n_1 & \cosh b & i\sinh b n_3 & -i\sinh b n_2 \\ i\sinh b n_2 & -i\sinh b n_3 & \cosh b & i\sinh b n_1 \\ i\sinh b n_3 & i\sinh b n_2 & -i\sinh b n_1 & \cosh b \end{vmatrix}, \tag{42}$$

and (for a time, the factor ε_0^{-1} is omitted; remember notation $f \equiv 1 - coshb$):

$$SJ = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-f(n_2^2+n_3^2) & -isinhbn_3 + fn_1n_2 & isinhbn_2 + fn_1n_3 \\ 0 & isinhbn_3 + fn_1n_2 & 1-f(n_3^2+n_1^2) & -isinhbn_1 + fn_2n_3 \\ 0 & -isinhbn_2 + fn_1n_3 & isinhbn_1 + fn_2n_3 & 1-f(n_1^2+n_2^2) \end{vmatrix} \begin{vmatrix} j^0 \\ ij^1 \\ ij^2 \\ ij^2 \end{vmatrix}$$

$$= \begin{vmatrix} j^0 \\ ij^1[1-f(n_2^2+n_3^2)] + ij^2[-isinhbn_3 + fn_1n_2] + ij^3[isinhbn_2 + fn_1n_3] \\ ij^1[isinhbn_3 + fn_1n_2] + ij^2[1-f(n_3^2+n_1^2)] + ij^3[-isinhbn_1 + fn_2n_3] \\ ij^1[-isinhbn_2 + fn_1n_3] + ij^2[isinhbn_1 + fn_2n_3] + ij^3[1-f(n_1^2+n_2^2)] \end{vmatrix} \qquad (43)$$

for ΔSJ, one should produce

$$\Delta SJ = \begin{vmatrix} j^{'0} \\ ij^{'1} \\ ij^{'2} \\ ij^{'3} \end{vmatrix}. \qquad (44)$$

First, we prove

$$j^{'0} = coshb\, j^0 + sinhb(n_1 j^1 + n_2 j^2 + n_3 j^3) = coshb\, j^0 + sinhb(nj). \qquad (45)$$

Now let us calculate the term

$$j^{'1} = +sinhbn_1 j^0 + j^1 + (coshb-1)n_1(n_1 j^1 + n_2 j^2 + n_3 j^3). \qquad (46)$$

It is evidently the first component of the vector formula is (see (2.56))

$$j' = +sinhbn\, j^0 + j + (coshb-1)n(nj). \qquad (47)$$

Let us calculate the term $j^{'2}$:

$$j^{'2} = +sinhbn_2 j^0 + j^2 + (coshb-1)n_2(n_1 j^1 + n_2 j^2 + n_3 j^3). \qquad (48)$$

In the same way, we prove the formula

$$j^{'3} = +sinhbn_3 j^0 + j^3 + (coshb-1)n_3(n_1 j^1 + n_2 j^2 + n_3 j^3). \qquad (49)$$

Thus, the matrix Maxwell equation $(-i\partial_0 + \alpha^i \partial_i)\Psi = J$ is invariant under an arbitrary Lorentzian boost:

$$S(ib,n) = \begin{vmatrix} 1 & 0 \\ 0 & O(ib,n) \end{vmatrix},$$

$$t' = cosh\beta t + sinh\beta nx, \quad x' = +n\, sinh\beta t + x + (cosh\beta - 1)n(nx),$$

$$\Delta(-i\partial_0 + S\alpha^i S^{-1}\partial_i)S\Psi = \Delta SJ \quad \Rightarrow \quad (\partial_0' + \alpha^i \partial_i')\Psi' = J'; \qquad (50)$$

the derivatives and current transform by the rules:

$$\partial_0' = coshb\partial_0 - sinhb(n\nabla), \quad \nabla' = -sinhbn\partial_0 + [\nabla + (coshb-1)n(n\nabla),$$

$$j^{'0} = coshb\, j^0 + sinhb(nj), \quad j' = +sinhbn\, j^0 + j + (coshb-1)n(nj). \qquad (51)$$

Invariance of the matrix equation under Euclidean rotations is achieved in a simpler way:

$$(-i\partial_0 + \alpha^i \partial_i)\Psi = J,$$

$$S(a,n) = \begin{vmatrix} 1 & 0 \\ 0 & O(a,n) \end{vmatrix}, \quad t' = t, \quad x' = R(a,n)x,$$

$$(-i\partial_0 + S\alpha^i S^{-1}\partial_i)S\Psi = SJ \quad \Rightarrow \quad (-i\partial_0' + \alpha^i \partial_i')\Psi' = J', \tag{52}$$

$$\partial_0' = \partial_0, \quad \nabla' = R(a,-n)\nabla, \quad j'^0 = j^0, \quad j' = R(a,n)\, j. \tag{53}$$

3. Modified Lorentz symmetry

Let us start with Maxwell equations in the uniform medium:

$$div\, cB = 0, \quad rot\, E = -\frac{\partial cB}{\partial ct}, \quad div\, E = \frac{\rho}{\varepsilon\varepsilon_0}, \quad rot\, cB = \mu\mu_0 cJ + \varepsilon\mu \frac{\partial E}{\partial ct}. \tag{54}$$

The coefficient $\varepsilon\mu$ can be factorized as follows

$$\varepsilon\mu = \sqrt{\varepsilon\mu}\sqrt{\varepsilon\mu} = \frac{1}{k^2}, \quad c' = \frac{1}{\sqrt{\varepsilon_0\varepsilon\mu_0\mu}} = kc; \tag{55}$$

then the above system can be presented differently

$$div\, kcB = 0, \quad rot\, E = -\frac{\partial kcB}{\partial kct}, \quad div\, E = \frac{\rho}{\varepsilon\varepsilon_0}, \quad rot\, kcB = \frac{1}{\varepsilon\varepsilon_0}\frac{J}{kc} + \frac{\partial E}{\partial kct}. \tag{56}$$

