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Moment-membrane theory of elastic cylindrical shells as a continual model of deformation of a single-layer carbon nanotube

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ABSTRACT

One of the key tasks in the study of two-dimensional nanomaterials (fullerenes, nanotubes, graphene) is to explore their mechanical properties: bending, vibrations, and stability. For the study of such problems, it is essential to construct both microscopic and macroscopic models of the deformation behavior of such materials. Based on the three-dimensional moment theory of elasticity, the moment-membrane theory of elastic cylindrical shells is constructed as a continual model of deformations of a single-layer carbon nanotube. The axisymmetric deformation of an elastic cylindrical shell is studied, and numerical results are presented. Further, a moment-membrane technical theory of elastic cylindrical shells is constructed, on the basis of which the stability of the initially axial compressed state of a carbon nanotube is studied, and the critical load value is numerically determined.

KEYWORDS

carbon nanotube • continual model • moment-membrane theory of cylindrical shells axisymmetric statics problem • stability problem • critical load value

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Introduction

The emergence of two-dimensional nanomaterials (fullerene [1], carbon nanotube [2], graphene [3]) has resulted in a need to construct continual models which enable the study of their deformations.

In [4–7], the consideration of the moment interaction between atoms in the crystal lattices of two-dimensional nanomaterials (and other nano-objects) has been justified. Also, the three-dimensional moment theory of elasticity (with independent fields of displacements and rotations) has been established as a continual model for deformations of the considered nanomaterials.

Studies [4–7] have naturally led to a topical problem concerning the construction of a model of thin plates or shells that can adequately describe the deformations of two-dimensional nanomaterials, based on the three-dimensional moment theory of elasticity.

We note that the general model of constructing the models of thin shells and plates based on the moment theory of elasticity are developed in [8,9], where a detailed review of the works in the field are presented. The general remark to these models is that the six elastic constants of the material based on the moment theory of elasticity have not been defined in them.

In this case we note that a new approach has been taken in [10] for the construction of an adequate model of two-dimensional nanomaterials based on the moment theory of elasticity.

In [10], the corresponding one-dimensional continual-beam model has been constructed, based on the study of the general deformation of a linear atomic chain, when the force interaction between its atoms is non-central and a moment interaction is in place. Considering the crystal graphene lattice, in the cell of its periodicity, by replacing interatomic interactions with the indicated beam model, a discrete-continual model of a graphene is constructed. Further, by passing the limit, a continual linear model of its deformation is constructed. In the same paper, it is stated that the constructed model of a graphene is completely identical to the so-called moment-membrane theory of elastic thin plates [11,12]. By comparing the two models, all elastic constants of the indicated theory of plates are numerically determined (through the physical parameters of the discrete model of a graphene). It is evident that the constructed moment-membrane linear theory of elastic thin plates (a) - plane stress state, (b) - transverse bending) with certain elastic constants is a continual model of graphene deformations ((a) - for its deformation in its own plane, (b) - for its transverse bending from its own plane). This opens a new avenue for studying various applied deformation problems of statics, dynamics and stability of a graphene sheet.

It is important to note that the main feature of the moment-membrane theory of elastic plates is the property of a uniform distribution of displacements, free rotations, stresses and moment stresses along the thickness of the plate. The property is characteristic for a thin membrane (it is no coincidence that in many publications (for example [13]), the term ‘graphene membrane’ is used for a ‘graphene’).

Considering that a single-layer carbon nanotube is an extended structure in the form of a hollow cylinder (consisting of one graphite layer or a graphene rolled into a tube [14], with a hexagonal organization of carbon atoms), it is appropriate to develop the ideas of works [10–12] and construct a moment-membrane theory of elastic cylindrical shells as a continual model of deformation of a single-layer carbon nanotube.

