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ON THE DEFORMATION OF A COMPOSITE ROD IN THE FRAMEWORK OF GRADIENT THERMOELASTICITY

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Abstract. The gradient thermoelasticity problem for a composite rod based on the applied one-parameter model is investigated. To find the Cauchy stresses, the Vishik-Lyusternik asymptotic approach is used, taking into account the presence of boundary-layer solutions in the vicinity of the rods' boundaries and interface. A new dimensionless parameter equal to the ratio of the second rod length and the gradient parameter are introduced. Simplified formulas are constructed in order to find the distribution of the Cauchy stresses depending on the new parameter. After finding the Cauchy stresses distribution, moment stresses, total stresses, displacements, and deformations are further calculated. The dependence of the Cauchy stress jump on the ratio of the rods' physical characteristics and the scale parameter is investigated. The analysis of the results provided is performed.

Keywords: composite rod, gradient model, thermoelasticity, Cauchy stresses, moment stresses, asymptotic approach, boundary layer

1. Introduction

An interest in studying the stress-strain state (SSS) of composite structures of small sizes is associated with the prospects for the development of microelectronics, nanostructures, aerospace systems, and highly sensitive equipment. In such structures, especially in coatings, the sizes of the studied elements may become comparable with the characteristic sizes of the material's microstructure. In addition, large stress concentrations can occur at the interface between dissimilar materials, which greatly affects the product strength. In classical mechanics, the constitutive equations do not include any scale parameters; therefore, they cannot be used in modeling scale effects. At present, gradient elasticity theories are commonly used to model the effects observed in ultrathin structures, as well as in nanostructured materials, geomaterials, and biomaterials which include length dimension parameters in the constitutive relations.

The gradient elasticity theory is a generalization of the classical theory of elasticity. It was formulated in the 60s of the last century in the works of Toupin [1] and Mindlin [2,3]. Subsequently, a large number of researchers dealt with the development of the gradient elasticity theory [4-27]. The scientific schools of Aifantis [4-9] and Lurie [13-20] made a particularly large contribution to the development of gradient mechanics. In the gradient elasticity theory, the strain energy density depends not only on the strain but also on the first strain gradient. The mathematical formulation of the gradient theory of elasticity in the general case is completely determined by the variational Lagrange principle. The equilibrium equations of the gradient theory with respect to displacements or stresses have a higher order of differential equations compared to the classical theory, and in order to construct a solution, it is required to satisfy additional boundary conditions. Note that the practical use of the

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model [3] raises the question of identifying five additional gradient modules. To overcome this difficulty, the applied gradient deformation models were proposed: the three-parameter model by D.C.C. Lam [23], the one-parameter models by E.C. Aifantis [7] and S.A. Lurie [13].

On the basis of the one-parameter model of the gradient elasticity theory, many problems have been recently solved, both one-dimensional and two-dimensional. In [10], the solutions of the gradient elasticity theory for a rod were obtained, both for static and dynamic statements. In [9], the problems of the gradient theory of elasticity for composite bodies are considered. The conditions of conjugation at the interface of materials modified in comparison with the classical theory are obtained. For the one-dimensional problem, the exact analytical solutions are obtained. In [16,17], a refined gradient theory on the bending of scale-dependent hyperfine rods was constructed. In [13], the equilibrium problem of a two-layer coating under the influence of localized normal load in the framework of the plane problem is numerically studied. The problem statement is given on the basis of the interfacial layer model, which is a one-parameter version of the gradient theory of elasticity. Based on the variational formulation, the authors obtained the equilibrium equations, boundary conditions, and conjugation conditions. The solution was carried out using the integral Fourier transform and its numerical inversion. The dependence of the stress distribution on the layer thickness and the gradient parameter of the model are investigated. In [26], based on a three-parameter gradient model, the static deformation of a two-layer microplate was studied. It was found that the Cauchy stresses break at the boundary of the layers. In [27], the problem of bending a microbeam with a partial coating was solved. To study the scale effects, an additional scale parameter was introduced - the ratio of the coating thickness to the gradient parameter. The effect of a decrease in the scale parameter on changes in the distribution of displacements, stresses, and the neutral line was studied. It is found out that gradient effects play an important role when the scale parameter is less than unity.

Starting from the 70s of the last century, gradient theories have begun to be applied to problems in the mechanics of coupled fields [12,14,15,18,19,21]. In [12], the formulation of the dynamic coupled problem of gradient thermoelasticity was obtained. Further, gradient models began to be employed to more accurately estimate the SSS of inhomogeneous thermoelastic bodies, including the layered ones made of functionally graded materials (FGM). FGM is a composite material manufactured by mixing different material components (e.g., ceramic and metal ones) and is characterized by a smooth change of properties along with the coordinate [28]. So, in [21], the SSS of a long thick-walled FGM cylinder under the influence of thermal and mechanical load is numerically studied. The material characteristics of the cylinder vary exponentially in the radial direction. The influence of the inhomogeneity parameter and the gradient parameter on the distribution of stresses and displacements is studied. In [14], the formulation of the unbound gradient thermoelasticity problem based on the model of the interfacial layer for the coating-substrate system is presented under the assumption of the one-dimensionality of the original problem. As a result of the numerical solution, graphs of the distribution of stresses and strains are built, taking into account the influence of both thermomechanical characteristics, and the gradient parameter.

In this work, we study the SSS of a composite rod under thermomechanical loading based on the applied one-parameter model [7]. We have chosen a one-dimensional problem to study due to the fact that for such a problem one can obtain simplified analytical solutions that can be further used to analyze the stress state of thin coatings. The study begins with finding the temperature distribution. Then, on the basis of the Vishik-Lyusternik asymptotic approach, simplified analytical expressions for the Cauchy stresses in a dimensionless form are obtained. After finding the Cauchy stresses distribution, we calculate moment stresses, total stresses, strains and displacements. A comparative analysis of the results obtained is performed.

2. Constitutive relations of gradient mechanics

In 1968, R.D. Mindlin and N.N. Eshel, put forward a position that the strain energy density is a function of not only the strain tensor but also the first strain gradient [3]. For a linear isotropic material, the expression for the strain energy density has the form:

$$w=\frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+c_1\varepsilon_{ij,j}\varepsilon_{ik,k}+c_2\varepsilon_{ii,k}\varepsilon_{kj,j}+c_3\varepsilon_{ii,k}\varepsilon_{jj,k}+c_4\varepsilon_{ij,k}\varepsilon_{ij,k}+c_5\varepsilon_{ij,k}\varepsilon_{kj,i}\,. \tag{1}$$

Here λ and μ are the Lame parameters, c_1, $c_2,\ldots,$ c_5 are the additional gradient parameters, $\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$ is the tensor of small deformations of an elastic body. Note that the practical use of this model raises the question of identifying additional modules.

To overcome this difficulty, B.S. Altan and E.C. Aifantis [7] proposed an applied one-parameter gradient deformation model based on a simplified form of the strain energy density. Putting in (1) $c_1=c_2=c_5=0$, $c_3=\frac{1}{2}\lambda l^2$, $c_4=\mu l^2$, we get:

$$w=\frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+\mu\varepsilon_{ij}\varepsilon_{ij}+l^2\left(\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k}+\mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right). \tag{2}$$

Here l is a gradient parameter with a length dimension and associated with sizes of microstructural inhomogeneities.