Introducing new variables

$$x^a = (x^0 = kct, x^i), \quad j^a = (j^0 = \rho, j = \frac{J}{kc}), \tag{57}$$

previous equations can be rewritten as

$$div\, kcB = 0, \quad rot\, E = -\frac{\partial kcB}{\partial x^0}, \quad div\, E = \frac{1}{\varepsilon\varepsilon_0} j^0, \quad rot\, kcB = \frac{1}{\varepsilon\varepsilon_0} j + \frac{\partial E}{\partial x^0}, \tag{58}$$

or in explicit component form

$$\partial_1 c'B^1 + \partial_2 c'B^2 + \partial_3 c'B^3 = 0, \quad \partial_2 E^3 - \partial_3 E^2 + \partial_0 c'B^1 = 0,$$

$$\partial_3 E^1 - \partial_1 E^3 + \partial_0 c'B^2 = 0, \quad \partial_1 E^2 - \partial_2 E^1 + \partial_0 c'B^3 = 0,$$

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = j^0 / \varepsilon\varepsilon_0, \quad \partial_2 cB^3 - \partial_3 cB^2 - \partial_0 E^1 = j^1 / \varepsilon\varepsilon_0,$$

$$\partial_3 cB^1 - \partial_1 cB^3 - \partial_0 E^2 = j^2 / \varepsilon\varepsilon_0, \quad \partial_1 cB^2 - \partial_2 cB^1 - \partial_0 E^3 = j^3 / \varepsilon\varepsilon_0. \tag{59}$$

Eqs. (58), (59) formally differ from eqs. (4) only in one change $c \Rightarrow c' = kc$ (and also $\varepsilon_0 \Rightarrow \varepsilon_0\varepsilon$); therefore, all analysis performed in Section 2 is applicable here as well:

$$\psi^k = E^k + ic'B^k; \tag{60}$$

eqs. (59) are combined into

$$\partial_1\Psi^1 + \partial_2\Psi^0 + \partial_3\Psi^3 = j^0 / \varepsilon\varepsilon_0, \quad -i\partial_0\psi^1 + (\partial_2\psi^3 - \partial_3\psi^2) = i\, j^1 / \varepsilon\varepsilon_0,$$

$$-i\partial_0\psi^2 + (\partial_3\psi^1 - \partial_1\psi^3) = i\, j^2 / \varepsilon\varepsilon_0, \quad -i\partial_0\psi^3 + (\partial_1\psi^2 - \partial_2\psi^1) = i\, j^3 / \varepsilon\varepsilon_0. \tag{61}$$

These ones are rewritten in the matrix form

$$(-i\partial_0 + \alpha^i \partial_i)\Psi = J; \tag{62}$$

where the same matrices are used. The given matrix form of the Maxwell theory in the uniform media proves the existence of the theory symmetry under a modified Lorentz group (see Rosen [71]) in which instead of the vacuum speed of light we are to use the modified speed of light:

$$c' = kc, \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \quad k = \frac{1}{\sqrt{\varepsilon\mu}}. \tag{63}$$

4. On Minkowski electrodynamics in moving bodies

In agreement with Minkowski approach [14-17], in the presence of the uniform media, we should introduce two electromagnetic tensors F^{ab} and H^{ab} that transform independently under the Lorentz group. At that, the known constitutive (or material) relations change their form in the moving reference frame.

In the rest reference frame the Maxwell equations are

$$F^{ab} = (E, cB), \quad div\, B = 0, \quad rot\, E = -\frac{\partial cB}{\partial ct}, \tag{64}$$

$$H^{ab} = (D, H/c), \quad div\, D = \rho, \quad rot \frac{H}{c} = \frac{J}{c} + \frac{\partial D}{\partial ct}, \tag{65}$$

with the constitutive relations

$$D = \varepsilon_0 \varepsilon E, \quad H = \frac{1}{\mu\mu_0} B. \tag{66}$$

The quantities with simple transformation properties under the Lorentz group are

$$f = E + icB, \quad h = \frac{1}{\varepsilon_0}(D + iH/c), \quad j^a = (j^0 = \rho, j = J/c); \tag{67}$$

where f, h are the complex 3-vector under complex orthogonal group $SO(3.C)$, the latter is isomorphic to the Lorentz group. One can combine eqs. (64), (65) into the following ones

$$div(\frac{D}{\varepsilon_0} + icB) = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0(\frac{D}{\varepsilon_0} + icB) + rot(E + i\frac{H/c}{\varepsilon_0}) = \frac{i}{\varepsilon_0} j. \tag{68}$$

Taking into account relationships

$$E = \frac{f + f^*}{2}, \quad i\frac{H/c}{\varepsilon_0} = \frac{h - h^*}{2}, \quad \frac{D}{\varepsilon_0} = \frac{h + h^*}{2}, \quad icB = \frac{f - f^*}{2}, \tag{69}$$

eqs. (68) can be rewritten in the form

$$div(\frac{h + h^*}{2} + \frac{f - f^*}{2}) = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0(\frac{h + h^*}{2} + \frac{f - f^*}{2}) + rot(\frac{f + f^*}{2} + \frac{h - h^*}{2}) = \frac{i}{\varepsilon_0} j. \tag{70}$$

Let us introduce two quantities

$$M = \frac{h + f}{2}, \quad N = \frac{h^* - f^*}{2}, \tag{71}$$

which are different 3-vectors under the group $SO(3.C)$:

$$M' = OM, \quad N' = O^* N; \tag{72}$$

with respect to Euclidean rotations, the identity $O^* = O$ holds; instead for Lorentzian boosts we have $O^* = O^{-1}$.

