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A MODEL FOR THE FLOW OF GRANULAR MATERIALS AND ITS APPLICATION TO INITIAL/BOUNDARY VALUE PROBLEMS

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Abstract. A plasticity type model is here considered for the flow of a dry granular material such as grain or sand. The physical and kinematic basis for the model is briefly summarised and the equations governing the model are presented in terms of the components of the deformation-rate, spin and stress tensors. The equations comprise a set of six first order partial differential equations of hyperbolic type for which there are five distinct characteristic directions. An idealised application to a hopper is considered for the flow in the vicinity of the upper free surface. A simple analytic solution is given in which (a) the velocity field is linear in the space coordinates and represents a dilatant or contractant shear, (b) two possible stress fields are proposed, one linear and one exponential in space, which satisfy the stress equilibrium equations, the yield condition and the traction-free condition at the free surface, (c) the density is homogeneous in space and exponential in time. Finally, a method is proposed for defining an intrinsic time-scale for the deformation, which enables a physically realistic density field to be obtained via a sequence of dilatant and contractant shearing motions. Full advantage is taken of the hyperbolic nature of the governing equations to allow the solution to have discontinuities in the field variables, or their derivatives, in crossing characteristic lines.

1. INTRODUCTION

The *static* analysis of granular materials such as dry grain or sand is generally agreed to be well expressed by the Coulomb yield condition together with the stress equilibrium equations. The text by V.V. Sokolovskii [1] gives an excellent treatment both of the theory and application to the solution of practical problems involving soil mass in a geotechnical engineering context (for example embankments, soil retained by a wall, wedge shaped soil mass). By way of contrast there is very little agreement concerning the equations governing the *flow* of such materials, indeed it is an open question as to whether or not continuum mechanics is a reasonable framework for such evidently discrete materials. It is the contention of the present author that this question may be answered in the affirmative and that the required equations are summarised in section 1.1 below. Their derivation and associated theory has been presented elsewhere, see Harris [2-6], and will not be repeated here.

The model presented is one of a class of plasticity models which assumes the existence of a yield condition limiting the stress states that the material may sustain but which replaces the standard flow rule obtained from a plastic potential by a flow rule based upon a kinematic hypothesis that prescribes the manner in which the material may flow. An early model of this type was presented by Geniev [7] and was based upon the notion of a single-slip direction with the material slipping on one of the two stress characteristic directions. A model based upon simultaneous slipping on both stress characteristic directions was proposed by de Josselin de Jong [8]. This model, originally proposed for incompressible materials, also incorporated the rotation-rate of the slip directions and was extended to dilatant materials in [9]. In order to overcome an

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indeterminacy in the above model, Spencer [10] interpreted the rotation-rate as that due to the rotation of the in-plane greater principal stress direction. This introduced the stress-rate into the model. No continuum model has been universally accepted as describing a good approximation to granular flow, each model having one or more disadvantages and this is the motivation for the present work. The model presented here is a single slip model incorporating a physical rotation of the slip systems and it has been constructed in a manner designed to avoid the disadvantages suffered by other models.

For the sake of understanding we shall briefly recapitulate the underlying physical and kinematic basis of the model. Consider a granular material confined, by virtue of a combination of containing boundaries and body force, to occupy a region R with piecewise smooth boundary ∂R. Contact between neighbouring grains is assumed to be of finite duration and contact forces are assumed non-impulsive. Relative motion of contacting grains is opposed by solid friction forces and there may be inter-granular cohesive forces. The grains are treated as rigid bodies subject to the kinematic constraint of non-overlap. The consequent reaction forces, invoked to ensure non-violation of this constraint, dominate the relative (and indeed absolute) motion of the grains.The existence of relative motion between grains requires the frictional forces opposing such relative motion to be surmounted and also the re-arrangement of the grain configuration to prevent grain overlap. This will involve dilatation for a densely packed configuration as grain re-arrangement makes room for a relative motion of the grains. In the case of a loosely packed configuration, for a flow regime that does not entail grain separation, relative motion of the grains will be accompanied by contraction as the grain structure collapses and grains move into the voids.