The constitutive relations for the components of the Cauchy stress tensor τ_{ij}, moment stress tensor m_{ijk}, and total stress tensor σ_{ij} have the form [7]:

$$\tau_{ij}=\frac{\partial w}{\partial\varepsilon_{ij}}, \tag{3}$$

$$m_{ijk}=\frac{\partial w}{\partial\varepsilon_{ij,k}}=l^2\tau_{ij,k}\,, \tag{4}$$

$$\sigma_{ij}=\tau_{ij}-m_{ijk,k}=(1-l^2\nabla^2)\tau_{ij}\,. \tag{5}$$

The mathematical formulation of the gradient theory of elasticity in the general case is completely determined by the variational Lagrange principle. By varying the functional compiled in [7], we obtain the equilibrium equation:

$$\sigma_{ij,j}=0, \tag{6}$$

and the natural static boundary conditions on the surface S bounding the region V are as follows:

$$m_{ijk,k}n_jn_k=q_i\,,\ \ \tau_{ij}n_j-m_{ijk,k}n_j-(m_{ijk,k}n_k)_{,j}+(m_{ijk,k}n_jn_k)_{,s}n_s=t_i\,. \tag{7}$$

Here t_i, q_i are the vectors of the given forces in the body volume and on its surface, n_i are the components of the unit normal vector to the body surface at the considered point. The formulation of the problem is supplemented by the kinematic boundary conditions: $u_i=\overline{u}_i$, $u_{i,l}n_l=\frac{\partial\overline{u}_i}{\partial n}$.

In the case of the problem of unbound thermoelasticity, according to [21], we will replace ε_{ij} with $\varepsilon_{ij}-\gamma T\delta_{ij}$ in the equation (2), where γ is the temperature stress coefficient, δ_{ij} is the Kronecker symbol. In addition, the equilibrium equation (6) and the mechanical boundary conditions (7) must be supplemented by the equation of classical thermal conductivity:

$$\left(k_{ij}T_{,i}\right)_{,j}=0 \tag{8}$$

and thermal boundary conditions

$$T\,|_{S_1}=0,\ T\,|_{S_2}=T_0. \tag{9}$$

Here $S=S_1+S_2$ is the body surface.

As an example, we consider the equation of equilibrium, thermal conductivity, and the constitutive relations of gradient thermoelasticity for an inhomogeneous rod:

$$\sigma'=0, \tag{10}$$

$$(k(x)T')'=0, \tag{11}$$

$$\sigma=\tau-l^2\tau'', \tag{12}$$

$$\tau=E(x)u'-\gamma(x)T(x), \tag{13}$$

$$m=l^2\tau'. \tag{14}$$

In the formulas (10) - (14), the prime sign denotes the derivative with respect to x.

3. Statement of the gradient thermoelasticity problem for a composite rod

Consider the equilibrium of a composite thermoelastic rod with a length H at the junction at the point $x=H_0$, under the influence of a combined thermo-mechanical load. One end of the rod $x=0$ is rigidly fixed and maintained at zero temperature; at the other end $x=H$ the force p_0 acts, and the temperature T_0 is maintained. The Young modulus E, the thermal conductivity k and the thermal stress coefficient γ are piecewise continuous functions of the coordinate x. Because the equilibrium equations in gradient theory have an increased order of differential equations compared to the classical theory, then the additional boundary conditions are also required. As additional boundary conditions, we take $u'(0)=0$, $m(H)=0$. In addition, according to [14,15], the interface conditions for temperature, heat flux, displacements, strains, total stresses, and moment stresses must be satisfied at the junction. Further in the formulas, we denote the functions and parameters corresponding to the first and second rod by the indices "1" and "2", respectively. To simplify the calculations, we assume that the gradient parameter is the same for each rod, i.e. $l_1=l_2=l$.

The original aim of the study was to find the distribution of the Cauchy stresses $\tau(x)$ along the length of the composite rod. For this, we express the total stresses $\sigma(x)$, moment stresses $m(x)$, and displacement gradients $u'(x)$ through the Cauchy stresses. Then the formulation of the thermoelasticity problem in terms of the Cauchy stresses will take the form:

$$\tau_1'-l^2\tau_1'''=0,\ \tau_2'-l^2\tau_2'''=0, \tag{15}$$

$$(k_1(x)T_1')'=0, \qquad (k_2(x)T_2')'=0, \tag{16}$$

$$T_1(0)=0,\ T_2(H)=T_0, \tag{17}$$

$$\tau_1(0)=0,\ \tau_2'(H)=0,\ \tau_2(H)-l^2\tau_2''(H)=p_0, \tag{18}$$

$$T_1(H_0)=T_2(H_0),\ k_1(H_0)T_1'(H_0)=k_2(H_0)T_2'(H_0), \tag{19}$$

$$\frac{\tau_1(H_0)+\gamma_1(H_0)T_1(H_0)}{E_1(H_0)}=\frac{\tau_2(H_0)+\gamma_2(H_0)T_2(H_0)}{E_2(H_0)}, \tag{20}$$

$$\tau_1'(H_0)=\tau_2'(H_0),\ \tau_1(H_0)-l^2\tau_1''(H_0)=\tau_2(H_0)-l^2\tau_2''(H_0). \tag{21}$$

Let us write out the dimensionless problem (15)-(21) by introducing the following dimensionless parameters and functions:

$$\xi=\frac{x}{H},\ h_0=\frac{H_0}{H},\ \alpha=\frac{1}{H},\ \beta_0=\frac{\gamma_0 T_0}{E_0},\ P_0=\frac{p_0}{E_0},\ W_i=\frac{\gamma_0 T_i}{E_0},\ \Omega_i=\frac{\tau_i}{E_0},\ M_i=\frac{m_i}{E_0 H},\ S_i=\frac{\sigma_i}{E_0},$$

$$\bar{s}_i=\frac{E_i}{E_0},\ \bar{k}_i=\frac{k_i}{k_0},\ \bar{\gamma}_i=\frac{\gamma_i}{\gamma_0},\ i=1,2,\ k_0=\max_{x\in[0,H]} k(x),\ \gamma_0=\max_{x\in[0,H]}\gamma(x),\ E_0=\max_{x\in[0,H]}E(x).$$

The dimensionless boundary value problem of thermoelasticity (15) - (21) takes the form:

$$\Omega_1'-\alpha^2\Omega_1'''=0,\ \Omega_2'-\alpha^2\Omega_2'''=0, \tag{22}$$

$$\left(\bar{k}_1(\xi)W_1'\right)'=0,\ \left(\bar{k}_2(\xi)W_2'\right)'=0, \tag{23}$$

$$W_1(0)=0,\ W_2(1)=\beta_0, \tag{24}$$

$$\Omega_1(0)=0,\ \Omega_2'(1)=0,\ \Omega_2(1)-\alpha^2\Omega_2''(1)=P_0, \tag{25}$$

$$W_1(h_0)=W_2(h_0),\ \bar{k}_1(h_0)W_1'(h_0)=\bar{k}_2(h_0)W_2'(h_0), \tag{26}$$

$$\frac{\Omega_1(h_0)+\bar{\gamma}_1(h_0)W_1(h_0)}{\bar{s}_1(h_0)}=\frac{\Omega_2(h_0)+\bar{\gamma}_2(h_0)W_2(h_0)}{\bar{s}_2(h_0)}, \tag{27}$$

$$\Omega_1'(h_0)=\Omega_2'(h_0),\ \Omega_1(h_0)-\alpha^2\Omega_1''(h_0)=\Omega_2(h_0)-\alpha^2\Omega_2''(h_0). \tag{28}$$

4. Solving the thermoelasticity problem for a rod

The solution of the thermoelasticity problem (22) - (28) begins with finding the temperature distribution along the length of the composite rod based on the solution of the classical heat conduction problem (23), (24), (26).