In terms of M and N eqs. (70) look as

$$div\, M + div\, N = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0 M + rot\, M - i\partial_0 N - rot\, N = \frac{i}{\varepsilon_0} j, \tag{73}$$

or in a matrix form

$$(-i\partial_0 + \alpha^i \partial_i) M + (-i\partial_0 + \beta^i \partial_i) N = J, \tag{74}$$

where

$$M = \begin{vmatrix} 0 \\ M \end{vmatrix}, \quad N = \begin{vmatrix} 0 \\ N \end{vmatrix}, \quad J = \frac{1}{\varepsilon_0}\begin{vmatrix} \rho \\ i\, j \end{vmatrix}.$$

The matrices α^i and β^i are taken in the form

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \beta^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}. \tag{75}$$

It should be noted that the main idea of Minkowski – to divide the Maxwell equations into equations for tensors F^{ab} and H^{ab} transforming independently under the Lorentz group – should correspond to immiscible equations for f, f^* and h, h^* respectively. Such a form can be readily found. Indeed, let us start with eqs. (70) and their conjugated:

$$div\,(\frac{h+h^*}{2} + \frac{f-f^*}{2}) = \frac{1}{\varepsilon_0}\rho, \quad div\,(\frac{h+h^*}{2} - \frac{f-f^*}{2}) = \frac{1}{\varepsilon_0}\rho;$$

$$-i\partial_0(\frac{h+h^*}{2} + \frac{f-f^*}{2}) + rot\,(\frac{f+f^*}{2} + \frac{h-h^*}{2}) = \frac{i}{\varepsilon_0} j,$$

$$+i\partial_0(\frac{h+h^*}{2} - \frac{f-f^*}{2}) + rot\,(\frac{f+f^*}{2} - \frac{h-h^*}{2}) = -\frac{i}{\varepsilon_0} j.$$

From these, it follows

$$div\frac{h+h^*}{2} = \frac{1}{\varepsilon_0}\rho, \quad div\frac{f-f^*}{2} = 0;$$

$$-i\partial_0\frac{f-f^*}{2} + rot\frac{f+f^*}{2} = 0, \quad -i\partial_0\frac{h+h^*}{2} + rot\frac{h-h^*}{2} = \frac{i}{\varepsilon_0} j.$$

Thus the needed form is

$$div\frac{f-f^*}{2} = 0, \quad -i\partial_0\frac{f-f^*}{2} + rot\frac{f+f^*}{2} = 0, \tag{76}$$

$$div \frac{h+h^*}{2} = \rho, \quad -i\partial_0 \frac{h+h^*}{2} + rot \frac{h-h^*}{2} = \frac{i}{\varepsilon_0} j. \qquad (77)$$

The quantities entering these equations behave under the Lorentz group in accordance with the rules:

$$f' = O f, \quad f^{*'} = O^* f^*, \quad h' = Oh, \quad h^{*'} = O^* h^*.$$

Comparing eqs. (76), (77) with eqs. (73), (74):

$$div M + div N = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0 M + rot M - i\partial_0 N - rot N = \frac{i}{\varepsilon_0} j,$$

$$(-i\partial_0 + \alpha^i \partial_i) M + (-i\partial_0 + \beta^i \partial_i) N = J,$$

and taking into account

$$div h + div h^* = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0 h + rot h - i\partial_0 h^* - rot h^* = \frac{i}{\varepsilon_0} j, \qquad (78)$$

$$div f - div f^* = 0, \quad -i\partial_0 f + rot f + i\partial_0 f^* + rot f^* = 0 \qquad (79)$$

we conclude that the matrix equations we need are

$$(-i\partial_0 + \alpha^i \partial_i) h + (-i\partial_0 + \beta^i \partial_i) h^* = J, \qquad (80)$$

$$(-i\partial_0 + \alpha^i \partial_i) f - (-i\partial_0 + \beta^i \partial_i) f^* = 0. \qquad (81)$$

In the following, we will work with the more simple matrix equation (74).

5. Minkowski relations in the complex 3-vector form

The symmetry of the matrix equation under the Lorentz transformations must exist because the symmetry exists for these equations written in the ordinary tensor form. Now, let us examine how the above constitutive relations behave under the Lorentz transformations.

Let us start with these relations in the rest reference frame

$$D = \varepsilon_0 \varepsilon E, \quad \frac{H}{c} = \frac{1}{\mu_0 \mu} \frac{1}{c^2} cB = \frac{\varepsilon_0}{\mu} cB. \qquad (82)$$

Allowing for (87), eqs. (82) can be rewritten as

$$\frac{h+h^*}{2} = \varepsilon \frac{f+f^*}{2}, \quad \frac{h-h^*}{2} = \frac{1}{\mu} \frac{f-f^*}{2}; \qquad (83)$$

whence it follows

$$2h = (\varepsilon + \frac{1}{\mu}) f + (\varepsilon - \frac{1}{\mu}) f^*, \quad 2h^* = (\varepsilon + \frac{1}{\mu}) f^* + (\varepsilon - \frac{1}{\mu}) f. \qquad (84)$$

This is a complex form of constitutive relations (82). It should be noted that eqs. (83) can be resolved under the variables f, f^* as well:

$$2f = (\frac{1}{\varepsilon} + \mu) h + (\frac{1}{\varepsilon} - \mu) h^*, \quad 2f^* = (\frac{1}{\varepsilon} + \mu) h^* + (\frac{1}{\varepsilon} - \mu) h; \qquad (85)$$

these are the same constitutive equations (84) but in other forms.

Now let us allow for the Lorentz transformations:

$$f' = O f, \quad f'^* = O^* f^*, \quad h' = Oh, \quad h'^* = O^* h^*,$$

then eqs. (83) become

$$\frac{O^{-1}h' + (O^{-1})^* h^{*\prime}}{2} = \varepsilon \frac{O^{-1}f' + (O^{-1})^* f^{*\prime}}{2}, \quad \frac{O^{-1}h' - (O^{-1})^* h^{*\prime}}{2} = \frac{1}{\mu} \frac{O^{-1}f' - (O^{-1})^* f^{*\prime}}{2}.$$

Multiplying both equations by O and summing (or subtracting) the results we get

$$h' = \varepsilon \frac{f' + O(O^{-1})^* f^{*\prime}}{2} + \frac{1}{\mu} \frac{f' - O(O^{-1})^* f^{*\prime}}{2}, \tag{86}$$

$$h^{*\prime} = \varepsilon \frac{O^* O^{-1} f' + f^{*\prime}}{2} - \frac{1}{\mu} \frac{O^* O^{-1} f' - f^{*\prime}}{2}. \tag{87}$$