The *local fabric* of the granular material is defined to be the network, in the vicinity of a point in the granular material, of contact points between grains, normals to the grain surfaces at contact points and also the directions of slip tangential to the grain surfaces at contact. The discrete local fabric of the real granular material is replaced, in the model, by a *continuum local fabric* at each point P which comprises a number (possibly infinite) of pairs of unit vector directions $\boldsymbol{t}_\beta$ and $\boldsymbol{t}_\gamma$ which we shall term the macroscopic and microscopic slip directions, respectively. At most one pair is allowed to be *active* at a given point and time (if no pair is active the material is rigid in the neighbourhood of the point) and the flow is assumed to be the resultant of the following three components

(a) a simple shear in the macroscopic slip direction $\boldsymbol{t}_\beta$,

(b) a dilatation/contraction in the normal direction $\boldsymbol{n}_\beta$ to $\boldsymbol{t}_\beta$ such that the resultant of the simple shear and the dilatation is a dilatant shear in the microscopic slip direction $\boldsymbol{t}_\gamma$,

(c) a local rigid spin of the macroscopic slip direction $\boldsymbol{t}_\beta$.

We shall refrain from expressing this kinematic hypothesis mathematically as this has been done elsewhere, see Harris [2-6] but merely content ourselves with writing down three of the four resulting second order tensor equations which the kinematic hypothesis gives rise to.

1.1. The equations governing the model.

Consider a planar deformation in the Ox_1x_2 plane. Let the velocity vector $\boldsymbol{v}$ have components v_i, the stress tensor σ have components σ_{ij} and let the bulk density be ρ. Let the deformation-rate tensor (i.e. the symmetric part of the velocity gradient tensor) be denoted by $\boldsymbol{d}$ with components d_{ij}, $1 \le i, j \le 2$. Let the spin tensor (i.e. the anti-symmetric part of the velocity gradient tensor) be denoted by $\boldsymbol{s}$ with components s_{ij}. Also, at each point P of R, let the active macroscopic and microscopic slip directions make angles $\theta = \theta(x_1,x_2,t)$ and $\chi = \chi(x_1,x_2,t)$, respectively, with the x_1-axis,

$$\boldsymbol{t}_\beta = (\cos\theta,\ \sin\theta),$$

$$\boldsymbol{t}_\gamma = (\cos\chi,\ \sin\chi).$$

It should be noted that

$$\chi = \theta + \nu,$$

where ν denotes the *angle of dilatancy*.

The first equation governing the model states that the dilatation-rate is proportional to the shear-rate, namely

$$d_{11} + d_{22} = \sin\nu[-(d_{11} - d_{22})\sin(2\theta + \nu) + 2d_{12}\cos(2\theta + \nu)], \qquad (1)$$

where the constant of proportionality is $\sin\nu$. The second equation relates the angle that the greater in-plane principal direction of the deformation-rate makes with the x_1-axis to the direction of the macroscopic slip direction, namely

$$(d_{11} - d_{22})\cos(2\theta + \nu) + 2d_{12}\sin(2\theta + \nu) = 0. \qquad (2)$$

The third equation relates the rotation-rate of $\boldsymbol{t}_\beta$ to the spin tensor,

$$2(s_{12} + \dot{\theta}) = \cos\nu[-(d_{11} - d_{22})\sin(2\theta + \nu) + 2d_{12}\cos(2\theta + \nu)], \quad (3)$$

where the superposed dot denotes the material derivative. This completes the equations governing the flow. The stress field is governed by the Cauchy equations of motion

$$\rho\frac{\partial v_1}{\partial t} + \rho v_1\frac{\partial v_1}{\partial x_1} + \rho v_2\frac{\partial v_1}{\partial x_2} - \frac{\partial \sigma_{11}}{\partial x_1} - \frac{\partial \sigma_{12}}{\partial x_2} - \rho F_1 = 0, \quad (4)$$

