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A NONCLASSICAL MODEL FOR CREEP-DAMAGE PROCESSES

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Abstract. The analysis of creep-damage processes is becoming more and more important in engineering practice due to the fact that the exploitation conditions like temperature and pressure are increasing while the weight of the structure should decrease. In the same time the safety standards are increasing too. The accuracy of the mechanical state estimation (stresses, strains and displacements) mainly depends on the introduced constitutive equations and on the chosen structural analysis model. For the first purpose an improved generalized phenomenological creep model is introduced and extended to the case of creep-damage coupling. In addition, a micromechanical-based model is discussed. For thin-walled structures under creep-damage conditions the advantages and the problems of different approaches are briefly discussed.

1. MOTIVATION

During the last hundred years there was published a lot of papers in Creep Mechanics mostly influenced by various approaches in establishing suitable constitutive equations or by the practical use of the proposed constitutive equations in structural analysis (thin-walled structures like tubes, discs, plates, shells, etc.). The state of art was reported by numerous authors in different papers, monographs or proceedings (e.g., [1-8]). In addition, with respect to the deformation mechanisms during the tertiary creep which can be related partly to damage processes leading to the material deterioration the development of the Continuum Damage Mechanics stimulates new investigations in this field (see, e.g., ([7-11]).

Due to the fact that till now an unique theory of creep-damage processes does not exist in dependence on the chosen variant of the constitutive equations we obtain in some cases different approximations of experimental observations even in the uniaxial case. Therefore we have to solve three problems:

- the establishing of suitable uniaxial creep equations (and if damage may be evolution equations) for a convenient description of uniaxial creep test observations,
- the definition of threedimensional constitutive equations which are generalizations of the uniaxial creep or creep-damage equations and
- the selection of relevant structural mechanics equations for the analysis of thin-walled structures reflecting adequate the structural and the constitutive behaviour.

With respect to the experimental observations the creep processes can be divided into three stages: the primary creep which is characterized by the increasing hardening and the decreasing softening tendencies in the material, the secondary or stationary creep which reflects the equilibrium between softening and hardening and the tertiary creep with a dominant increasing of the material deterioration. These three stages can be obtained for all materials at elevated temperatures (in comparison with the melting temperature), but in dependence on the temperature level, the loading rate, etc. these stages may be more or less significant.

Simple phenomenological constitutive equations were proposed during the first thirty years of this century [1]. The Norton-Bailey creep law was mostly used in practice

$$\dot{\varepsilon} = K\sigma^n,$$

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where K, n are material properties following from uniaxial tests. ε, σ denote strains and stresses, respectively, the dot is the derivation with respect to the time. The Norton-Bailey law is a suitable description for the stationary creep. This relation can be modified for the primary creep introducing an explicit dependence on the time t or for the tertiary creep introducing a damage variable ω and defining a damage evolution law [2].

Multiaxial creep-damage states which are usually realized in engineering structures require a suitable extension of creep-damage uniaxial equations. The starting point is the introduction of a creep potential Φ which is a function of the stress state and of additional parameters describing, for instance, the rheological behaviour or the damage of the material. Due to the demand that the uniaxial and the multiaxial states should be comparable (all material properties are following from simple tests which are mostly uniaxial) an equivalent quantity, for instance the equivalent stress σ_{eq}, has to be introduced by engineering assumptions. There are various possibilities, but mostly used is up to now the so-called J_2-theory based on the assumption that $\sigma_{eq} \equiv \sigma_{vM}$ is a quadratic function of the stress deviator. In this case the hydrostatic stress state has no influence on the creep behaviour and only the deviatoric stresses control the creep strain rates. A similar assumption was used by von Mises in the theory of plasticity [12]. The J_2-theory is easy in handling, but ignore, in general, the influence of the kind of stress state (no differences in tension and compression, etc.).

In this paper a generalization of the classical creep and creep-damage equations is introduced and classified. In accordance to the classical monograph of Rabotnov [2] these equations can be formulated from an unique point of view. They are more complicated in comparison with the classical models, but on the other hand, they are able to reflect the material softening (damaging) or hardening or the dependence on the kind of stress state. The generalized creep or creep-damage equation can be applied to structure mechanics problems. Due to this purpose different structure mechanics assumptions presumed in the modelling the mechanical behaviour of thin-walled structures are briefly discussed. Finally, on experiences in applications will be reported.

2. CREEP-DAMAGE CONSTITUTIVE EQUATIONS

Creep-damage constitutive equations are mostly based on creep equations for stationary creep. Taking into account the possibility to extend the creep equations for stationary creep proposed by Kachanov and Rabotnov with the help of one damage variable the tertionary creep can be modelled. In this section different approaches are presented. All approaches are able to describe more or less the experimental observations. The advantages/disadvantages of each proposal can be obtained from the handling of practical problems (for instance, in the case of thin-walled structures).

2.1. Creep constitutive equation

The starting point for various threedimensional phenomenological creep equations is the introduction of a creep potential as a function of the stress state (cp., [2] e.g.,). Due to the necessity that the threedimensional equations have to be identified by uniaxial tests an equivalent stress will be defined which represents the multiaxial stress state. Different proposals for an equivalent stress expression can be find in the literature (see, e.g., [13]). They are mostly based on some mathematical manipulations and the experimental proof. For example, if the material is isotropic the creep potential depends on the stress tensor invariants. In the general case we have to take into account three linear independent invariants, and finally using the associated flow rule for the derivation the creep strain rate tensor as the derivative of the creep potential with respect to the stress tensor we obtain a tensorial nonlinear constitutive equation.