Main hypotheses. Kinematic model of deformations of moment-membrane theory of cylindrical shells. Equations of strain compatibility

As a basis, we will consider equations and boundary conditions of the three-dimensional moment linear theory of elasticity with independent fields of displacements and rotations [15] in the region of a cylindrical shell with thickness $2h$. A cylindrical coordinate system x, θ, z is considered, where x, θ represent the lines of the main curvatures of the middle surface of the shell ($z = 0$), and the rectilinear axis $z$is directed along the normal to this surface. In this case, for Lame coefficients of the middle surface, we take: $A_1 = 1, A_2 = R$.

To study the deformations of a single-layer carbon nanotube, below, based on the three-dimensional moment theory of elasticity, as well as based on certain hypotheses, we will consider the basic equations of the moment-membrane theory of elastic cylindrical shells.

The hypotheses adopted below can be considered as kinematic and static:
1. The essence of the kinematic hypothesis is the assumption about the constancy of all components of the displacement and free rotation vectors along the coordinate z (i.e., their uniform distribution over the thickness of the shell):

$$V_i = u_i(x,\theta),\ \ V_3 = w(x,\theta),\ \ \omega_i = \Omega_i(x,\theta),\ \ \omega_3 = \Omega_3(x,\theta),\ \ \ i = 1{,}2. \tag{1}$$

2. In the physical relations of the moment theory of elasticity [15], stresses $\sigma_{33}, \sigma_{3i}$and moment stresses μ_{33}, μ_{3i} can be neglected with respect to $\sigma_{ii}, \sigma_{i3}, \mu_{ii}, \mu_{i3} (i = 1{,}2)$.
3. It is accepted that the shell is thin $\left(\frac{2h}{R} << 1\right)$.

We note that the formulated hypotheses correspond to the initial approximation of the result of the asymptotic method of integration of the three-dimensional boundary value problem of the moment elasticity theory in a thin shell region [16,17].

Based on hypotheses (1) and (3), from the corresponding geometric relations of the moment theory of elasticity [15] (in the case of cylindrical coordinates) for the components of the tensors of deformations and bending torsions, we have:

$$\begin{gathered}
\gamma_{11}(x,\theta,z) = \Gamma_{11}(x,\theta),\ \ \gamma_{22}(x,\phi,z) = \Gamma_{22}(x,\phi),\ \ \gamma_{12}(x,\phi,z) = \Gamma_{12}(x,\phi),\\
\gamma_{21}(x,\phi,z) = \Gamma_{21}(x,\phi),\ \ \gamma_{13}(x,\phi,z) = \Gamma_{13}(x,\phi),\ \ \gamma_{23}(x,\phi,z) = \Gamma_{23}(x,\phi),\\
\chi_{11}(x,\phi,z) = k_{11}(x,\phi),\ \ \chi_{22}(x,\phi,z) = k_{22}(x,\phi),\ \ \chi_{12}(x,\phi,z) = k_{12}(x,\phi),\\
\chi_{21}(x,\phi,z) = k_{21}(x,\phi),\ \ \chi_{13}(x,\phi,z) = k_{13}(x,\phi),\ \ \chi_{23}(x,\phi,z) = k_{23}(x,\phi),\\
\chi_{31} = \chi_{32} = \chi_{33} = 0,\ \ \ \gamma_{31} = \gamma_{32} = \gamma_{33} = 0,
\end{gathered} \tag{2}$$

where

$$\begin{gathered}
\Gamma_{11} = \frac{\partial u_1}{\partial x},\ \ \Gamma_{22} = \frac{1}{R}\frac{\partial u_2}{\partial \theta} + \frac{w}{R},\ \ \Gamma_{12} = \frac{\partial u_2}{\partial x} - \Omega_3,\ \ \Gamma_{21} = \frac{1}{R}\frac{\partial u_1}{\partial \theta} + \Omega_3,\\
\Gamma_{13} = \frac{\partial w}{\partial x} + \Omega_2,\ \ \Gamma_{23} = \frac{1}{R}\frac{\partial w}{\partial \theta} - \frac{u_2}{R} - \Omega_1,\\
k_{11} = \frac{\partial \Omega_1}{\partial x},\ \ k_{22} = \frac{1}{R}\frac{\partial \Omega_2}{\partial \theta} + \frac{\Omega_3}{R},\ \ k_{12} = \frac{\partial \Omega_2}{\partial x},\ \ k_{21} = \frac{1}{R}\frac{\partial \Omega_1}{\partial \theta},\\
k_{13} = \frac{\partial \Omega_3}{\partial x},\ \ k_{23} = \frac{1}{R}\frac{\partial \Omega_3}{\partial \theta} - \frac{\Omega_2}{R}.
\end{gathered} \tag{3}$$