$$\rho\frac{\partial v_2}{\partial t} + \rho v_1\frac{\partial v_2}{\partial x_1} + \rho v_2\frac{\partial v_2}{\partial x_2} - \frac{\partial \sigma_{12}}{\partial x_1} - \frac{\partial \sigma_{22}}{\partial x_2} - \rho F_2 = 0, \quad (5)$$

together with the yield condition given by the inequality (6) below, which is closely related to the Coulomb yield condition. Let τ_χ, σ_χ denote the normal and tangential components of the traction vector at a point *P* across the line element along the microscopic slip direction $\boldsymbol{t}_\gamma$ then we shall assume that the stress at *P* satisfies the inequality

$$|\tau_x| \le k - |\sigma_x|\tan\phi_v, \quad (6)$$

where φ_ν denotes the *angle of mobilised internal friction* (and takes into account both the resistance to the grain on grain friction and the resistance to separating the particles during dilatation) and *k* denotes the *cohesion*. If ϕ_0 denotes the angle of internal friction when the state of the material and the stress admits an isochoric flow then

$$\phi_\nu = \phi_0 + \nu.$$

The bulk density of the material ρ is determined by the continuity equation

$$\rho\frac{\partial v_1}{\partial x_1} + \rho\frac{\partial v_2}{\partial x_2} + \frac{\partial \rho}{\partial t} + v_1\frac{\partial \rho}{\partial x_1} + v_2\frac{\partial \rho}{\partial x_2} = 0. \quad (7)$$

1.2. The characteristic directions

The system of first order partial differential equations presented in the previous subsection are hyperbolic in nature, for a proof of this see Harris [2-4,6]. There are five distinct characteristic directions defined at each point *P* of the granular material.

1. The kinematic equations (1) and (2) give rise to the following pair of velocity characteristic directions, the α-characteristic direction m_α,

$$m_\alpha = \frac{dx_2}{dx_1} = \tan\left(\chi + \frac{1}{2}\pi\right), \quad (8)$$

i.e. the direction $\boldsymbol{n}_\gamma$, perpendicular to the microscopic slip direction, and the β-characteristic direction m_β,

$$m_\beta = \frac{dx_2}{dx_1} = \tan\theta, \quad (9)$$

i.e. the macroscopic slip direction $\boldsymbol{t}_\beta$; the angle between the α- and β- characteristic directions is $(1/2)\pi+\nu$.

2. The equations of motion (4)-(6) give rise to the following pair of stress characteristic directions, the γ-characteristic direction m_γ,

$$m_\gamma = \frac{dx_2}{dx_1} = \tan\chi, \quad (10)$$

i.e. the microscopic slip direction $\boldsymbol{t}_\gamma$ and the δ-characteristic direction m_δ,

$$m_\delta = \frac{dx_2}{dx_1} = \tan\left(\chi \pm \phi + \frac{1}{2}\pi\right), \quad (11)$$

i.e. an angle ±ϕ to the perpendicular to the microscopic slip direction; the angle between the γ and δ directions is $(1/2)\pi\pm\phi$. The ambiguity of sign is removed by the condition of non-negative work-rate.

3. The kinematic equation (3) and continuity equation (7) both give rise to the same characteristic direction, which is determined by the pair of ordinary differential equations

$$\frac{dx_1}{dt} = v_1, \quad \frac{dx_2}{dt} = v_2 \quad (12)$$

together with the fact that the characteristic is embedded in the flow. These are parametric equations for the streamline characteristic direction, m_ε, in the case of time dependent flow and the equation

$$m_\varepsilon = \frac{v_2}{v_1} = \tan\chi \quad (13)$$

in the case of time independent flow.