In case when both rods are made of inhomogeneous materials, the solution to the problem of thermal conductivity (23), (24), (26) has the form:

$$W_1(\xi)=\beta_0\frac{f_1(\xi)}{f_1(h_0)+f_2(1)},\ W_2(\xi)=\beta_0\frac{f_1(h_0)+f_2(\xi)}{f_1(h_0)+f_2(1)},\ f_1(\xi)=\int_0^{\xi}\frac{d\eta}{\bar{k}_1(\eta)},\ f_2(\xi)=\int_{h_0}^{\xi}\frac{d\eta}{\bar{k}_2(\eta)}. \tag{29}$$

In case both rods are made of homogeneous materials, by setting in (29) $\bar{k}_1=const$ and $\bar{k}_2=const$, we obtain:

$$W_1(\xi)=\beta_0\frac{\bar{k}_2\xi}{\bar{k}_1+h_0(\bar{k}_2-\bar{k}_1)},\ W_2(\xi)=\beta_0\frac{\bar{k}_1\xi+h_0(\bar{k}_2-\bar{k}_1)}{\bar{k}_1+h_0(\bar{k}_2-\bar{k}_1)}. \tag{30}$$

After finding the temperature distribution, further, in order to find the Cauchy stresses, it is necessary to solve the boundary-value problem (22), (25), (27), (28). The accurate analytical solutions were obtained in the work when both rods were made of homogeneous materials. These solutions are cumbersome and therefore are not presented here: they are used to evaluate the accuracy of the approximate analytical solution.

The problem (22), (25), (27), (28) contains the differential equations (22) with a small parameter in the highest derivative and is singularly perturbed. We obtain the approximate analytical solution to the boundary value problem (22), (25), (27), (28) based on the Vishik-Lyusternik method [29,30].

According to the scheme of the Vishik-Lyusternik method, we construct the first iterative process. To do this, we present solutions for each of the equations (22) in the form of an expansion for the small parameter α in the form:

$$\Omega_1(\xi,\alpha)\cong G_1(\xi,\alpha)=g_1^{(0)}(\xi)+\alpha g_1^{(1)}(\xi)+\alpha^2 g_1^{(2)}(\xi)+\dots, \tag{31}$$

$$\Omega_2(\xi,\alpha) \cong G_2(\xi,\alpha) = g_2^{(0)}(\xi) + \alpha g_2^{(1)}(\xi) + \alpha^2 g_2^{(2)}(\xi) + \dots . \quad (32)$$

Substituting the expansions (31), (32) into (22) and performing the splitting by α powers, we obtain the sequence of boundary value problems:

$$\begin{cases} g_1^{(0)\prime}(\xi) = 0, g_2^{(0)\prime}(\xi) = 0, g_2^{(0)}(1) = 1, g_1^{(0)}(h_0) = g_2^{(0)}(h_0), \\ g_1^{(1)\prime}(\xi) = 0, g_2^{(1)\prime}(\xi) = 0, g_2^{(1)}(1) = 0, g_1^{(1)}(h_0) = g_2^{(1)}(h_0), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \quad (33)$$

Obviously, by solving each of the problems (33), it is impossible to satisfy all the boundary and conjugation conditions. It is necessary to build on additional boundary layer solutions that should quickly fade away with distance from the border.

According to the scheme of the Vishik-Lyusternik method, we construct the second iterative process. For the first rod, the boundary layers are localized in the vicinity of the attachment point $\xi = 0$ and the interface point $\xi = h_0$ with the second rod. For the second rod, the boundary layers are in the vicinity of $\xi = h_0$ and $\xi = 1$. We introduce the tensile coordinates in the vicinity of the boundaries $\eta_1 = \frac{\xi}{\alpha}$, $\eta_2 = \frac{\xi - h_0}{\alpha}$, $\eta_3 = \frac{h_0 - \xi}{\alpha}$, $\eta_4 = \frac{\xi - 1}{\alpha}$. Thus, the expressions for the Cauchy stresses of each rod can be represented as:

$$\Omega_1(\xi,\alpha) \cong G_1(\xi,\alpha) + Z_1\left(\frac{\xi}{\alpha},\alpha\right) + Z_2\left(\frac{\xi - h_0}{\alpha},\alpha\right), \quad (34)$$

$$\Omega_2(\xi,\alpha) \cong G_2(\xi,\alpha) + Z_3\left(\frac{h_0 - \xi}{\alpha},\alpha\right) + Z_4\left(\frac{\xi - 1}{\alpha},\alpha\right), \quad (35)$$

where $Z_1\left(\frac{\xi}{\alpha},\alpha\right) = z_1^{(0)}\left(\frac{\xi}{\alpha}\right) + \alpha z_1^{(1)}\left(\frac{\xi}{\alpha}\right) + \alpha^2 z_1^{(2)}\left(\frac{\xi}{\alpha}\right) + \dots,$

$$Z_2\left(\frac{\xi - h_0}{\alpha},\alpha\right) = z_2^{(0)}\left(\frac{\xi - h_0}{\alpha}\right) + \alpha z_2^{(1)}\left(\frac{\xi - h_0}{\alpha}\right) + \alpha^2 z_2^{(2)}\left(\frac{\xi - h_0}{\alpha}\right) + \dots,$$

$$Z_3\left(\frac{h_0 - \xi}{\alpha},\alpha\right) = z_3^{(0)}\left(\frac{h_0 - \xi}{\alpha}\right) + \alpha z_3^{(1)}\left(\frac{h_0 - \xi}{\alpha}\right) + \alpha^2 z_3^{(2)}\left(\frac{h_0 - \xi}{\alpha}\right) + \dots,$$

$$Z_4\left(\frac{\xi - 1}{\alpha},\alpha\right) = z_4^{(0)}\left(\frac{\xi - 1}{\alpha}\right) + \alpha z_4^{(1)}\left(\frac{\xi - 1}{\alpha}\right) + \alpha^2 z_4^{(2)}\left(\frac{\xi - 1}{\alpha}\right) + \dots .$$