Let us assume the irreducible stress tensor invariants [14]:

$$I_1 = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{I},$$
$$I_2 = \boldsymbol{\sigma} \cdot\cdot \boldsymbol{\sigma},$$
$$I_3 = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) \cdot\cdot \boldsymbol{\sigma}$$

with $\boldsymbol{I}$ as the second rank unit tensor and $\boldsymbol{\sigma}$ as the stress tensor. Combining these invariants as new linear, quadratic and cubic invariants

$$\sigma_1 = \mu_1 I_1,$$
$$\sigma_2^2 = \mu_2 I_1^2 + \mu_3 I_2,$$
$$\sigma_3^3 = \mu_4 I_1^3 + \mu_5 I_1 I_2 + \mu_6 I_3$$

a generalized equivalent stress may be introduced as a linear combination of the invariants σ_i, $i = 1, 2, 3$.

$$\sigma_{eq} = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3,$$

and the creep potential will be defined as follows

$\Phi = \Phi(\sigma_{eq})$

The explicit dependence on the temperature is dropped when the temperature is fixed. Using the associated flow law (normality rule) [2]

$$\dot{\varepsilon} = \dot{\lambda} \frac{\partial \Phi}{\partial \boldsymbol{\sigma}} \tag{1}$$

and the chain rule for derivation the creep strain rate tensor can be calculated as

$$\dot{\varepsilon} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma_{eq}} \frac{\partial \sigma_{eq}}{\partial \boldsymbol{\sigma}} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma_{eq}} \left(\alpha \frac{\partial \sigma_1}{\partial \boldsymbol{\sigma}} + \beta \frac{\partial \sigma_2}{\partial \boldsymbol{\sigma}} + \gamma \frac{\partial \sigma_3}{\partial \boldsymbol{\sigma}} \right). \tag{2}$$

Finally, taking into account the expressions of σ_1, σ_2, σ_3 from Eq. (2) with respect to the tensor calculus [15] follows

$$\dot{\varepsilon} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma_{eq}} \left[\alpha\mu_1 \boldsymbol{I} + \beta \frac{\mu_2 I_1 \boldsymbol{I} + \mu_3 \boldsymbol{\sigma}}{\sigma_2} + \gamma \frac{(\mu_4 I_1^2 + (\mu_5/3) I_2) \boldsymbol{I} + (2\mu_5/3) I_1 \boldsymbol{\sigma} + \mu_6 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}{\sigma_3^2} \right]. \tag{3}$$

Eq. (3) contains 6 material constants μ_i (i = 1, …,6) which should be determined by tests. The coefficients α, β and γ allow a "weighting" of the different parts of the equivalent stress expression. The introduction of such weighting coefficients is well-known from the literature. For example, in the Leckie-Hayhurst criterion in Damage Mechanics [16] which is able to reflect different damage mechanisms in dependence on the kind of material and loading conditions similar coefficients are introduced.

From Eq. (3) various special cases can be deduced. Setting $\alpha = \gamma = 0$, $\beta = 1$ and $\mu_2 = -1/2$, $\mu_3 = 3/2$ we get the so-called von Mises-type creep equation. The changing of these values and the additional introduction of more or less nonzero values for μ_i (i = 1,4,5,6) and α, γ allows the description of the influence of the kind of stress state on the creep behaviour.

The flow law (1) establishs the dependence of the creep strain rates on the stresses using an additional factor $\dot{\lambda}$. This factor can be estimated from the assumption of the identical specific dissipation power in the uniaxial and the multiaxial state

$$P_{diss} = \sigma_{eq} \dot{\varepsilon}_{eq} = \boldsymbol{\sigma} \cdot\cdot \dot{\varepsilon}, \tag{4}$$

where ε_{eq} is a formal introduced equivalent strain (note that ε_{eq} is not defined like σ_{eq} as a quadratic expression of the strain deviator). Finally, after some calculations we obtain the tensorial nonlinear creep equation as follows

$$\dot{\varepsilon} = \varepsilon_{eq} \left[\alpha\mu_1 \boldsymbol{I} + \beta \frac{\mu_1 I_1 \boldsymbol{I} + \mu_3 \boldsymbol{\sigma}}{\sigma_2} + \gamma \frac{(\mu_4 I_1^2 + (\mu_5/3) I_2) \boldsymbol{I} + (2/3)\mu_5 I_1 \boldsymbol{\sigma} + \mu_6 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}}{\sigma_3^2} \right]. \tag{5}$$

It should be underlined that in Eq. (5) the equivalent strain rate $\dot{\varepsilon}_{eq}$ results from suitable approximations of uniaxial creep tests. Setting for this approximation Norton's power law and considering the von Mises-type creep equation with $\Phi(\sigma_{vM}) = \sigma_{vM}^2$ we get

$$\dot{\varepsilon} = \frac{3}{2} K \sigma_{vM}^{n-1} s \tag{6}$$

with

$$\sigma_{vM} = \sqrt{\frac{3}{2} s \cdot\cdot s}$$

and

$$s = \boldsymbol{s} - \frac{1}{3} \boldsymbol{s} \ddot{Y}\ddot{Y} II,$$

which is presented in the classical monographs (e.g., [1]). This creep equation is tensorial linear and does not reflect any non-classical material behaviour. Other examples of uniaxial creep test approximations are presented, for instance, in [2,4].