Eqs. (86), (87) can be presented as

$$2h' = (\varepsilon + \frac{1}{\mu}) f' + (\varepsilon - \frac{1}{\mu}) O(O^{-1})^* f'^*, \quad 2h'^* = (\varepsilon + \frac{1}{\mu}) f'^* + (\varepsilon - \frac{1}{\mu}) O^* O^{-1} f'. \tag{88}$$

Analogously, starting from eqs. (85) we can produce

$$2f' = (\frac{1}{\varepsilon} + \mu) h' + (\frac{1}{\varepsilon} - \mu) O(O^{-1})^* h'^*, \quad 2f'^* = (\frac{1}{\varepsilon} + \mu) h'^* + (\frac{1}{\varepsilon} - \mu) O^* O^{-1} h'. \tag{89}$$

Eqs. (88), (89) represent the constitutive relations after changing the reference frame. At this point, one should distinguish between two cases: Euclidean rotation and Lorentzian boosts. Indeed, for any Euclidean rotations, we have the identities

$$O^* = O, \quad \Rightarrow \quad O(O^{-1})^* = I, \quad O^* O^{-1} = I;$$

and therefore eqs. (88), (89) take the form of (84), (85); in other words, at Euclidean rotations, the constitutive relations do not change their form. However, for any pseudo-Euclidean rotations, we have different relations

$$O^* = O^{-1} \quad \Rightarrow \quad O(O^{-1})^* = 0^2, \quad O^* O^{-1} = O^{*2};$$

therefore eqs. (88), (89) look like

$$2h' = (\varepsilon + \frac{1}{\mu}) f' + (\varepsilon - \frac{1}{\mu}) O^2 f'^*, \quad 2h'^* = (\varepsilon + \frac{1}{\mu}) f'^* + (\varepsilon - \frac{1}{\mu}) O^2 f'; \tag{90}$$

$$2f' = (\frac{1}{\varepsilon} + \mu) h' + (\frac{1}{\varepsilon} - \mu) O^{*2} h'^*, \quad 2f'^* = (\frac{1}{\varepsilon} + \mu) h'^* + (\frac{1}{\varepsilon} - \mu) O^{*2} h'. \tag{91}$$

In the complex 3-vector form these relations seem to be shorter than in the real 3-vector form:

$$D' + i\frac{H'}{c} = \varepsilon_0 \varepsilon \frac{(E' + icB') + O(O^{-1})^* (E' - icB')}{2} + \frac{\varepsilon_0}{\mu} \frac{(E' + icB') - O(O^{-1})^* (E' - icB')}{2},$$

$$D' - i\frac{H'}{c} = \varepsilon_0 \varepsilon \frac{(E' - icB') + O^* O^{-1} (E' + icB')}{2} + \frac{\varepsilon_0}{\mu} \frac{(E' - icB') - O^* O^{-1} (E' + icB')}{2},$$

or

$$2D' = \varepsilon_0 \varepsilon [E' + \frac{O(O^{-1})^* + O^* O^{-1}}{2} E' + \frac{O(O^{-1})^* - O^* O^{-1}}{2i} cB']$$

$$+ \frac{\varepsilon_0}{\mu} [E' - \frac{O(O^{-1})^* + O^* O^{-1}}{2} E' - \frac{O(O^{-1})^* - O^* O^{-1}}{2i} cB'], \tag{92}$$

$$2\frac{H'}{c} = \varepsilon_0 \varepsilon [cB' - \frac{O(O^{-1})^* + O^* O^{-1}}{2} cB' + \frac{O(O^{-1})^* - O^* O^{-1}}{2i} E']$$

$$+\frac{\varepsilon_0}{\mu}[cB'+\frac{O(O^{-1})^*+O^*O^{-1}}{2}cB'-\frac{O(O^{-1})^*-O^*O^{-1}}{2i}E']. \qquad (93)$$

These are the constitutive equations for electromagnetic field in the uniform media according to Minkowski in a moving reference frame. For any Euclidean rotation, $O^*=O$ and eqs. (92), (93) coincide with the initial ones

$$D'=\varepsilon_0\varepsilon E', \quad H'/c=\varepsilon_0\frac{1}{\mu}cB' \Rightarrow H'=\frac{1}{\mu_0\mu}B'.$$

For any Lorentzian boosts $O^*=O^{-1}$, eqs. (92), (93) read

$$2D'=\varepsilon_0\varepsilon[(I+\frac{OO+O^*O^*}{2})E'+\frac{OO-O^*O^*}{2i}cB']+\frac{\varepsilon_0}{\mu}[(I-\frac{OO+O^*O^*}{2})E'-\frac{OO-O^*O^*}{2i}cB'], \quad (94)$$

$$2H'/c=\varepsilon_0\varepsilon[(I-\frac{OO+O^*O^*}{2})cB'+\frac{OO-O^*O^*}{2i}E']+\frac{\varepsilon_0}{\mu}[(I+\frac{OO+O^*O^*}{2})cB'-\frac{OO-O^*O^*}{2i}E']. (95)$$

They can be written differently

$$D'=\frac{\varepsilon_0}{2}\left\{[(\varepsilon+\frac{1}{\mu})+(\varepsilon-\frac{1}{\mu})Re O^2]E'+(\varepsilon-\frac{1}{\mu})Im O^2 cB']\right\}, \qquad (96)$$

$$\frac{H'}{c}=\frac{\varepsilon_0}{2}\left\{[(\varepsilon+\frac{1}{\mu})-(\varepsilon-\frac{1}{\mu})Re O^2]cB'+(\varepsilon-\frac{1}{\mu})Im O^2 E']\right\}. \qquad (97)$$