Here $\Gamma_{11}, \Gamma_{22}, \Gamma_{12}, \Gamma_{21}, \Gamma_{13}, \Gamma_{23}$-are components of deformation tensor, $k_{11}, k_{22}, k_{12}, k_{21}, k_{13}, k_{23}$-components of the bending-torsion tensor in the moment-membrane theory of thin cylindrical shells.

Based on Eqs. (1)-(3), it is easy to verify that the components of the displacement and free rotation vectors, the components of the deformation and bending-torsion tensors are constant by $z$coordinate functions (i.e. they are all uniformly distributed over the thickness of the cylindrical shell).

It should be noted that Eqs. (1)-(3) determine the kinematic model of deformations in the moment-membrane theory of thin cylindrical shells.

Twelve quantities $\Gamma_{11}, \Gamma_{22}, \Gamma_{12}, \Gamma_{21}, \Gamma_{13}, \Gamma_{23}, k_{11}, k_{22}, k_{12}, k_{21}, k_{13}, k_{23}$are expressed through six functions $u_1, u_2, w, \Omega_1, \Omega_2, \Omega_3$ by Eq. (3). These twelve quantities are interconnected by six equations, which are called the equations of strain compatibility of the middle surface in the moment-membrane theory of thin cylindrical shells (which can be obtained from Eq. (3), excluding the quantities $u_1, u_2, w, \Omega_1, \Omega_2, \Omega_3$):

$$\begin{gathered}
\frac{1}{R}\frac{\partial \Gamma_{11}}{\partial \theta} - \frac{\partial \Gamma_{21}}{\partial x} + k_{13} = 0,\ \ \frac{\partial \Gamma_{22}}{\partial x} - \frac{1}{R}\frac{\partial \Gamma_{12}}{\partial \theta} - \frac{\Gamma_{13}}{R} - k_{23} = 0,\\
\frac{\partial \Gamma_{23}}{\partial x} - \frac{1}{R}\frac{\partial \Gamma_{13}}{\partial \theta} + \frac{\Gamma_{12}}{R} + k_{11} + k_{22} = 0,
\end{gathered} \tag{4}$$

$$\begin{gathered}
\frac{1}{R}\frac{\partial k_{11}}{\partial \theta} - \frac{\partial k_{21}}{\partial x} = 0,\ \ \frac{\partial k_{22}}{\partial x} - \frac{1}{R}\frac{\partial k_{12}}{\partial \theta} - \frac{k_{13}}{R} = 0,\\
\frac{\partial k_{23}}{\partial x} - \frac{1}{R}\frac{\partial k_{13}}{\partial \theta} + \frac{k_{12}}{R} = 0.
\end{gathered} \tag{5}$$

Unit forces and moments of the moment-membrane theory of elastic cylindrical shells. Elasticity relations. Equilibrium equations

We consider an element, with the lengths of the arcs of the shell middle surface, extracted from the shell by normal sections drawn in the direction of the curvature lines. In these sections, there are tangential normal $\sigma_{11},\ \sigma_{22}$, tangential tangents $\sigma_{12},\ \sigma_{21}$ and transverse tangents stresses σ_{13},σ_{23}, as well as corresponding moment stresses $\mu_{11},\ \mu_{22},\ \mu_{12},\ \mu_{21},\ \mu_{13},\ \mu_{23}$, which will be distributed uniformly over the thickness of the shell. This is easy to verify if we consider the physical relations of the moment theory of elasticity [15], hypothesis (2) and geometric relations (2)-(3):