2.2. Identification of the generalized creep model

Let us discuss the identification procedure for the generalized creep model considering the "equal weight" of the invariants σ_i. In this case we set $\alpha = \beta = \gamma = 1$. After this simplification the proposed generalized isotropic creep Eq. (5) contains 6 unknown parameters μ_i only which should be identified by tests. Various possibilities of identification procedures are known. A simple procedure can be presented with the help of creep tests by constant loading and temperature. This approach is widely used in engineering applications, but should be carefully handled because the material dependent parameters are determined for a given constant load level. The use of the state equations with these parameters is allowed only in a small range of stresses in the neighbourhood of the given load level. Other possibilities are discussed, for example, in [17].

The choice of the tests depends on the experimental facilities and on the possibility to achieve analytical solutions for comparison with the experimental results. Assuming a Norton-type creep law with a creep exponent n which is constant with respect to the kind of loading (this was established experimentally [4]) let us suggest the following tests

• uniaxial tension ($\sigma_{11} > 0$)

$$\dot{\varepsilon}_{11}^{cr} = A_+ \sigma_{11}^n,$$
$$\dot{\varepsilon}_{22}^{cr} = \dot{\varepsilon}_{33}^{cr} = A_T \sigma_{11}^n;$$

• uniaxial compression ($\sigma_{11} < 0$)

$$\dot{\varepsilon}_{11}^{cr} = -A_- |\sigma_{11}|^n \, ;$$

• simple shear ($\sigma_{12} = \sigma_{21} \neq 0$)

$$2\dot{\varepsilon}_{12}^{cr} = 2\dot{\varepsilon}_{21}^{cr} = A_S \sigma_{12}^n,$$
$$\dot{\varepsilon}_{11}^{cr} = A_{PS} \sigma_{12}^n;$$

• hydrostatic pressure ($\sigma_{11} = \sigma_{22} = \sigma_{33} < 0$)

$$\dot{\varepsilon}_{11}^{cr} = \dot{\varepsilon}_{22}^{cr} = \dot{\varepsilon}_{33}^{cr} = -3A_P |\sigma_{11}|^n \, .$$

The A_+, A_-, A_T, A_S, A_{PS} and A_P are material characteristics following from the tests. Providing analytical solutions for these simple stress states the comparison with the tests results in the following coefficients μ_i [18]

$$\mu_3 = \frac{1}{2} A_S^{2r},$$
$$\mu_1 = \frac{A_{PS}}{(\sqrt{2\mu_3})^n},$$
$$\mu_2 = X^2 - \mu_3,$$
$$6\mu_4 = \left(\sqrt{9\mu_2 + 3\mu_3} - 3\mu_1 - A_P^r\right)^3 - 3(T - \mu_1)^3$$
$$+18\left(\frac{\mu_2}{\sqrt{\mu_2 - \mu_3}} + \mu_1 + A_T A_+^{-nr}\right)(T - \mu_1)^2,$$
$$2\mu_5 = 3(T - \mu_1)^3 - \left(\sqrt{9\mu_2 + 3\mu_3} - 3\mu_1 - A_P^r\right)^3 \tag{7}$$
$$-24\left(\frac{\mu_2}{\sqrt{\mu_2 - \mu_3}} + \mu_1 + A_T A_+^{-nr}\right)(T - \mu_1)^2,$$
$$\mu_6 = (T - \mu_1)^3 - \mu_4 - \mu_5$$

with $\;\; = \frac{1}{2}(A_+^r - A_-^r), \;\; X = \frac{1}{2}(A_+^r + A_-^r), \;\; r = \frac{1}{n+1}.$

The proposed model takes into account various nonclassical effects. For instance, if $A_{PS} \neq 0$ the Poynting-Swift effect [19] can be modelled and volumetric creep strain rates take place as a result of shear loads (similar to the Kelvin effect in elasticity [20], if $A_T \neq 0$ – the independent transverse contraction can be described. On the other hand, these effects are partly second order effects [21]. Their experimental observation is limited due to the difficulties to provide suitable tests. Another non-classical effect is the neglecting the creep incompressibility. The classical model is based on the assumption that $\dot{\varepsilon}_{ii}^{cr} = 0$. At the same time A_P tends to 0.