2. APPLICATION: HOPPER FLOW NEAR THE FREE SURFACE

In this section we shall consider an application of the model to a case in which there is a free surface, for example the emptying of a hopper, represented schematically in Fig. 1. Granular material is confined between the vertical walls AC and BD. The upper surface of the granular material is initially the straight line EFG, where F denotes the mid-point of EG. The arrangement whereby the material is let out of the hopper, somewhere below CD, is not shown and we will not attempt to model the geometry or the flow in the exit region. Instead we shall concentrate attention on the region in the vicinity of EFG and will present a simple analytic solution of the governing equations which represents a simplified flow in the upper portion of the hopper sufficiently far from the exit. As the hopper empties the material flows downward and the free surface descends, changing shape as the flow continues until it reaches a steady state. Let H denote the mid-point of the straight line segment joining C and D. A corner often appears on the free surface on the axis of symmetry FH of the hopper and this is shown as E'F'G' in Fig. 1. Choose Cartesian co-ordinate axes Ox_1x_2 (where the origin O coincides with the mid-point F in the initial configuration) with the x_1-axis horizontal and to the right, the x_2-axis vertically upward. The material properties of the granular material are assumed to be such that the angle of internal friction ϕ_0 is constant and that the cohesion c is zero. In this section the angle of dilatancy ν will be assumed everywhere constant. We shall assume a solution in which the active microscopic direction $\boldsymbol{t}_\gamma$ in CEFGDH is everywhere parallel to the hopper walls $\overrightarrow{AC}$ and $\overrightarrow{BD}$, i.e the angle χ is everywhere constant and equal to $(-1/2)\pi$ relative to the Ox_1x_2-axes, then

$$\theta = -\frac{1}{2}\pi - \nu.$$

The α-, β-, γ- and δ-chracteristic directions are shown in Fig. 1 as the directions $\overrightarrow{PL}$, $\overrightarrow{PI}$, $\overrightarrow{PJ}$ and $\overrightarrow{PK}$, respectively. If the material in CEFGDH is nowhere in a state of yield then, as material below CD exits the hopper, the material in CEFGD will move vertically down the hopper as a rigid body. If the material in CEFGD is in a state of yield then it should be noted that the material placed infinitesimally close either side of the centre line FH will have no tendency to shear past each other and so the material on the vertical centre line is not in a state of yield. Thus the solutions in the two regions in yield CEFH, DGFH may be treated separately and the two solutions patched together along FH. It should be noted that these are not the only two possibilities. Any member of the family of γ-characteristics may separate a rigid region from a yielding region and so the region CEFGDH may be divided up into an arbitrary number of alternating rigid and deforming regions separated by γ-characteristics. This ambiguity may be removed by consideration of the exit conditions below CD.

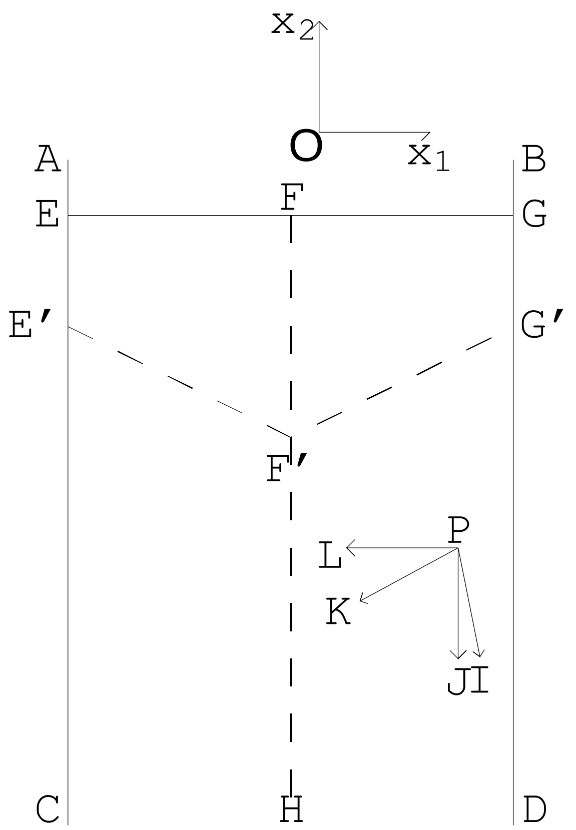

Fig.1. Hopper flow.