In the expansions (34), (35) we restrict ourselves to only zero approximations. Then approximate solutions can be represented as:

$$\Omega_1(\xi,\alpha) \cong g_1^{(0)}(\xi) + z_1^{(0)}\left(\frac{\xi}{\alpha}\right) + z_2^{(0)}\left(\frac{\xi - h_0}{\alpha}\right), \quad (36)$$

$$\Omega_2(\xi,\alpha) \cong g_2^{(0)}(\xi) + z_3^{(0)}\left(\frac{h_0 - \xi}{\alpha}\right) + z_4^{(0)}\left(\frac{\xi - 1}{\alpha}\right). \quad (37)$$

The functions $g_1^{(0)}(\xi)$, $g_2^{(0)}(\xi)$ coincide with the solution of the problem for a composite rod obtained on the basis of the classical model of thermoelasticity and have the form:

$$g_1^{(0)}(\xi) = g_2^{(0)}(\xi) = P_0 . \quad (38)$$

To find the first boundary-layer solution $z_1^{(0)}(\eta_1)$, given that $\frac{d^2}{d\xi^2}=\frac{1}{\alpha^2}\frac{d^2}{d\eta_1^2}$, $\frac{d^4}{d\xi^4}=\frac{1}{\alpha^4}\frac{d^4}{d\eta_1^4}$, we obtain the equation

$$z_1^{(0)\prime}-z_1^{(0)\prime\prime\prime}=0, \tag{39}$$

with the solution

$$z_1^{(0)}=C_1+C_2e^{\eta_1}+C_3e^{-\eta_1}. \tag{40}$$

Since the boundary-layer solution $z_1^{(0)}(\eta_1)$ must asymptotically tend to zero for $\eta_1\to\infty$, we assume $C_1=C_2=0$ in (40). Therefore, $z_1^{(0)}=C_3e^{-\frac{\xi}{\alpha}}$.

To find the second boundary-layer solution $z_2^{(0)}(\eta_2)$, we have the equation:

$$z_2^{(0)\prime}-z_2^{(0)\prime\prime\prime}=0, \tag{41}$$

which solution has the form:

$$z_2^{(0)}=C_4+C_5e^{\eta_2}+C_6e^{-\eta_2}. \tag{42}$$

As far as the boundary-layer solution $z_2^{(0)}(\eta_2)$ also has to tend asymptotically to zero for $\eta_2\to-\infty$, we assume in (42) $C_4=C_6=0$. So, $z_2^{(0)}=C_5e^{\frac{\xi-h_0}{\alpha}}$. The expressions for the Cauchy stress in the first rod will take the form:

$$\Omega_1(\xi,\alpha)=P_0\left(1-e^{-\frac{\xi}{\alpha}}\right)+C_5e^{\frac{\xi-h_0}{\alpha}}. \tag{43}$$

To find C_3, we proceed to satisfy the boundary condition on the left end of the rod $\xi=0$: $\Omega_1(0)=g_1^{(0)}(0)+z_1^{(0)}(0)=P_0+C_3=0$. From this, we have $C_3=-P_0$. Here we take into account that when $\xi=0$, only the first boundary-layer solution manifests itself $z_1^{(0)}$ since the value h_0 is such that the influence of the second boundary layer solution $z_2^{(0)}$ can be omitted due to the small size of $e^{-\frac{h_0}{\alpha}}$.

Then the Cauchy stress in the first rod will take the form:

$$\Omega_1(\xi,\alpha)\cong P_0\left(1-e^{-\frac{\xi}{\alpha}}\right)+C_5e^{\frac{\xi-h_0}{\alpha}}. \tag{44}$$

Given the physical meaning, we similarly determine the third and fourth boundary-layer solutions in the form $z_3^{(0)}=C_7e^{\frac{h_0-\xi}{\alpha}}$, $z_4^{(0)}=C_8e^{\frac{\xi-1}{\alpha}}$. From the condition $\Omega_2'(1)=0$ follows the relationship between the constants C_7 and C_8 in the form $C_8=C_7e^{\frac{h_0-1}{\alpha}}$. The expressions for the stress in the second rod will take the form:

$$\Omega_2(\xi,\alpha)\cong P_0+C_7\left(e^{\frac{h_0-\xi}{\alpha}}+e^{\frac{h_0-1}{\alpha}}e^{\frac{\xi-1}{\alpha}}\right). \tag{45}$$

The Unknowns C_5 and C_7 are determined from the boundary conditions (27), (28), assuming $1-e^{-\frac{h_0}{\alpha}}\cong 1$ in the calculations.

Then the expressions for the dimensionless Cauchy stresses, denoting by $\delta_0 = \dfrac{1-h_0}{\alpha}$ the ratio of the length of the second rod and the gradient parameter α, can be represented as:

$$\Omega_1(\xi,\alpha) \cong P_0\left(1-e^{-\frac{\xi}{\alpha}}\right) + \mathrm{K}\left(1-e^{-2\delta_0}\right)e^{\frac{\xi-h_0}{\alpha}}, \tag{46}$$

$$\Omega_2(\xi,\alpha) \cong P_0 - \mathrm{K}\left(e^{\frac{h_0-\xi}{\alpha}} + e^{-\delta_0}e^{\frac{\xi-1}{\alpha}}\right), \tag{47}$$

where

$$\mathrm{K} = \frac{P_0\left(\overline{s}_1(h_0)-\overline{s}_2(h_0)\right) + W_1(h_0)\left(\overline{s}_1(h_0)\overline{\gamma}_2(h_0) - \overline{s}_2(h_0)\overline{\gamma}_1(h_0)\right)}{\overline{s}_1(h_0)+\overline{s}_2(h_0)+(\overline{s}_1(h_0)-\overline{s}_2(h_0))e^{-2\delta_0}}. \tag{48}$$

If we put in (46), (47) $\alpha = 0$, then we obtain the expressions for stresses corresponding to the classical thermoelasticity: $\Omega_1 = \Omega_2 = P_0$.

From the formulas (46), (47) it follows that at the point $\xi = h_0$ there is a stress jump. Neglecting the magnitude $P_0 e^{-\frac{h_0}{\alpha}}$ compared with $2K$, we get the expression for the stress jump:

$$\Delta\Omega = \Omega_1 - \Omega_2 \cong 2K. \tag{49}$$

The value of the stress jump, according to (48), (49), is determined by the mechanical stress P_0, the temperature $W_1(h_0)$, and the relation between thermoelastic characteristics and the parameter δ_0. From the formula (48) it follows that if a continuous change in the thermomechanical characteristics through the junction of the rods is ensured, then there will be no Cauchy stress jump.

If $\delta_0 \le 1$ (the relative length of the second rod is comparable to or less than the value of the gradient parameter α), then scale effects will appear, consisting in the dependence of the Cauchy stress jump on the value of the parameter δ_0. When $\delta_0 = 0$ the Cauchy stress jump is minimal; with the increase δ_0 from 0 to $\delta_0 \sim 1$ comes the exponential increase of $\Delta\Omega$. At $\delta_0 = 0$, the value $\Delta\Omega$ for a rod made of homogeneous parts, in the case of mechanical loading, is determined by the formula

$$\Delta\Omega \cong P_0\left(1-\frac{\overline{s}_2}{\overline{s}_1}\right). \tag{50}$$

If the elastic modulus of the first rod is greater than that of the second one, we have the following in the dimensionless form: $\overline{s}_1 = 1$, $\overline{s}_2 \in [0,1)$. The maximum stress jump, equal to $\Delta\Omega \cong P_0$, will be at $\overline{s}_2 = 0$. If the elastic modulus of the first rod is less than the second one, we have: $\overline{s}_2 = 1$, $\overline{s}_1 \in [0,1)$. The absolute value of the maximum stress jump $|\Delta\Omega| \to +\infty$ will be at $\overline{s}_1 \to 0$.