Let us discuss special cases of the generalized creep equation with respect to the identified μ_i-values. The special cases are important for the practical use of the proposed generalized model. In practice the necessary information on creep tests are mostly uncomplete, and therefore special cases of the Eq. (5) have to be used. The above discussed von Mises-type theory we obtain from the following considerations. At first we assume that the behaviour in tension and compression is the same ($A_+ = A_- \equiv K$) and we get $T = 0$ and $X = K^r$. The classical theory cannot reflect the Poynting-Swift effect ($A_{PS} = 0$) and is based on the second axiom of rheology [22] - the volumetric strains are purly elastic (A_P=0). Finally we get

$$\mu_3 = \frac{1}{2} A_S^{2r},$$
$$\mu_1 = 0,$$
$$\mu_2 = K^{2r} - \mu_3.$$

Taking into account the values of μ_2 and μ_3 in Sect. 2.1 which are based on the assumption of the Norton creep law with the material characteristics K and n we estimate that the classical creep theory is restricted by the relation $A_S^{2r} = 3K^{2r}$. For the last three values μ_4, μ_5, μ_6 we conclude: $\mu_6 = 0$ (the von Mises theory is tensorial linear), at the same time we find $\mu_4 = \mu_5 = 0$. In this case $3\mu_2 = \mu_3$, and we get again the restriction $A_S^{2r} = 3K^{2r}$.

Another simplification can be introduced by $\mu_1 = \mu_4 = \mu_5 = \mu_6 = 0$. This extended von Mises-type theory is restricted by $A_{PS} = 0$ (no Poynting-Swift effect), $T = 0$ (identical behaviour in tension and compression with $A_+ = A_- \equiv K$) and $9K^{2r} - 3A_S^{2r} = A_P^{2r}$. This simplified theory is based on two tests: tension and torsion or tension and transverse contraction. The first possibility was prefered in [18].

Finally let us discuss variants of the generalized creep law based on three material parameters. The following possibilities can be introduced. A tensorial linear creep law can be formulated with the help of the assumptions $\mu_4 = \mu_5 = \mu_6 = 0$. From these assumptions follow the restrictions for the material characteristics

$$= \frac{1}{2}(A_+^r - A_-^r) = A_{PS} A_S^{-rn},$$

$$9(A_+^r + A_-^r)^2 - 12A_S^{2r} = 4(A_P^r + 3A_{PS}A_S^{-rn})^2.$$

Another tensorial linear creep law we obtain if $\mu_1 = \mu_4 = \mu_6 = 0$. On the other hand, a tensorial nonlinear creep law can be established by $\mu_1 = \mu_4 = \mu_5 = 0$. Providing similar calculations as in the first case the restrictions for the material characteristics for the two last cases can be deduced. A detailed description of the possibilities to formulate simplified equations with three parameters starting from the generalized creep law is presented in [13].

2.3 Creep-damage constitutive equations

Considering creep-damage processes in the material simple extensions of the creep constitutive equations presented in the previous subsection can be proposed using the Kachanov-Rabotnov approach in Continuum Damage Mechanics. The starting point of this approach is the introduction of a damage (or continuity) variable. In addition, the assumption of some equivalence principle [11] leads to modified creep equations which allow the modelling of creep-damage processes. Following this approach we present here another modification of the stationary creep equations. Assuming the creep potential as a function of the stress state and a set of internal variables we can reformulate the associated creep law

$$\dot{\varepsilon} = \dot{\lambda}\frac{\partial\Phi(\sigma_{eq}, H_i, \omega_k)}{\partial\sigma}, \tag{8}$$

H_i, ω_k denote two sets of hardening and damaging variables, respectively. This generalized creep-damage law allows the reflecting of different damage and hardening processes. It can be used for the primary creep, too.

In addition to the associated creep law, we must formulate the evolution equations for the hardening and the damage variables

$$H_i = H_i(\sigma_{eq}^H, H_p, \omega_q, \ldots),$$

$$\omega_k = \omega_k(\sigma_{eq}^\omega, H_p, \omega_q, \ldots).$$

Note that we have now three equivalent stresses σ_{eq}, σ_{eq}^H and σ_{eq}^ω which are in general different and allow to reflect various deformation mechanisms influenced by different parts of the stress state.

Numerous examples of creep equations and additional evolution equation are presented in the literature. One of the first was introduced by Rabotnov assuming only creep-damage coupling. The starting point of this approach is the creep equation (6) modified as follows

$$\dot{\varepsilon} = \frac{3}{2}K\sigma_{vM}^{n-1}\frac{s}{(1-\omega^r)^k}.$$

Here ω $(0 \le \omega \le 1)$ denotes a damage variable. r, k are material parameters. For the damage evolution law an expression similar to the Norton law can be formulated

$$\dot{\omega} = \frac{B(\sigma_{eq}^\omega)^m}{(1-\omega^r)^l},$$

where m, l are material parameters and $\sigma_{eq}^\omega = \sigma_{vM}$. The Rabotnov creep-damage equations enable an extension of the classical creep equation for stationary creep to the tertiary creep. The advantage of this approach is the simple form easy for handling. The main problems are connected with the neglecting of primary creep and the formal introduction of the damage parameter ω that leads to a range of this parameter from 0 (undamaged state) to 1 (fully damaged state) and which seems to be not realistic. In addition, the damage law does not reflect the different damage behaviour, for instance, resulting from tensile or compressive loads.