Lorentzian complex vector boosts are given by the matrix (remembering $f=1-cosh b$)

$$O=O(b,n)=\begin{vmatrix} 1-f(n_2^2+n_3^2) & -i\,sinh\,bn_3+fn_1n_2 & i\,sinh\,bn_2+fn_1n_3 \\ i\,sinh\,bn_3+fn_1n_2 & 1-f(n_3^2+n_1^2) & -i\,sinh\,bn_1+fn_2n_3 \\ -sinh\,bn_2+fn_1n_3 & i\,sinh\,bn_1+fn_2n_3 & 1-f(n_1^2+n_2^2) \end{vmatrix}. \qquad (98)$$

The square of O^2 is (let $F=1-cosh\,2b$)

$$O^2=\begin{vmatrix} cosh2b+(1-cosh2b)n_1^2 & Fn_1n_2-i\,sinh\,2bn_3 & Fn_3n_1+i\,sinh\,2bn_2 \\ Fn_1n_2+i\,sinh\,2bn_3 & cosh\,2b+Fn_2^2 & Fn_2n_3-i\,sinh\,2bn_1 \\ Fn_3n_1-i\,sinh\,2bn_2 & Fn_2n_3+i\,sinh\,2bn_1 & cosh\,2b+Fn_3^2 \end{vmatrix}. \qquad (99)$$

We have obtained the result that should be expected: eq. (99) differs from eq. (98) only in one change $b\to 2b$. These expressions for O^2 ought to be substituted into the formulas:

$$2h'=(\varepsilon+\frac{1}{\mu})f'+(\varepsilon-\frac{1}{\mu})O^2 f'^*, \quad 2h'^*=(\varepsilon+\frac{1}{\mu})f'^*+(\varepsilon-\frac{1}{\mu})O^2 f',$$

$$2f'=(\frac{1}{\varepsilon}+\mu)h'+(\frac{1}{\varepsilon}-\mu)O^{*2}h'^*, \quad 2f'^*=(\frac{1}{\varepsilon}+\mu)h'^*+(\frac{1}{\varepsilon}-\mu)O^{*2}h',$$

or

$$D'=\frac{\varepsilon_0}{2}\left\{[(\varepsilon+\frac{1}{\mu})+(\varepsilon-\frac{1}{\mu})Re O^2]E'+(\varepsilon-\frac{1}{\mu})Im O^2 cB']\right\},$$

$$\frac{H'}{c}=\frac{\varepsilon_0}{2}\left\{[(\varepsilon+\frac{1}{\mu})-(\varepsilon-\frac{1}{\mu})Re O^2]cB'+(\varepsilon-\frac{1}{\mu})Im O^2 E']\right\}.$$

All the formulas become much more simple for rotations in the planes $(0-1),(0-2),(0-3)$:

$$n=(1,0,0),\quad O^2=\begin{vmatrix}1 & 0 & 0\\ 0 & \cosh 2b & +i\sinh 2b\\ 0 & -i\sinh 2b & \cosh 2b\end{vmatrix},$$

$$n=(0,1,0),\quad O^2=\begin{vmatrix}\cosh 2b & 0 & -i\sinh 2b\\ 0 & 1 & 0\\ +i\sinh 2b & 0 & \cosh 2b\end{vmatrix}, \qquad (100)$$

$$n=(0,0,1),\quad O^2=\begin{vmatrix}\cosh 2b & +i\sinh 2b & 0\\ -i\sinh 2b & \cosh 2b & 0\\ 0 & 0 & 1\end{vmatrix}.$$

The previous results can be easily extended to more general media, let us restrict ourselves to linear media. Arbitrary linear media are characterized by the following constitutive equations:

$$D=\varepsilon_0\varepsilon(x)E+\varepsilon_0 c\alpha(x)B,\quad H=\varepsilon_0 c\beta(x)E+\frac{1}{\mu_0}\mu(x)B, \qquad (101)$$

where $\varepsilon(x),\mu(x),\alpha(x),\beta(x)$ are (3×3)-dimensionless matrices. Eqs. (101) should be rewritten in terms of complex vectors f,h:

$$\frac{h+h^*}{2}=\varepsilon(x)\frac{f+f^*}{2}+\alpha(x)\frac{f-f^*}{2i},\quad \frac{h-h^*}{2i}=\beta(x)\frac{f+f^*}{2}+\mu(x)\frac{f-f^*}{2i}. \qquad (102)$$

From (101), (102) it follows

$$h=[(\varepsilon(x)+\mu(x))+i(\beta(x)-\alpha(x))]f+[(\varepsilon(x)-\mu(x))+i(\beta(x)+\alpha(x))]f^*, \qquad (103)$$

$$h^*=[(\varepsilon(x)+\mu(x))-i(\beta(x)-\alpha(x))]f^*+[(\varepsilon(x)-\mu(x))-i(\beta(x)+\alpha(x))]f. \qquad (104)$$

Under Lorentz transformations, relations (5.20) will take the form

$$O^{-1}h'=\big[(\varepsilon(x)+\mu(x))+i(\beta(x)-\alpha(x))\big]O^{-1}f'+\big[(\varepsilon(x)-\mu(x))+i(\beta(x)+\alpha(x))\big](O^{-1})^*f'^*,$$

$$(O^{-1})^*h'^*=\big[(\varepsilon(x)+\mu(x))-i(\beta(x)-\alpha(x))\big](O^{-1})^*f'^*$$

$$+\big[(\varepsilon(x)-\mu(x))-i(\beta(x)+\alpha(x))\big](O^{-1})f', \qquad (105)$$

or

$$h'=\varepsilon_0\big[(\varepsilon(x)+\mu(x))+i(\beta(x)-\alpha(x))\big]f'+\big[(\varepsilon(x)-\mu(x))+i(\beta(x)+\alpha(x))\big][O(O^{-1})^*]f'^*,$$

$$h'^*=\varepsilon_0\big[(\varepsilon(x)+\mu(x))-i(\beta(x)-\alpha(x))\big]f'^*+\big[(\varepsilon(x)-\mu(x))-i(\beta(x)+\alpha(x))\big][O^*(O^{-1})]f'. \qquad (106)$$

For Euclidean rotation, we have

$$[O(O^{-1})^*]=I,\quad [O^*(O^{-1})]=I,$$

and the constitutive relations preserve their form. For Lorentz boosts we have

$$[O(O^{-1})^*]=O^2,\quad [O^*(O^{-1})]=O^{*2},$$

and the constitutive equations in a moving reference frame change their form in accordance with the relations

$$h' = \left[(\varepsilon(x) + \mu(x)) + i(\beta(x) - \alpha(x)) \right] f' + \left[(\varepsilon(x) - \mu(x)) + i(\beta(x) + \alpha(x)) \right] O^2 f'^*,$$

$$h'^* = \left[(\varepsilon(x) + \mu(x)) - i(\beta(x) - \alpha(x)) \right] f'^* + \left[(\varepsilon(x) - \mu(x)) - i(\beta(x) + \alpha(x)) \right] O^{*2} f'. \qquad (107)$$

They are the constitutive relations for arbitrary linear media in the moving reference frame.