At $\delta_0 \gg 1$ (the relative length of the second rod is much larger than the gradient parameter α), the exponents $e^{-\delta_0}$ are very small quantities. Then in the expressions (46)-(48) one can put $1-e^{-2\delta_0} \cong 1$, $e^{-\delta_0}e^{\frac{\xi-1}{\alpha}} \cong 0$, $\overline{s}_1(h_0)+\overline{s}_2(h_0)+(\overline{s}_1(h_0)-\overline{s}_2(h_0))e^{-2\delta_0} \cong \overline{s}_1(h_0)+\overline{s}_2(h_0)$.

In this case, the value of the stress jump $\Delta\Omega$ is independent of the specific parameter value δ_0, and it is determined by the ratio of thermomechanical characteristics. The value $\Delta\Omega$

for a rod made of homogeneous parts, in the case of mechanical loading, at $\delta_0 \gg 1$, is determined by the formula

$$\Delta\Omega \cong 2P_0 \frac{\overline{s}_1 - \overline{s}_2}{\overline{s}_1 + \overline{s}_2}. \tag{51}$$

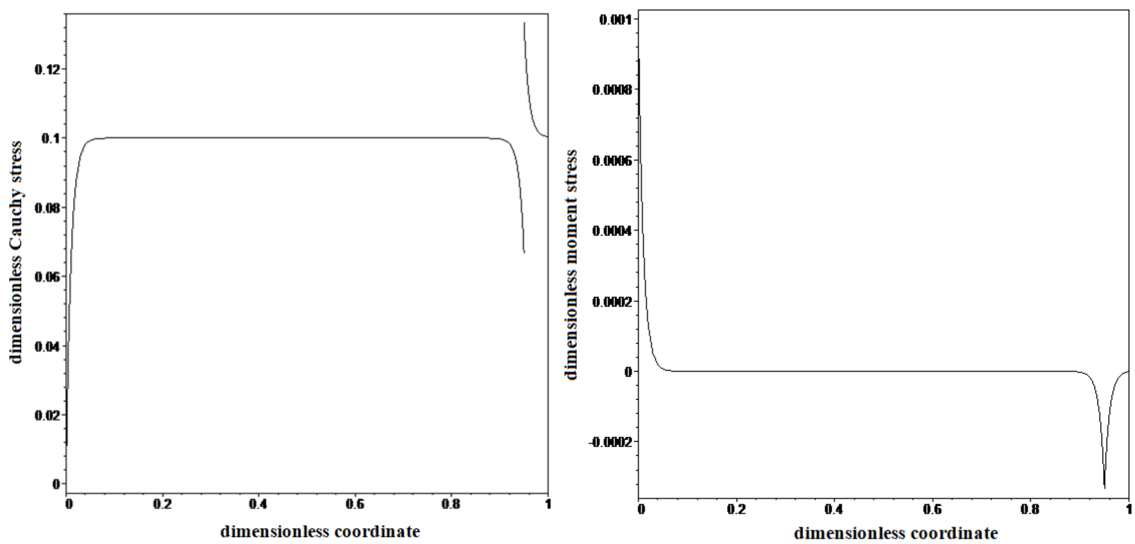
The absolute value of the maximum stress jump in this case is $|\Delta\Omega| \cong 2P_0$ and will be reached for $\overline{s}_1 = 1$, $\overline{s}_2 = 0$ or $\overline{s}_2 = 1$, $\overline{s}_1 = 0$.

After finding the laws of distribution of the dimensionless Cauchy stresses along the coordinate ξ, we further calculate the dimensionless moment stresses $M_i = \alpha^2 \Omega_i'$, and the dimensionless strains $\mathrm{E}_i = \frac{1}{\overline{s}_i}\left(\Omega_i + \overline{\gamma}_i W_i\right)$, $i = 1, 2$. The total stresses $S_i = \Omega_i - M_i'$, based on boundary conditions (25), (28), are the same and equal $S_1(\xi) = S_1(\xi) = P_0$ regardless of the material and gradient characteristics of the rods. The displacements distributions U_i, $i = 1, 2$ by the coordinate ξ are found by integrating the expressions for strains E_i, given the boundary condition $U_1(0) = 0$ and the conjugation condition $U_1(h_0) = U_2(h_0)$.

5. Computation results

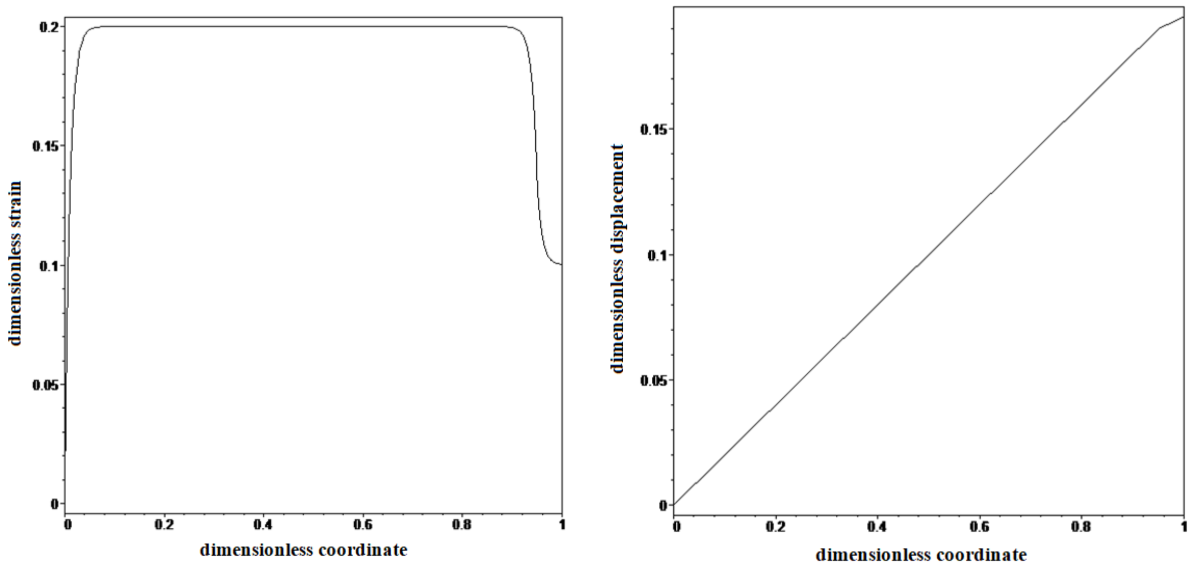
This section presents the results of calculations on finding the distribution of dimensionless Cauchy stresses, moment stresses, total stresses, strains, and coordinate displacements for both mechanical and thermal loading.