One possible variation of the Rabotnov creep-damage equation is given by the introduction of other expressions for the equivalent stress σ_{eq}^ω. With the help of the modification of the equivalent stress damage mechanisms controlled not only by the deviatoric stresses can be introduced. A suitable generalized equivalent stress expression was proposed, for instance, in [23]. Starting from the Novozhilov's stress tensor invariants [24] the equivalent stress can be presented as follows

$$\begin{aligned}\sigma_{eq} &= \lambda_1\sigma_{vM}\sin\xi + \lambda_2\sigma_{vM}\cos\xi + \lambda_3\sigma_{vM} + \\ &\lambda_4 I_1 + \lambda_5 I_1 \sin\xi + \lambda_6 I_1\cos\xi\end{aligned} \tag{9}$$

with

$$\sin 3\xi = -\frac{27}{2}\frac{\det \boldsymbol{s}}{\sigma_{vM}^3}.$$

This criterion contains several special cases with a reduced number of parameters λ_i. For practical purposes mostly the proposals of Sdobyrev [25] and Leckie-Hayhurst [16] are in use. Both proposals take into account the different influence on the damage process following from the first principal stress, the deviatoric stresses and from the hydrostatic stress. A different modification was proposed by Lemaitre [11]. While the damage process is connected only with tensile stresses and compressive stresses lead to the closing of the voids, etc. (but not "healing") there was introduced a model which reflect the different damage behaviour in tension and compression. Obtaining tensile stresses damage occurs even for compressive stresses the damage process is interrupted.

Another possibility of creep-damage modelling can be proposed using the creep constitutive equation (5), which reflects the dependences on the kind of stress state, and the Rabotnov-Kachanov approach based on one scalar damage variable. Then following [26] we can introduce again the specific dissipation power (4) and integrating the dissipation power over the time t we find a measure of the damage process

$$\varphi = \int_0^{t_*} P_{diss} dt.$$

It is presumed that at the starting point $t = 0$ no damage is observed in the material and the final state $t = t_*$ is characterized by macroscopic failure. From these follows $\varphi(t{=}0){=}0$ and $\varphi(t = t_*) = \varphi_*$. The value φ_* is a material constant which was established experimentally [27].

At fixed temperatures we can define a state equation for creep-damage processes

$$P_{diss} = f(\sigma_{eq}^{\omega}, \varphi).$$

Separating the function f as a product of two functions which are depending only on one variable

$$f(\sigma_{eq}^{\omega}, \varphi) = \chi(\sigma_{eq}^{\omega}) \frac{\varphi_*^p}{(\varphi_* - \varphi)^p},$$

p is a material parameter and $\chi(\sigma_{eq}^{\omega})$ should be a special selected equivalent stress function. It is easy to show that Rabotnov's damage parameter ω is connected with φ by $\omega = \varphi/\varphi_*$. The dissipation power can be estimated from the uniaxial tests. Assuming once more the identical behaviour in the uniaxial and the multiaxial cases we obtain $\dot{\varepsilon}_{eq} = P_{diss}/\sigma_{eq}$. The equivalent creep strain rate can be calculated by the use of the expression for the dissipation power. Combining all equations we get

$$\dot{\varepsilon}_{eq} = \frac{\sigma_{eq}^{\omega}}{(1-\omega)^p \sigma_{eq}}.$$

The creep state equation for load dependent material behaviour under the consideration of creep-damage coupling can be proposed if we take the creep state equation (2) into account and setting instead of the equivalent creep strain rate the modified expression including the damage variable and an additional equivalent stress for the damage process

$$\dot{\boldsymbol{\varepsilon}} = \frac{\sigma_{eq}^{\omega}}{(1-\omega)^p \sigma_{eq}} \left[\alpha\mu_1 \boldsymbol{I} + \beta \frac{\mu_1 I_1 \boldsymbol{I} + \mu_3 \boldsymbol{\sigma}}{\sigma_2} + \gamma \frac{(\mu_4 I_1^2 + (\mu_5/3) I_2)\boldsymbol{I} + (2/3)\mu_5 I_1 \boldsymbol{\sigma} + \mu_6 \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}{\sigma_3^2} \right].$$

The load dependent creep equation can be extended also by other damage variables definitions, by introducing more internal variables or by other equivalent stress formulations. The difficulties of the parameter identification are increasing significant when the number of parameters is increasing. In addition, till now we discussed only constitutive equations reflecting initially isotropic creep and damage behaviour. During the last years several anisotropic models were proposed, but the experimental efforts for the parameter identification are increasing rapidly.

During the last years another set of creep-damage equations starts to be very popular. The starting point was the necessity to describe primary, secondary and tertiary creep for metallic materials. Hayhurst and co-workers have proposed the following set of creep constitutive and evolution equations for different internal variables and recommended these equations for an aluminium alloy [28]