6. Symmetry properties of the matrix equation in media

Maxwell equations in media are written in two groups

$$F^{ab}, \quad div\, f - div\, f^* = 0, \quad -i\partial_0 f + rot\, f + i\partial_0 f^* + rot\, f^* = 0; \qquad (108)$$

$$H^{ab}, \quad div\, h + div\, h^* = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0 h + rot\, h - i\partial_0 h^* - rot\, h^* = i\frac{1}{\varepsilon_0} j. \qquad (109)$$

Their matrix forms respectively read

$$(-i\partial_0 + \alpha^i \partial_i) f - (-i\partial_0 + \beta^i \partial_i) f^* = 0, \qquad (110)$$

$$(-i\partial_0 + \alpha^i \partial_i) h + (-i\partial_0 + \beta^i \partial_i) h^* = J. \qquad (111)$$

Equations (108), (109) can be presented in a more short form (see (91), (92)):

$$div\, M + div\, N = \frac{1}{\varepsilon_0}\rho, \quad -i\partial_0 M + rot\, M - i\partial_0 N - rot\, N = \frac{i}{\varepsilon_0} j; \qquad (112)$$

with the corresponding matrix form

$$(-i\partial_0 + \alpha^i \partial_i) M + (-i\partial_0 + \beta^i \partial_i) N = J. \qquad (113)$$

The matrices α^i and β^i are given by

$$\alpha^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \alpha^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \alpha^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$

$$\beta^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \beta^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}. \qquad (114)$$

We will consider eq. (113). The terms with matrices α^j were examined above, the terms with matrices β^j are new. We restrict ourselves to demonstrating the Lorentz symmetry of eq. (113) under two simplest transformations.

First, let us consider the Euclidean rotation in the plane $(1-2)$, and examine only the term with β-matrices:

$$S\beta^1 S^{-1} = \begin{vmatrix} 0 & \cos a & -\sin a & 0 \\ -\cos a & 0 & 0 & \sin a \\ \sin a & 0 & 0 & \cos a \\ 0 & -\sin a & -\cos a & 0 \end{vmatrix} = \cos a \beta^1 - \sin a \beta^2 = \beta^j O_{j1}, \qquad (115)$$

$$S\beta^2 S^{-1} = \begin{vmatrix} 0 & \sin a & \cos a & 0 \\ -\sin a & 0 & 0 & -\cos a \\ -\cos a & 0 & 0 & \sin a \\ 0 & \cos a & -\sin a & 0 \end{vmatrix} = \sin a\beta^1 + \cos a\beta^2 = \beta^j O_{j2}, \qquad (116)$$

$$S\beta^3 S^{-1} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} = \beta^3 = \beta^j O_{j3}. \qquad (117)$$

We conclude that eq. (113) is symmetrical under Euclidean rotations in accordance with the relations

$$(-i\partial_0 + S\alpha^i S^{-1}\partial_i)M' + (-i\partial_0 + S\beta^i S^{-1}\partial_i)N' = +SJ$$

$$\Rightarrow (-i\partial_0 + \alpha^i \partial_i')M' + (-i\partial_0 + \beta^i \partial_i')N' = +J'. \qquad (118)$$

For the Lorentz boost in the plane $(0-3)$ we have

$$M' = SM, \quad N' = S^* N = S^{-1}N, \quad S^* = S^{-1};$$

and eq. (113) takes the form (the additional transformation $\Delta = \Delta_{(\alpha)}$ is combined in terms of α^j)

$$\Delta_{(\alpha)} S\Big[(-i\partial_0 + \alpha^i \partial_i)S^{-1}M' + (-i\partial_0 + \beta^i \partial_i)SN' \Big] = \Delta SJ, \qquad (119)$$

or

$$\Delta_{(\alpha)}\Big[(-i\partial_0 + S\alpha^i S^{-1}\partial_i)M' + S^2(-i\partial_0 + S^{-1}\beta^i S\partial_i)N' \Big] = J', \qquad (120)$$

and further

$$(-i\partial_0' + \alpha^i \partial_i')M' + \Delta_{(\alpha)} S^2(-i\partial_0 + S^{-1}\beta^i S\partial_i)N' = J'. \qquad (121)$$

It remains to prove the relationship

$$\Delta_{(\alpha)} S^2(-i\partial_0 + S^{-1}\beta^i S\partial_i)N' = (-i\partial_0' + \beta^i \partial_i')N'. \qquad (122)$$

By simplicity reason one may expect two identities:

$$\Delta_{(\alpha)} S^2 = \Delta_{(\beta)} \quad \Leftrightarrow \quad \Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1}, \qquad (123)$$

$$\Delta_{(\beta)}(-i\partial_0 + S^{-1}\beta^i S\partial_i)N' = (-i\partial_0' + \beta^i \partial_i')N'. \qquad (124)$$