Example 1. Consider the case of mechanical loading of a composite rod ($\beta = 0$, $P_0 = 0.1$), the parts of which are made of homogeneous materials with the following characteristics: $\delta_0 = 5$, $\overline{s}_1 = 0.5$, $\overline{s}_2 = 1$. The influence of the gradient parameter α magnitude on the accuracy of the calculation of the dimensionless Cauchy stresses by the asymptotic formulas (46), (47) is studied. During the calculations, it was found that the error in the approximate calculation of the dimensionless Cauchy stresses does not exceed 1% at $\alpha \le 0.02$.



a) b)

Fig. 1. Distribution graphs along the coordinate ξ: a) dimensionless Cauchy stresses; b) dimensionless moment stresses under mechanical loading



a) b)

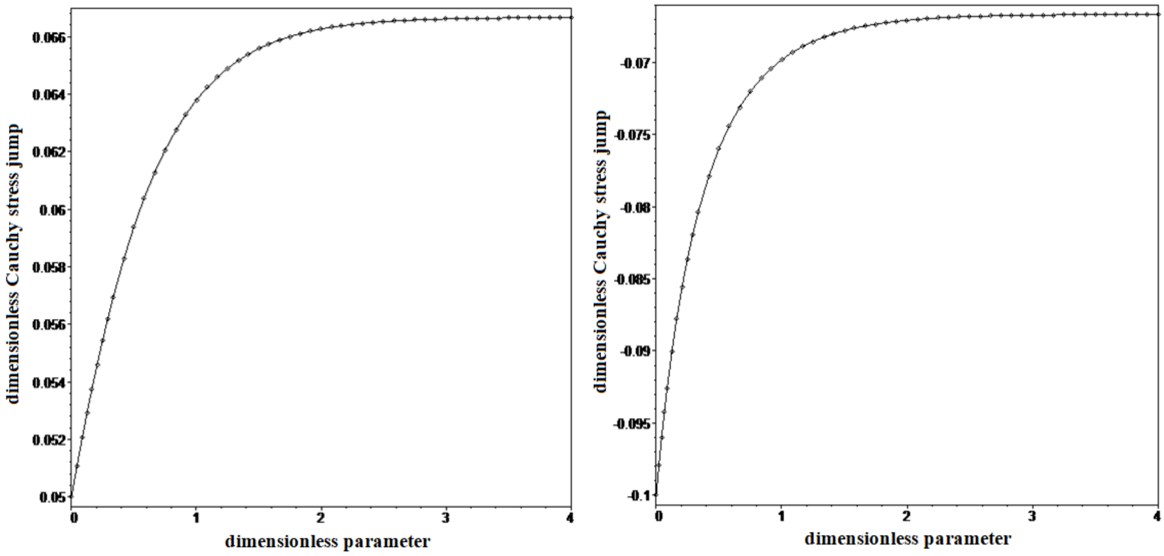
Fig. 2. Distribution graphs along the coordinate ξ : a) dimensionless strains; b) dimensionless displacements during mechanical loading

Figures 1 and 2 show images of the distribution of the dimensionless Cauchy stresses (Fig. 1a), the moment stresses (Fig. 1b), the strains (Fig. 2a), and the displacements (Fig. 2b) at $\alpha = 0.01$. The value h_0 is determined from the expression $h_0 = 1 - \delta_0\alpha$.

From Figure 1a it follows that the Cauchy stresses: 1) near the end face $\xi = 0$ decay exponentially to zero in accordance with the boundary condition $\Omega_1(0) = 0$; 2) experience a jump at the point $\xi = h_0$, which values, according to (51), is determined by the ratio of the elastic modulus of the rods. From Figure 1b it follows that the moment stresses equal to zero, except for the vicinity of the fixing and conjugation points, and reach a peak at the point of contact of the rods.

Figure 2a depicts the strains and displacements.

In the case of mechanical loading, we study the dependence of the jump of the Cauchy stresses $\Delta\Omega$ at the point $\xi = h_0$, calculated by the formula (49), on the value of the parameter δ_0 at $P_0 = 0.1$, $\alpha = 0.01$ and various ratios of the elastic modulus. Figure 3 presents the results of calculations of the dependence of the stress jump on the parameter δ_0 for: 1) $\overline{s}_1 = 1$, $\overline{s}_2 = 0.5$ (Fig. 3a); 2) $\overline{s}_1 = 0.5$, $\overline{s}_2 = 1$ (Fig. 3b). In this case, the solid line shows the dependence $\Delta\Omega(\delta_0)$, obtained in the course of the exact analytical solution, and the dots – on the basis of the formula (49).

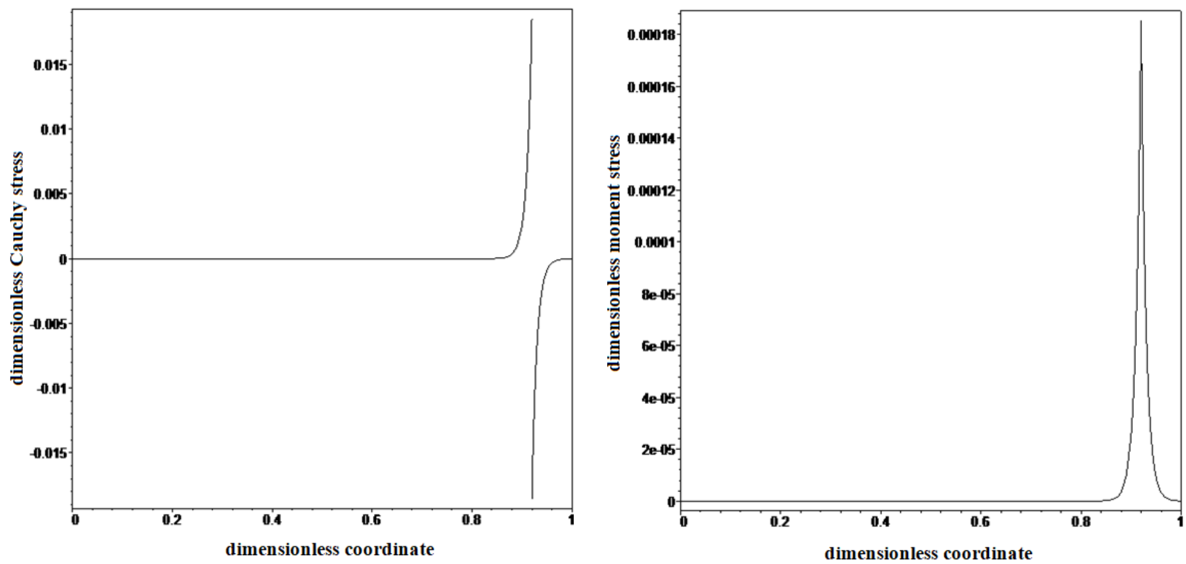


a) b)

Fig. 3. Graph of the dependence of the jump in the dimensionless Cauchy stresses on the parameter δ_0 under mechanical loading