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^{cr} &= \frac{3}{2} \frac{A}{(1-\omega_2)^n} \frac{\boldsymbol{s}}{\sigma_{eq}} \sinh\left[\frac{B\sigma_{eq}(1-H)}{1-\Phi}\right], \\
\dot{H} &= \frac{h}{\sigma_{eq}} \frac{A}{(1-\omega_2)^n} \sinh\left[\frac{B\sigma_{eq}(1-H)}{1-\Phi}\right]\left(1-\frac{H}{H^*}\right), \\
\dot{\Phi} &= \frac{K_c}{3}(1-\Phi)^4, \\
\dot{\omega}_2 &= \frac{DA}{(1-\omega_2)^n}\left(\frac{\sigma_I}{\sigma_{eq}}\right)^{\mu} N \sinh\left[\frac{B\sigma_{eq}(1-H)}{1-\Phi}\right], \qquad (10) \\
n &= \frac{B\sigma_{eq}(1-H)}{1-\Phi} \coth\left[\frac{B\sigma_{eq}(1-H)}{1-\Phi}\right],
\end{aligned}$$

where A, B, h, H^*, K_c and D are material parameters, σ_i is the maximum principal stress, $N = 1$ for $\sigma_i > 0$ and $N = 0$ for $\sigma_i < 0$ and m denote the multiaxial stress state index. The internal state variables are introduced as follows: H, $0 \le H \le H_*$ is the hardening variable which describes primary creep; Φ, $0 \le \Phi \le 1$ and σ_2, $0 \le \omega_2 \le 0.3$ are damage variables which characterize two major mechanisms of material softening - ageing of the particular microstructure and grain boundary cavitation. The damage process is taken into account while the principal stress σ_i is active and positive (only tensile stresses lead to the damage). The including of ageing into the model reflect the temperature influence of the longtime behaviour even the model is related to the creep at elevated temperatures. The creep constitutive equations (10) are motivated in [28] and [29]. In these papers the problem of the materials parameter identification is briefly discussed. As was shown in [28] the modified Rabotnov model and the set of equations (10) result to similar approximations of uniaxial creep curves from tests.

3. APPLICATION OF CREEP MODELS TO PLATES AND SHELLS

The introduced material models may be used for the simulation of creep processes occurring in thin-walled structures at elevated temperatures.The use of the generalized creep equation is connected with great experimental efforts, so in many practical cases we have to work with creep equations containing a reduced number of parameters. On the other hand, the proposed creep-damage models are in some situations incomplete and do not reflect the real damage behaviour. The reason for this unaccuracy is that we deal, for example, only with scalar internal variables. They allow only the description of isotropic material behaviour.

The mechanical state of thin-walled structures can be described on the base of various structure mechanics models formulated with the help of different assumptions. In the case of plates the principles of formulations are discussed, e.g., in [30]. For example, the classical plate and shell equations are grounded on the following assumptions (see, e.g., [31]):

- The plate or the shell are thin in comparison to lateral dimension or minimum radius of curvature.
- The displacements are small in comparison to the thickness (the equilibrium can be described with respect to the undeformed geometry).
- Transverse shearing strains are neglected.
- Normal stresses acting on planes parallel to the middle surface are dropped.

The equations following from these assumptions correspond to the Kirchhoff's plate theory or the Kirchhoff - Love's approximation of the shell theory. With respect to the coupled creep-damage behaviour the assumptions should be modified. Firstly, time-dendent material behaviour can result in large deflections even in the case of infinitesimal strains. Secondly, the question of the influence of the transverse shearing strains with respect to the assumed creep-damage behaviour is not enough investigated.

The following kinematical assumptions can be proposed as the first improvement:

- All geometrical relations are formulated by the assumption that the strains are infinitesimal quantities. The squares of the rotation angles of the normal vector are of the same order as the strains. In addition, some nonlinearities are introduced to the kinematics: quadratic terms are considered only in the strains of the middle surface, the curvature changes are linear. This extension has an influence on the strain-displacement relations and on the equilibrium equations. A corresponding shell theory is discussed, e.g., in [32].
- The strain-displacement relations are extended by two additional relations describing the transverse shearing strains. This extension has no influence on the equilibrium equations, but we have to formulate additional constitutive equations.

Supplementary assumptions can be introduced with respect to the constitutive and the loading model:

- The plate is assumed to be uniformly heated without thermal stresses.
- The plate is loaded by external stationary surface loads.
- The constitutive behaviour can be splitted into an elastic and a creep part. With respect to the geometrical assumptions this splitting is additive. Let us suppose that the elastic strains are small and the damaging, softening, hardening, etc. have an influence only on the creep part.
- The equilibrium equations can be formulated in terms of stress resultants. The forces can be computed by averaging the stresses over the thickness, the couples – by averaging the stresses multiplied with the thickness coordinate over the thickness.

For simplicity we restrict our discussions presuming Cartesian coordinates x, y in the middle surface and $z = 0$ (the coordinate in the thickness direction is limited by $-h/2 \leq z \leq z/2$ with h as the plate thickness). Further we use the so-called Timoshenko's kinematics [33] with independent rotations. This model is used, e.g., in the case of thin elastic plates in [34] and of thin elastic shells in [35].

The kinematics of the plate continuum with respect to our assumptions can be approximated by the following components of the displacement vector

$$\begin{aligned} \tilde{u}_x(x,y,z) &= u_x(x,y) + \psi_x(x,y)z, \\ \tilde{u}_y(x,y,z) &= u_y(x,y) + \psi_y(x,y)z, \\ \tilde{w}(x,y,z) &= w(x,y), \end{aligned} \qquad (11)$$

u_i are the tangential displacements in the middle surface, w is the transverse displacement, ψ_i are the independent rotations of the normal to the middle surface in the xz- or the yz-planes. The approximation of the strains of the plate is

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial \psi_x}{\partial x} z, \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial \psi_y}{\partial y} z, \\ \varepsilon_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) z, \\ 2\varepsilon_{xz} &= \psi_x + \frac{\partial w}{\partial x}, \\ 2\varepsilon_{yz} &= \psi_y + \frac{\partial w}{\partial y}. \end{aligned} \qquad (12)$$

The normal strains $\varepsilon_{zz} \approx 0$. This variant of the plate theory leads to constant transverse shear strains over the thickness h. Other proposals, e.g., [36], result in varying strains. The special case of von Kármán's theory follows from $\varepsilon_{iz} = 0$, the Kirchhoff theory we get by dropping the quadratic terms.