$$\begin{gathered}\sigma_{11}=\frac{E}{1-\nu^2}(\Gamma_{11}+\nu\Gamma_{22}),\ \ \sigma_{22}=\frac{E}{1-\nu^2}(\Gamma_{22}+\nu\Gamma_{11}),\ \ \sigma_{12}=(\mu+\alpha)(\Gamma_{12}+\eta_1\cdot\Gamma_{22}),\\ \sigma_{21}=(\mu+\alpha)(\Gamma_{21}+\eta_1\cdot\Gamma_{12}),\ \ \sigma_{13}=\tilde{G}\cdot\Gamma_{12},\ \ \sigma_{23}=\tilde{G}\cdot\Gamma_{23},\ \ \eta_1=\frac{\mu-\alpha}{\mu+\alpha},\\ \mu_{11}=(\gamma+\varepsilon)[(1+2\eta_2)k_{11}+\eta_2\cdot k_{22}],\\ \mu_{22}=(\gamma+\varepsilon)[(1+2\eta_2)k_{22}+\eta_2\cdot k_{11}],\ \ \eta_2=\frac{\gamma-\varepsilon}{\gamma+\varepsilon},\\ \mu_{12}=(\gamma+\varepsilon)(k_{12}+\eta_2\cdot k_{21}),\\ \mu_{21}=(\gamma+\varepsilon)(k_{21}+\eta_2\cdot k_{12}),\\ \mu_{13}=B\cdot k_{13},\ \ \mu_{23}=B\cdot k_{23},\end{gathered}\tag{6}$$

where [10]:

$$E,\ \nu,\ \mu=\frac{E}{2(1+\nu)},\ \alpha,\ \tilde{G}=\frac{4\mu\alpha}{\mu+\alpha},\ \gamma,\ \varepsilon,\ \beta=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}2\gamma,\ B=\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\tag{7}$$

are the elastic constants of the moment theory of elasticity for two-dimensional nanomaterials.

Now, instead we introduce integral characteristics - forces and moments - statically equivalent to the components of the stress tensor and the moment stress tensor (6). Considering the thinness of the shell, we have:

$$\begin{gathered}T_{11}=\int_{-h}^{h}\sigma_{11}dz=2\,\sigma_{11}h,\ \ T_{22}=\int_{-h}^{h}\sigma_{22}dz=2\,\sigma_{22}h,\ \ S_{12}=\int_{-h}^{h}\sigma_{12}dz=2\,\sigma_{12}h,\\ S_{21}=\int_{-h}^{h}\sigma_{21}dz=2\,\sigma_{21}h, N_{13}=\int_{-h}^{h}\sigma_{13}dz=2\,\sigma_{13}h,\ \ N_{23}=\int_{-h}^{h}\sigma_{23}dz=2\,\sigma_{23}h,\\ L_{11}=\int_{-h}^{h}\mu_{11}dz=2\,\mu_{11}h,\ \ L_{22}=\int_{-h}^{h}\mu_{22}dz=2\,\mu_{22}h,\ \ L_{12}=\int_{-h}^{h}\mu_{12}dz=2\,\mu_{12}h,\\ L_{21}=\int_{-h}^{h}\mu_{21}dz=2\,\mu_{21}h,\ L_{13}=\int_{-h}^{h}\mu_{13}dz=2\,\mu_{13}h,\ \ L_{23}=\int_{-h}^{h}\mu_{23}dz=2\,\mu_{23}h.\end{gathered}\tag{8}$$