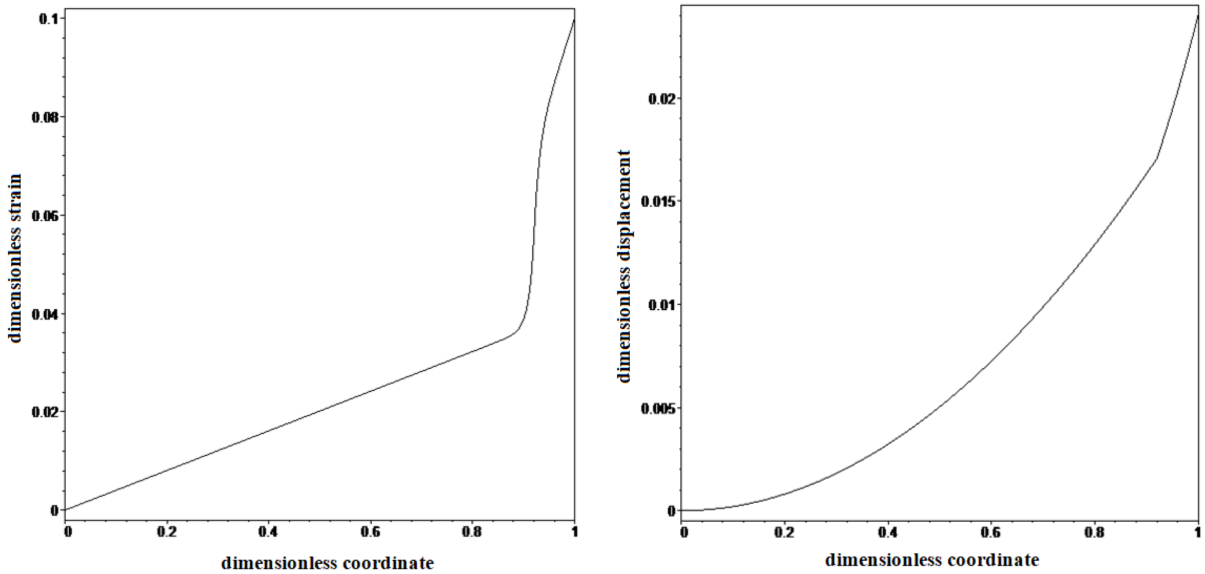
From Figure 3 it follows that the minimum Cauchy stress jump occurs when $\delta_0 \ll 1$, i.e. when the length of the second rod is much less than the gradient parameter α. As you increase δ_0, the stress jump increases exponentially. Starting from $\delta_0 > 3$, i.e. when the length of the second rod becomes 3 times greater than the gradient parameter α, $\Delta\Omega$ almost reaches a stationary value.

Example 2. Consider the case of thermal loading of a composite rod ($\beta_0 = 0.1$, $P_0 = 0$), the parts of which are made of homogeneous materials with the following characteristics: $\alpha = 0.01$, $\delta_0 = 8$, $\bar{s}_1 = 1$, $\bar{s}_2 = 1$, $\bar{k}_1 = 1$, $\bar{k}_2 = 0.25$, $\bar{\gamma}_1 = 0.5$, $\bar{\gamma}_2 = 1$.To find the temperature at the point h_0, we use the first formula (30).



a) b)

Fig. 4. Distribution graphs along the coordinate ξ: a) dimensionless Cauchy stresses; b) dimensionless moment stresses under thermal loading



a) b)

Fig. 5. Distribution graphs along the coordinate ξ : a) dimensionless strains; b) dimensionless displacements during mechanical loading

Figures 4-5 show images of the distribution of the dimensionless functions: the Cauchy stresses (Fig. 4a), the moment stresses (Fig. 4b), the strains(Fig. 5a), and the displacements (Fig. 5b). The error in calculating the distribution of the dimensionless functions was less than 1%.

From Figure 4a, it follows that the Cauchy stresses are equal to zero, with the exception of the vicinity of the junction of the rods, where a stress jump occurs, due to the difference in the coefficients of the thermal stresses of the rods. From Figure 4b it follows that the moment stresses are equal to zero, with the exception of the vicinity of the junction of the rods and reach a peak at the junction point of the rods.

The magnitude of the Cauchy stress jump during the thermal way of loading the rod made of homogeneous parts is:

$$\Delta\Omega \cong 2\frac{W_1(h_0)\left(\overline{s}_1\overline{\gamma}_2 - \overline{s}_2\overline{\gamma}_1\right)}{\overline{s}_1 + \overline{s}_2 + (\overline{s}_1 - \overline{s}_2)e^{-2\delta_0}} . \tag{52}$$

For the case of thermal loading, we study the dependence of the Cauchy stress jump $\Delta\Omega$ at a point $\xi = h_0$ on the parameter δ_0 at $\beta_0 = 0.1$, $\overline{k}_1 = 1$, $\overline{k}_2 = 0.25$, $\overline{s}_1 = 0.5$, $\overline{s}_2 = 1$, $\overline{\gamma}_1 = 1$, $\overline{\gamma}_2 = 0.8$, $\alpha = 0.01$. The value h_0, necessary to find the temperature, is determined from the expression $h_0 = 1 - \delta_0\alpha$. In Figure 6, the solid line shows the dependence obtained in the course of the exact analytical solution, and the dots – on the basis of approximate formulas.

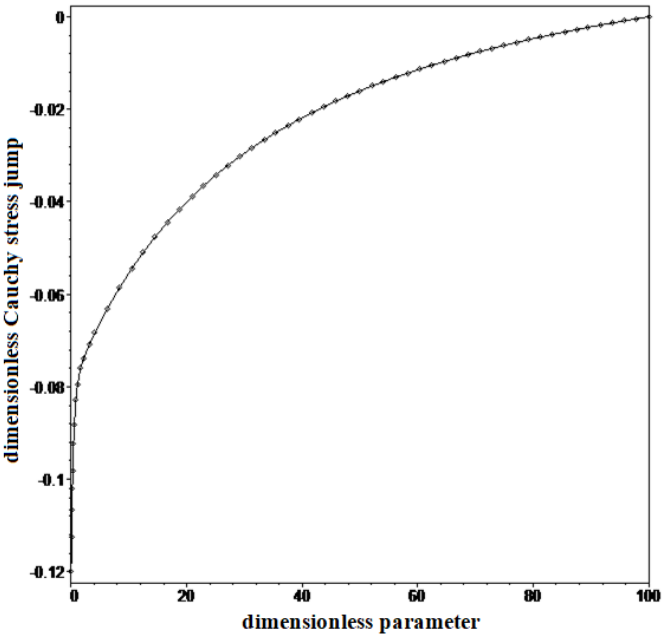


Fig. 6. Graph of the dependence of the jump in the dimensionless Cauchy stresses on the parameter δ_0 under thermal loading

From Figure 6 it follows that in the case of thermal loading, the maximum rate of change of the function $\Delta\Omega(\delta_0)$ is observed at $\delta_0 < 1$, i.e. when the length of the second rod is less than the gradient parameter α. For $\delta_0 > 1$, with an increase of δ_0, a smooth change in function $\Delta\Omega(\delta_0)$ is observed.