The constitutive behaviour description is based on the Hooke law

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{cr}); \; i,k = x,y; \; j,l = x,y,z. \qquad (13)$$

C_{ijkl} is the Hooke tensor. Introducing the stress resultants ($i,j = x,y$),

$$\begin{aligned} N_{ij} &= \int_h \sigma_{ij} dz, \\ M_{ij} &= \int_h \sigma_{ij} z dz, \\ Q_i &= \int_h \sigma_{iz} dz \end{aligned} \qquad (14)$$

(N_{ij}, M_{ij}, Q_i denote the in-plane forces, the couples and the transverse shear forces, respectively) and assuming isotropic homogeneous material behaviour the stress resultants contain an elastic and a creep part

$$\begin{aligned} N_{ij} &= C_{ijkl} h e_{kl} - N_{kl}^{cr}, \\ M_{ij} &= C_{ijkl} \frac{h^3}{12} \mu_{kl} - M_{kl}^{cr}, \\ Q_i &= kGhe_{iz} - Q_{cr}^{i} \end{aligned}$$

with

$$\begin{aligned} C_{xxxx} &= C_{yyyy} = \frac{E}{1-\nu^2}, \\ C_{xxyy} &= C_{yyxx} = \frac{\nu E}{1-\nu^2}, \\ C_{xyxy} &= C_{yxyx} = G. \end{aligned}$$

E, ν, $G = E/2(1+\nu)$ are the isotropic elastic constants, k denotes the shear correction factor. The creep parts of the stress resultants can be computed by

$$\begin{aligned} N_{ij}^{cr} &= \int_h C_{ijkl} \varepsilon_{kl}^{cr} dz, \\ M_{ij}^{cr} &= \int_h C_{ijkl} \varepsilon_{kl}^{cr} z dz, \\ Q_i^{cr} &= \int_h kG \varepsilon_{i3}^{cr} dz. \end{aligned}$$

In the case of the von Karman theory the quasistatic equilibrium equations are

$$\begin{aligned} &\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \\ &\frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0, \\ &\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_{xx} w_{,xx} + N_{yy} w_{,yy} + 2N_{xy} w_{,xy} + q_z = 0, \\ &\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \\ &\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \end{aligned} \qquad (15)$$

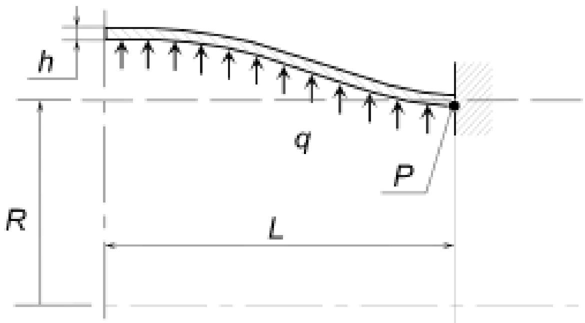

Fig.1. Clamped cylindrical shell (*L* length of the shell, *R* shell radius, *h* thickness).

q_z is the transverse surface load.

In addition to the sets of differential equations (kinematical and equilibrium equations) and the constitutive equations we must formulate initial/ boundary conditions. We can prescribe kinematical and/or statical boundary conditions for the coordinate lines *x, y*. The initial conditions follows from the elastic solution.

In [37] the equations for a shell theory under the considerations of moderate rotations, small strains and neglected transverse shear are presented. The corresponding equations for the shear deformable shell theory can be deduced in a similar way.

Another constitutive model is presented in [38]. This model is based on the assumption that the damage evolution influences the Hooke law as follows:

$$\sigma_{ij} = C_{ijkl}(1-\zeta\omega)(\varepsilon_{kl} - \varepsilon_{kl}^{cr}).$$

The ζ control the damage evolution, ω is a scalar damage variable. Repeating all steps of formulation a plate theory including creep-damage effects we get a new set of creep-damage plate equations. As in the theory of composite plates [39] in the general case the plate state and the plane stress state are coupled by the constitutive equations when the creep-damage distribution over the thickness is unsymmetric. This conclusion is true even in the case of applying the Kirchhoff assumptions.

4. EXAMPLE

Numerical investigations of creep problems are usually performed by time-step discretisation methods and by the solution of physical linearized boundary-value problems with fixed fictitious creep-load components at each time step [40,41]. Creep loads are defined from the creep strain field assumed to be known at each time step. Details of the numerical procedures and some results of numerical tests (numerical stability, etc.) are discussed, e.g., in [42-44].