If the material interior to the regions CEFH, DGFH is everywhere in a state of yield then we shall model the flow in DGFH by a simple linear velocity field

$$v_1 = 0, \tag{14}$$

$$v_2 = -k_*(x_1 - x_2 \tan\nu) + A, \tag{15}$$

which expresses a dilatant shear of the material vertically downwards through the hopper. Here $k_* = k_*(t) > 0$ is a constant characterising the magnitude of the shear. The arbitrary constant $A = A(t)$ is determined by the conditions at the exit of the hopper. This velocity field satisfies the flow rule (1)-(3). The flow in CEFH is assumed to be the mirror image of the flow in DGFH. We turn now to the continuum equations of motion in Cartesian form relative to the Ox_1x_2-axes, then taking into account the above velocity field, the equations (4) and (5) become

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = 0,$$

$$\rho \frac{\partial \nu_2}{\partial t} + v_2 \frac{\partial v_2}{\partial x_2} - \frac{\partial \sigma_{12}}{\partial x_1} - \frac{\partial \sigma_{22}}{\partial x_2} + \rho g = 0.$$

The convection terms are $O(k_*^2)$ and $O(k_* A)$ and for sufficiently small k_*, A and $\dot{k}_*$, $\dot{A}$ it is usual to neglect the inertia terms in comparison with the equilibrium stresses. This practice is almost universal in plasticity theory and we adopt this assumption here. In order that the rate of working of the stresses be positive for the velocity field (14) and (15) it follows that, relative to the Ox_1x_2-axes, $\tau_\chi < 0$ and so the yield condition becomes

$$\tau_\chi = \sigma_\chi \tan(\phi + \nu).$$

Relative to Ox_1x_2-axes, the yield condition takes the simple form

$$\sigma_{12} \cos(\phi + \nu) - \sigma_{11} \sin(\phi + \nu) = 0$$

since the x_2-axis is aligned along the γ-direction.

Now, the upper surface EG is traction free,

$$\boldsymbol{t}_{EG} = \boldsymbol{\sigma n}_{EG} = \boldsymbol{0}$$

where $\boldsymbol{t}_{EG}$ denotes the traction vector across EG, σ denotes the stress tensor and $\boldsymbol{n}$ = (-sinη, cosη), where η is the angle the upper surface makes with the x_1-axis.

Thus,

$$\sigma_{12} \cos\eta = \sigma_{11} \sin\eta,$$

$$\sigma_{22} \cos\eta = \sigma_{12} \sin\eta$$

and, in the case that $\eta \neq \phi + \nu$, this is consistent with the free surface EFG being in a state of yield only if

$$\sigma_{11} = \sigma_{12} = \sigma_{22} = 0.$$

In the case that $\eta = \phi + \nu$, there are infinitely many states of stress that satisfy the traction-free boundary condition and the yield condition. If FG is a straight line inclined at an angle $\phi + \nu$ to the x_1-axis, another slip direction inclined at angle ν to FG becomes active, and a secondary flow of material from the vicinity of the hopper walls towards the centre F' is activated. We now consider two forms for the solution of the stress equilibrium equations.

Case (a). Linear stress field.

Suppose that

$$\sigma_{11} = Ax_1 + Bx_2 + C,$$

$$\sigma_{12} = Dx_1 + Ex_2 + F,$$

$$\sigma_{22} = Gx_1 + Hx_2 + I.$$

Substituting into the yield condition gives

$$D = A\tan(\phi + \nu),$$

$$E = B\tan(\phi + \nu),$$

$$F = C\tan(\phi + \nu).$$

Substituting into the equilibrium equations gives

$$A = -B\tan(\phi + \nu),$$

$$H = \rho g - A\tan(\phi + \nu).$$

Substituting into the traction free condition gives $\eta = \phi + \nu$ and

$$G = -B\tan^3(\phi + \nu),$$

$$I = C\tan^2(\phi + \nu).$$

Hence, the stress field is

$$\sigma_{11} = B[x_2 - \tan(\phi + \nu)x_1] + C,$$

$$\sigma_{12} = \{B[x_2 - \tan(\phi + \nu)x_1] + C\}\tan(\phi + \nu),$$

$$\sigma_{12} = \{B[x_2 - \tan(\phi + \nu)x_1] + C\}\tan^2(\phi + \nu) +$$
$$\rho g[x_2 - \tan(\phi + \nu)x_1].$$

Let the equation of the free surface F'G', as it descends, be

$$x_2 = x_1 \tan(\phi + \nu) - h$$

then F'G' is stress-free if $C = Bh$.