Let us prove them for a Lorentzian boost in the plane $(0-3)$:

$$S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh b & -i\sinh b & 0 \\ 0 & i\sinh b & \cosh b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad S^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh b & -i\sinh b & 0 \\ 0 & i\sinh b & \cosh b & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Allowing for (115) – (117) we readily get

$$S^{-1}\beta^1 S = \cosh b\beta^1 - i\sinh b\beta^2 = \beta^j O_{j1}^{-1},$$

$$S^{-1}\beta^2 S = i\sinh b\beta^1 + \cosh b\beta^2 = \beta^j O_{j2}^{-1}, \quad S^{-1}\beta^3 S = \beta^3 = \beta^j O_{j3}^{-1}. \qquad (125)$$

To verify the identity $\Delta_{(\alpha)} S = \Delta_{(\beta)} S^{-1}$, or $(\cosh b - i \sinh b \alpha^3) S = (\cosh b - i \sinh b \beta^3) S^{-1}$, let us calculate separately the left-hand part

$$(\cosh b - i \sinh b \alpha^3) S = \begin{vmatrix} \cosh b & 0 & 0 & -i \sinh b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i \sinh b & 0 & 0 & \cosh b \end{vmatrix};$$

and the right-hand part

$$(\cosh b - i \sinh b \beta^3) S^{-1} = \begin{vmatrix} \cosh b & 0 & 0 & -i \sinh b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i \sinh b & 0 & 0 & \cosh b \end{vmatrix}.$$

They coincide with each other, so eq. (123) holds. It remains to prove the relation (124). Allowing for the properties of β–matrices $(\beta^0)^2 = -I, (\beta^1)^2 = -I, \beta^1\beta^2 = -\beta^3, \beta^2\beta^1 = +\beta^3$, we readily find

$$\Delta_{(\beta)}(-i\partial_0 + S^{-1}\beta^i S \partial_i) N' =$$
$$= [-i(\cosh b \partial_0 - \sinh b \partial_3) + \beta^3(-\sinh b \partial_0 + \cosh b \partial_3) + \beta^1 \partial_1 + \beta^2 \partial_2] N', \qquad (126)$$

that is

$$\Delta_{(\beta)}(-i\partial_0 + S^{-1}\beta^i S \partial_i) N' = (-i\partial_0' + \beta^1 \partial_1 + \beta^2 \partial_2 + \beta^3 \partial_3') N'. \qquad (127)$$

Thus, the symmetry of the matrix Maxwell equation in media under the Lorentz group is proved.

7. Dirac matrices and electromagnetic field

Let us shortly discuss two points relevant to the above matrix formulation of the Maxwell theory. First, let us write down explicit form for Dirac matrices:

$$\gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad \gamma^5 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix},$$

$$\gamma^1 = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \gamma^2 = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \gamma^3 = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

Bearing in mind the expressions for α^i and β^i, we immediately see the identities

$$\alpha^1 = i\gamma^0\gamma^2, \quad \alpha^2 = \gamma^0\gamma^5, \quad \alpha^3 = i\gamma^5\gamma^2, \quad \beta^1 = -\gamma^3\gamma^1, \quad \beta^2 = -\gamma^3, \quad \beta^3 = -\gamma^1, \qquad (128)$$

so the Maxwell matrix equation in media takes the form

$$(-i\partial_0 + i\gamma^0\gamma^2\partial_1 + \gamma^0\gamma^5\partial_2 + i\gamma^5\gamma^2\partial_3) M + (-i\partial_0 - \gamma^3\gamma^1\partial_1 - \gamma^3\partial_2 - \gamma^1\partial_3) N = J. \qquad (129)$$

This Dirac matrix-based form does not seem to be very useful to apply in Maxwell's theory, it does not prove much similarities with ordinary Dirac equation (though that analogy was often discussed in the literature).

Now, starting from the electromagnetic tensor $F_{\alpha\beta}$ and dual to it $\tilde{F}_{\rho\sigma}$:

$$\tilde{F}_{\rho\sigma} = \frac{1}{2}\varepsilon_{\rho\sigma\alpha\beta}F^{\alpha\beta}, \quad F_{\alpha\beta} = -\frac{1}{2}\varepsilon_{\alpha\beta\rho\sigma}\tilde{F}^{\rho\sigma},$$

Let us introduce two electromagnetic 4-vectors (below u^{α} is any 4-vector, which in general can not coincide with 4-velocity of a reference frame)

$$e^{\alpha} = u_{\beta}F^{\alpha\beta}, \quad b^{\alpha} = u_{\beta}\tilde{F}^{\alpha\beta}, \quad u^{\alpha}u_{\alpha} = 1; \tag{130}$$

the inverse formulas are

$$F^{\alpha\beta} = (e^{\alpha}u^{\beta} - e^{\beta}u^{\alpha}) - \varepsilon^{\alpha\beta\rho\sigma}b_{\rho}u_{\sigma}, \quad \tilde{F}^{\alpha\beta} = (b^{\alpha}u^{\beta} - b^{\beta}u^{\alpha}) + \varepsilon^{\alpha\beta\rho\sigma}e_{\rho}u_{\sigma}. \tag{131}$$

Such electromagnetic 4-vectors are presented in the literature on the electrodynamics of moving bodies, from the very beginning, see Minkowski [14-17], Gordon [18], Tamm and Mandelstam [19-21]; for instance, see Yépez, Brito, and Vargas [108]. The interest in these variables gets renewed after Esposito paper [116] in 1998.

In 3-dimensional notation $E^1 = -E_1 = F^{10}, cB^1 = cB_1 = \tilde{F}^{10} = -F_{23}, \textit{ and so on}$, the formulas (130) take the form

$$e^0 = uE, \quad e = u^0 E + cu \times B, \quad b^0 = cuB, \quad b = cu^0 B - u \times E, \tag{132}$$

or symbolically $(e,b) = U(u)(E,B)$; the inverse formulas look like

$$E = eu^0 - e^0 u + b \times u, \quad cB = bu^0 - b^0 u - e \times u, \tag{133}$$

or in a symbolical form

$$(E,B) = U^{-1}(u)(e,b).$$

The above possibility is often used to produce a special form of the Maxwell equations. For simplicity, let us consider the vacuum case:

$$\partial_{\alpha}F_{\beta\gamma} + \partial_{\beta}F_{\gamma\alpha} + \partial_{\gamma}F_{\alpha\beta} = 0, \quad \partial_{\alpha}F^{\alpha\beta} = \varepsilon_0^{-1}j^{\beta},$$

or differently with the help of the dual tensor:

$$\partial_{\beta}\tilde{F}^{\beta\alpha} = 0, \quad \partial_{\alpha}F^{\alpha\beta} = \varepsilon_0^{-1}j^{\beta}. \tag{134}$$