6. Conclusion

A statement of the gradient thermoelasticity problem for a composite rod based on the one-parameter Aifantis model is given. After finding the temperature distribution from the solution of the classical heat conduction problem, simplified analytical expressions for finding the Cauchy stresses are obtained on the basis of the asymptotic Vishik-Lyusternik approach. The cases of thermal and mechanical loading of the rod are considered. A new scale parameter is introduced, equal to the ratio of the length of the second rod and the gradient parameter. After finding the distribution of the Cauchy stresses, moment stresses, total stresses, displacements, and deformations are calculated. It was revealed that, within the framework of the gradient theory, the deformations are continuous at the point of contact of the rods. This fact explains the jump in the Cauchy stresses in the vicinity of the point of the rod conjugation. The magnitude of the Cauchy stress jump depends on both the ratio of thermomechanical characteristics and the value of the scale parameter. The dependence of the Cauchy stress jump on the scale parameter is investigated. It was found out that the stress jump function changes most rapidly at values of the scale parameter less than the length of the second rod. The moment stresses are continuous, equal to zero, except for the vicinity of the fixing and conjugation points, and reach a peak at the rods' conjugation point. The total stresses, which are a combination of the Cauchy stresses and the first gradient of the moment stresses, are continuous in each rod and equal to the value of the mechanical load at the rod's end.

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References

[1] Toupin RA. Elastic materials with couple stresses. *Arch. Rational Mech. Anal*. 1962;11(1): 385-414.
[2] Mindlin RD. Micro-structure in linear elasticity. *Arch. Rational Mech. Anal*. 1964;16: 51-78.
[3] Mindlin RD, Eshel NN. On first strain-gradient theories in linear elasticity. *International Journal of Solids and Structures*. 1968;4(1): 109-124.
[4] Aifantis EC. On the role of gradients in the localization of deformation and fracture. *Int. J. Eng. Sei*. 1992;30(10): 1279-1299.
[5] Aifantis EC. Gradient effects at the macro, micro and nano scales. *J. Mech. Behav. Mater*. 1994;5(3): 335-353.
[6] Ru CQ, Aifantis EC. A simple approach to solve boundary value problems in gradient elasticity. *Acta Mech*. 1993;101: 59-68.
[7] Altan BS, Aifantis EC. On some aspects in the special theory of gradient elasticity. *J. Mech. Behav. Mater*. 1997;8(3): 231-282.
[8] Aifantis KE, Willis JR. The role of interfaces in enhancing the yield strength of composites and polycrystals. *J. Mech. Phys. Solids*. 2005;53(5): 1047-1070.
[9] Aifantis K, Askes H. Gradient elasticity with interfaces as surfaces of discontinuity for the strain gradient. *J. Mech. Behav. Mater*. 2007;18(4): 283-306.
[10] Tsepoura KG, Papargyri-Beskou S, Polyzos D, Beskos DE. Static and dynamic analysis of a gradient-elastic bar in tension. *Archive of Applied Mechanics*. 2002;72: 483-497.
[11] Duan HL, Wang J, Huang ZP, Karihaloo BL. Size-dependent effective elastic constants of solids containing nano-inhomogeneities with interface stress. *J. Mech. Phys. Solids*. 2005;53(7): 1574-1596.
[12] Ahmadi G, Firoozbakhsh K. First strain gradient theory of thermoelasticity. *International Journal of Solids and Structures*. 1975;11(3): 339-345.
[13] Lurie SA, Solyaev YO, Rabinsky LN, Kondratova UN, Volov MI. Simulation of the stress-strain state of thin composite coating based on solutions of the plane problem of strain-gradient elasticity for layer. *PNRPU Mechanics Bulletin*. 2013;1: 161-181. (In Russian)
[14] Lurie SA, Pham T, Soliaev JO. Gradient model of thermoelasticity and its application for the modeling of thin layered structures. *Journal of composite mechanics and design*. 2012; 18(3): 440-449. (In Russian)
[15] Lurie SA, Solyaev YO, Nguyen DK, Medvedsky AL, Rabin LN. Investigation of local effects in the distribution of temperature stresses on the contact boundaries of layered media. *Electronic journal "Transactions of Moscow Aviation Institute"*. 2013;71: 1-22. (In Russian)
[16] Lurie SA, Kuznetsova EL, Rabinskii LN, Popova EI. Refined gradient theory of scale-dependent superthin rods. *Mechanics of Solids*. 2015;50(2): 135-146. (In Russian)
[17] Lurie S, Solyaev Y. Revisiting bending theories of elastic gradient beams. *International Journal of Engineering Science*. 2018;126: 1-21.
[18] Lurie SA, Belov PA, Volkov-Bogorodskii DB. Variational models of coupled gradient thermoelasticity and thermal conductivity. *Materials Physics and Mechanics*. 2019;42(5): 564-581.
[19] Lurie S, Solyaev Y. On the formulation of elastic and electroelastic gradient beam theories. *Continuum Mech. Thermodyn*. 2019;31: 1601-1613.
[20] Lomakin EV, Lurie SA, Rabinskiy LN, Solyaev YO. Refined stress analysis in applied elasticity problems accounting for gradient effects. *Doklady Physics*. 2019; 64(12): 482-486. (In Russian).
[21] Sadeghi H, Baghani M, Naghdabadi R. Strain gradient thermoelasticity of functionally graded cylinders. *Scientia Iranica B*. 2014;21(4): 1415-1423.

[22] Alnefaie K. A note on gradient thermoelasticity. *J. Mech behave Mater*. 2014;23(3-4): 107-108.
[23] Lam DCC, Yang F, Chong ACM, Wang J, Tong P. Experiments and theory in strain gradient elasticity. *J Mech Phys Solids*. 2003;51(8): 1477-508.
[24] Zhang NH, Meng WL, Aifantis EC. Elastic bending analysis of bilayered beams containing a gradient layer by an alternative two-variable method. *Compos. Struct.* 2011;93(12): 3130-3139.
[25] Li A, Zhou S, Wang B. A size-dependent bilayered microbeam model based on strain gradient elasticity theory. *Compos. Struct.* 2014;108: 259-266.
[26] Li A, Zhou S, Wang B. A size-dependent model for bi-layered Kirchhoff micro-plate. *Compos. Struct.* 2014;113: 272-280.
[27] Guangyang F, Shenjuie Z, Lu Q. The size-dependent static bending of a partially covered laminated microbeam. *Int. J. Mech. Sci.* 2019;152: 411-419.
[28] Vatulyan A, Nesterov S, Nedin R. Regarding some thermoelastic models of «coating-substrate» system deformation. *Continuum Mech Thermodyn.* 2020;32(4): 1173-1186.
[29] Vishik MI, Lyusternik LA. Regular degeneracy and boundary layer for linear differential equations with a small parameter. *Advances in Mathematical Sciences*. 1957;12(5): 3-122. (In Russian)
[30] Bauer SM, Smirnov AL, Tovstik PE, Filippov SB. *Asymptotic methods in solid mechanics*. Moscow: Institute for Computer Research; 2007. (In Russian)