Case (b). Exponential stress field.

Consider a stress field of the form

$$\sigma_{11} = A + B\exp(k_1x_1 + k_2x_2),$$

$$\sigma_{12} = C + D\exp(k_1x_1 + k_2x_2),$$

$$\sigma_{22} = E + F\exp(k_1x_1 + k_2x_2) +$$
$$\rho g[x_2 - \tan(\phi + \nu)x_1],$$

where A, B, C, D, E, F, k_1, k_2 are constants to be determined. Substituting into the yield condition gives

$$C = A\tan(\phi + \nu),$$

$$D = B\tan(\phi + \nu).$$

Substituting into the equilibrium equations gives

$$k_1 = -k_2 \tan(\phi + \nu),$$

$$F = B\tan^2(\phi + \nu).$$

Substituting into the traction free condition gives $\eta = \varphi + \nu$ and

$$E = A\tan^2(\phi + \nu).$$

Hence,

$$\sigma_{11} = A + B\exp\{k[x_2 - \tan(\phi + \nu)x_1]\},$$

$$\sigma_{12} = [\{A + B\exp\{k[x_2 - \tan(\phi + \nu)x_1]\}]\tan(\phi + \nu),$$

$$\sigma_{12} = [\{A + B\exp\{k[x_2 - \tan(\phi + \nu)x_1]\}]\tan^2(\phi + \nu) + \rho g[x_2 - \tan(\phi + \nu)x_1],$$

where the constants A, B, k are arbitrary. Note that for the choice

$$B = -A$$

the upper free boundary is stress free. The remaining constants A and k are determined by the conditions at the exit to the hopper.

Finally, we consider the density field. Relative to axes $Ox_1'x_2'$ in which the x_1'-axis is directed along the α-characteristic line the velocity field becomes

$$v_1' = k_* x_2',$$

$$v_2' = k_* x_2 \tan\nu$$

then the continuity equation becomes

$$\frac{\partial\rho}{\partial t} + (k_* x_2')\frac{\partial\rho}{\partial x_1'} + (k_* x_2' \tan\nu)\frac{\partial\rho}{\partial x_2'} + k_*\rho\tan\nu = 0$$

which is equivalent to the set of ordinary differential equations

$$\frac{dt}{1} = \frac{dx_1'}{k_* x_2'} = \frac{dx_2'}{k_* x_2' \tan\nu} = \frac{d\rho}{-k_*\rho\tan\nu}.$$

Hence

$$\rho = \rho_0 \exp(-k_* t \tan\nu)$$

along the characteristic lines

$$x_2' - x_1' \tan\nu = c$$

which become

$$x_1 = c$$

relative to the Ox_1x_2-axes. Hence, for a dilating material (i.e. $\nu > 0$) $\rho \to 0$ as $t \to \infty$, whereas for a compacting material (i.e. $\nu < 0$) $\rho \to \infty$ as $t \to \infty$.

3. AN INTRINSIC TIME SCALE

The bulk density ρ of the granular material is bounded above by the density ρ_g of the grains which make up the material and below by the bulk density ρ_s at separation packing. In any given flow it is unlikely that either of these bounds will be attained. The actual maximum and minimum bulk densities attained during the flow will depend upon several factors, for example, the initial state of the material, grain size, confining boundaries and the confining pressure. Let ρ_{max}, ρ_{min}, respectively, denote these quantities, then

$$\rho_s \le \rho_{min} \le \rho \le \rho_{max} \le \rho_g.$$

At time $t = 0$ let $\rho = \rho_{min}$ then during a deformation governed by equations (14) and (15), the material must initially densify, i.e. $\nu < 0$. Suppose the material continues to densify at a constant angle of dilatancy ν_c until $\rho = \rho_{max}$ at time τ_c, say, then

$$\rho_{max} = \rho_{min} \exp(-k_* \tau_c \tan\nu)$$

i.e

$$\tau_c = \frac{1}{k_* \tan\nu_c} \ln\left(\frac{\rho_{max}}{\rho_{min}}\right).$$