These can be transformed into variables e^{α}, b^{α}:

$$\partial_{\alpha}(b^{\alpha}u^{\beta} - b^{\beta}u^{\alpha} + \varepsilon^{\alpha\beta\rho\sigma}e_{\rho}u_{\sigma}) = 0, \quad \partial_{\alpha}(e^{\alpha}u^{\beta} - e^{\beta}u^{\alpha} - \varepsilon^{\alpha\beta\rho\sigma}b_{\rho}u_{\sigma}) = \varepsilon_0^{-1}j^{\beta}. \tag{135}$$

They can be combined into the equations for complex field function

$$\Phi^{\alpha} = e^{\alpha} + ib^{\alpha}, \quad \partial_{\alpha}[\Phi^{\alpha}u^{\beta} - \Phi^{\beta}u^{\alpha} + i\varepsilon^{\alpha\beta\rho\sigma}\Phi_{\rho}u_{\sigma}] = \varepsilon_0^{-1}j^{\beta},$$

or differently

$$\partial_{\alpha}[\delta^{\alpha}_{\gamma}u^{\beta} - \delta^{\beta}_{\gamma}u^{\alpha} + i\varepsilon^{\alpha\beta\rho\sigma}g_{\rho\gamma}u_{\sigma}]\Phi^{\gamma} = \varepsilon_0^{-1}j^{\beta}. \tag{136}$$

This is Esposito's representation [116] of the Maxwell equations. One may introduce four matrices depending on 4-vector u^{α}:

$$(\Gamma^{\alpha})^{\beta}_{\ \gamma} = \delta^{\alpha}_{\gamma}u^{\beta} - \delta^{\beta}_{\gamma}u^{\alpha} + i\varepsilon^{\alpha\beta\rho\sigma}g_{\rho\gamma}u_{\sigma}, \tag{137}$$

then eq. (136) becomes

$$\partial_{\alpha}(\Gamma^{\alpha})^{\beta}_{\ \gamma}\Phi^{\gamma} = \varepsilon_0^{-1}j^{\beta}, \quad \textit{or} \quad \Gamma^{\alpha}\partial_{\alpha}\Phi = \varepsilon_0^{-1}j. \tag{138}$$

In the rest reference frame, when $u^{\alpha} = (1, 0, 0, 0)$, the matrices Γ^{α} become simpler, and $\Phi = \Psi$:

$$\Gamma^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \Gamma^1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, \Gamma^2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \Gamma^3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

correspondingly, eq. (138) takes the form

$$\begin{vmatrix} 0 & \partial_1 & \partial_2 & \partial_3 \\ 0 & -\partial_0 & i\partial_3 & -i\partial_2 \\ 0 & -i\partial_3 & -\partial_0 & i\partial_1 \\ 0 & i\partial_2 & -i\partial_1 & -\partial_0 \end{vmatrix} \begin{vmatrix} 0 \\ E^1 + icB^1 \\ E^2 + icB^2 \\ E^3 + icB^3 \end{vmatrix} = \varepsilon_0^{-1} \begin{vmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{vmatrix} = \varepsilon_0^{-1} j, \qquad (139)$$

or

$$div(E + icB) = \varepsilon_0^{-1} \rho, \quad -\partial_0 (E + icB) - i\, rot(E + icB) = \varepsilon_0^{-1} j.$$

Whence we get equations

$$div\, cB = 0, \quad rot\, E = -\frac{\partial cB}{\partial ct}, \quad div\, E = \frac{\rho}{\varepsilon_0}, \quad rot\, cB = \frac{j}{\varepsilon_0} + \frac{\partial E}{\partial ct},$$

which coincides with the Maxwell equations. Relations (139) correspond to a special choice of α-matrices:

$$\beta(-i\alpha^0) = \Gamma^0, \beta\alpha^j = \Gamma^j, \; where \;\; \beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{vmatrix}. \qquad (140)$$

Esposito's representation of the Maxwell equation 4-vector u^{α} can be easily related to the matrix equation of Riemann–Silberstein–Majorana–Oppenheimer:

$$(-i\alpha^0 \partial_0 + \alpha^j \partial_j)\Psi = J, \qquad (141)$$

indeed

$$(-i\alpha^0 \partial_0 + \alpha^j \partial_j) U^{-1} (U\Psi) = J, -i\alpha^0 U^{-1} = \beta\Gamma^0, \quad \alpha^j U^{-1} = \beta\Gamma^j, \quad U\Psi = \Phi,$$

$$\beta(\Gamma^0 \partial_0 + \Gamma^j \partial_j)\Phi = J, \quad \beta^{-1} J = \varepsilon_0^{-1} (j^a), \quad (\Gamma^0 \partial_0 + \Gamma^j \partial_j)\Phi = \varepsilon_0^{-1} j. \qquad (142)$$

Eq. (142) is a matrix representation of the Maxwell equations in Esposito's form

$$\partial_{\alpha} [\delta_{\gamma}^{\alpha} u^{\beta} - \delta_{\gamma}^{\beta} u^{\alpha} + i\varepsilon^{\alpha\beta\rho\sigma} g_{\rho\gamma} u_{\sigma}] \Phi^{\gamma} = \varepsilon_0^{-1} j^{\beta}. \qquad (143)$$

Evidently, eqs. (141) and (143) are equivalent to each other. There is no ground to consider the form (143) obtained through the trivial use of identity $I = U^{-1}(u)U(u)$ as having a certain especially profound sense. Our point of view contrasts with the claim by Ivezić [121-125] that eq. (143) has a status of a true Maxwell equation in a moving reference frame (at that u^{α} is identified with 4-velocity).

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