From the equations discussed above several special cases can be deduced. Results based on these variants of the proposed constitutive and damage evolution equations and shell or plate theories can be compared with experimental data or finite element calculations. For example, the comparison with experimental data show the necessity of including geometrical nonlinear terms [42]. On the other hand the predictions are in a good agreement with finite element calculations.

The influence of the transverse shear deformation on the stress state and the damage in a cylindrical shell of moderate thickness (Fig. 1) was investigated. The calculations were performed for h= 0.2 m, R=1 m and L = 2 m. The shell was loaded by inner pressure q = 32 MPa and made from the aluminium alloy D16AT. In [37] the experimental data of uniaxial creep at 300° C are discussed. The uniaxial creep behaviour was described by

$$\dot{\varepsilon}^{cr} = F(\sigma)G(\omega), \qquad \dot{d} = H(\sigma)R(\omega) \tag{16}$$

with

$$F(\sigma) = K\sigma^n,$$
$$G(\omega) = (1-\omega^r)^{-n},$$
$$H(\sigma) = b\sigma^k,$$
$$R(\omega) = (1-\omega^r)^{-k},$$
$$K = 0.335 \cdot 10^{-7} MPa^{-n}/h,$$
$$b = 1.9 \cdot 10^{-7} MPa^{-k}/h,$$
$$n = k = 3,\ \ r = 14,\ \ d_* = 0.8.$$

For the generalization to the complex stress state the creep rate tensor follows from the von Mises flow rule. The damage evolution law was taken in the following form [16]

$$\dot{\omega} = r\left[\langle\chi(\sigma)\rangle, \omega\right] = r\left[\langle\sigma_{vM}\rangle, \omega\right].$$

The elastic material constants are

$E = 0.65{\cdot}10^5$ MPa, n = 0,3.

The refined theory takes into account the transverse shear strains dropped in the Kirchhoff-Love theory. On Fig. 2 the stress relaxation and the damage evolution are shown.The Kirchhoff-Love theory leads to an overestimation of the damage state. The reason is that in this case the equivalent stress is

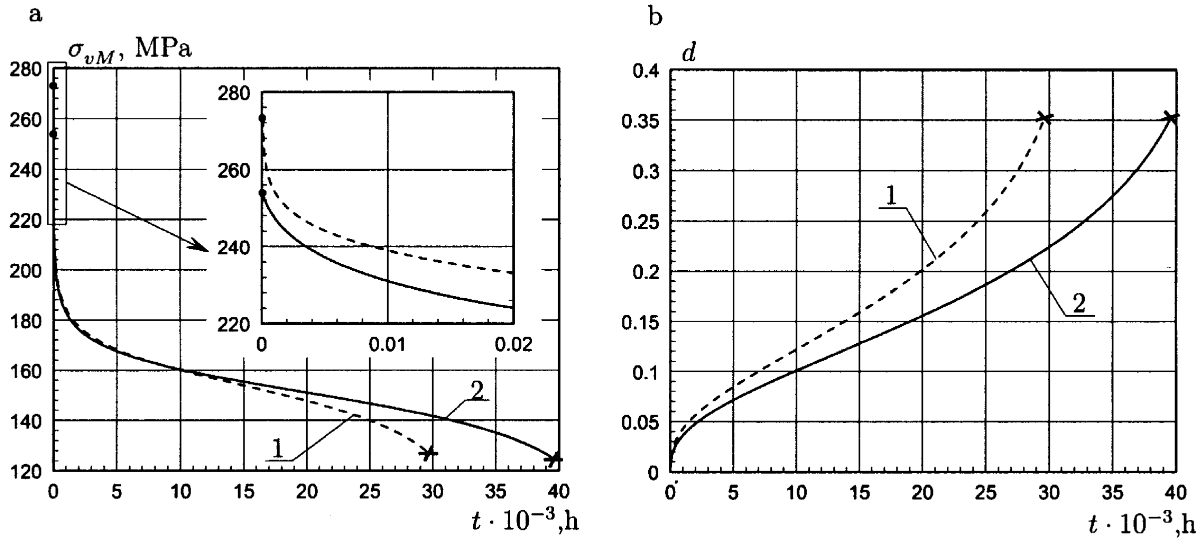

Fig. 2. Stress relaxation and damage evolution in the point *P*, cf. Fig. 1, of the cylindrical shell of moderate thickness. a: von Mises stress vs. time; b: damage parameter vs. time; 1 - Kirchhoff-Love's theory; 2 - refined theory.

calculated without shear stresses, which leads to qualitative differences of the predicted stress state.

5. CONCLUSIONS

Various inelastic behaviour equations with respect to some generalized creep models taking into account the dependency on the kind of loading and the different damage evolution laws can be proposed. The models of thin-walled structures should be selected carefully. From own calculations we can conclude that:

- the complex character of the creep strains in the thickness direction demands the introduction of the transverse shear strains in the shell or plate models,
- the kind of loading have a significant influence on the creep behaviour, especially, on tertiary creep and
- introducing geometrical nonlinearities a better agreement between experimental data and calculation can be obtained.

Further investigations should be directed to the development of a general shell theory, which is able to describe the complex creep behaviour by the proposed creep models. For this purpose new experiments must be realized for comparison with numerical simulations. In addition, the proposed models should be connected with fracture mechanics approaches for a better understanding and prediction of structural failure.

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