We shall call τ_c the *intrinsic contraction time*. Similarly, a dilatant deformation at constant angle of dilatancy ν_d from $\rho = \rho_{max}$ to $\rho = \rho_{min}$ gives rise to an *intrinsic dilatation time*

$$\tau_d = \frac{1}{k_* \tan\nu_d} \ln\left(\frac{\rho_{max}}{\rho_{min}}\right).$$

We now propose that the velocity field in the real granular material may be approximated by a sequence of flows of the type considered in the previous section and alternating between contraction and dilatation. For definiteness let the initial density $\rho = \rho_0$ be such that the initial flow is contractant with angle of dilatation $\nu_c < 0$, followed by a dilatant flow with angle of dilatancy $\nu_d > 0$. Then the material oscillates between two states, with the change of state occurring at intervals of time τ_c and τ_d. In Fig. 2, AB denotes the value of ρ_{min}, CD the value of ρ_{max}, OE the value of τ_c and EF the value of τ_d. The deformation is clearly non-smooth, with discontinuities

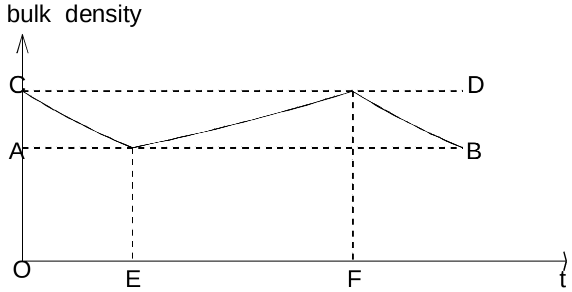

Fig.2. Intrinsic time scale.

in both the velocity and stress fields at times when the material flips from a dilatant to a contractant state and vice versa. Note that, in particular, the yield condition differs in the two states, with the yield strength greater in a dilatant state than in a contractant state i.e.

$$|\sigma_{12}| \le |\sigma_{11}| \tan(\phi + \nu_d),$$
$$|\sigma_{12}| \le |\sigma_{11}| \tan(\phi + \nu_c).$$

It is a well-known experimental observation that flow in a hopper may suffer from various discontinuities, for example the pressure at the walls is known to be discontinuous in time and the flow may either "pulsate" or exhibit "stick-slip". Such phenomena may be explained in terms of the above mechanism.

4. CONCLUSIONS

We have stated the equations of a novel physically based model for the flow of a granular material and have presented a simple analytic solution based upon a linear velocity field and a choice of either a linear or exponential (in space) stress field, together with an exponential (in time) density field. Despite the simplicity of the solution it is able to satisfy the yield condition, stress equilibrium equations, and conditions are given when it may also satisfy a traction-free or stress-free boundary. The solution has a natural application to hopper flow in the vicinity of the upper free surface. The non-physical asymptotic behaviour of the density field, i.e. the fact that dilatation/contraction cannot continue indefinitely suggests that it is possible to alternate the two possible states and consider a sequence of initial value problems in which the material successively dilates and contracts, the final value in a given state giving rise to the initial conditions of the succeeding state. This idea takes advantage of the discrete nature of a granular material by relaxing the conditions of differentiability and continuity usually present and is possible because of the hyperbolic nature of the governing equations which allow discontinuities in field variables and their gradients in crossing the characteristics. In future, it may prove possible to refine the notion of intrinsic time introduced here and, combining this concept with a typical velocity magnitude, obtain an intrinsic microscopic length scale associated with the granular material. It may eventually prove possible for this microscopic length scale to play the role of a particle length scale. This idea is speculative but may be worth pursuing. On a more practical level, the next stage in the development of the model is to overcome the restrictions of a simple analytic solution and to solve the equations numerically to obtain numerical approximations to more realistic problems. This work is now in progress.

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