With the help of Eq. (6), as well as Eq. (8), we obtain the physical relations of elasticity of the moment-membrane theory of cylindrical shells:

$$\begin{gathered}T_{11}=\tilde{E}_*(\Gamma_{11}+\nu\cdot\Gamma_{22}),\ \ T_{22}=\tilde{E}_*(\Gamma_{22}+\nu\cdot\Gamma_{11}),\\ S_{12}=D_2(\Gamma_{12}+\eta_1\cdot\Gamma_{21}),\ \ S_{21}=D_2(\Gamma_{21}+\eta_1\cdot\Gamma_{12}),\ N_{13}=D_1\cdot\Gamma_{13},\\ N_{23}=D_1\cdot\Gamma_{23}, L_{11}=D'[(1+2\eta_2)k_{22}+\eta_2\cdot k_{11}],\ \ L_{22}=D'[(1+2\eta_2)k_{22}+\eta_2\cdot k_{11}],\\ L_{12}=D'(k_{12}+\eta_2\cdot k_{21}),\ \ L_{21}=D'(k_{21}+\eta_2\cdot k_{12}), L_{13}=B_*\cdot k_{13},\ \ L_{23}=B_*\cdot k_{23},\end{gathered}\tag{9}$$

where

$$E_*=2Eh,\ \ \tilde{E}_*=\frac{E_*}{1-\nu^2},\ \ D_2=(\mu+\alpha)\cdot 2h,\ \ D_1=2\tilde{G}h,\ D'=(\gamma+\varepsilon)\cdot 2h,\ \ B_*=2Bh.\tag{10}$$

It should be noted that expressions (10) represent the stiffness characteristics of the moment-membrane theory of elastic cylindrical shells, the numerical values of which (in the current form) are determined in [10].

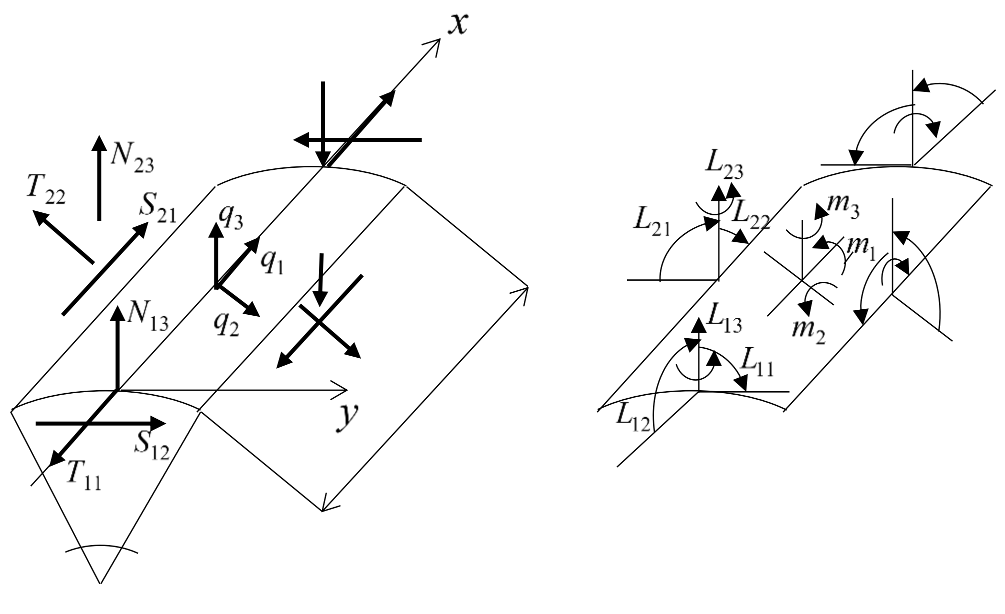

Fig. 1. Forces and moments acting on the shell element

We now consider the equilibrium of an element of the middle surface of a cylindrical shell (Fig. 1), loaded with internal forces, moments and an externally distributed load. Based on the condition that the main vector and the main moment of the indicated forces, acting on the element, are equal to zero and after switching to scalar equations, the six equilibrium equations of the moment-membrane theory of elastic thin cylindrical shells are obtained:

$$\begin{gathered}\frac{\partial T_{11}}{\partial x}+\frac{1}{R}\frac{\partial S_{21}}{\partial \theta}=-q_1,\quad \frac{1}{R}\frac{\partial T_{22}}{\partial \theta}+\frac{\partial S_{12}}{\partial x}+\frac{N_{23}}{R}=-q_2,\\ \frac{T_{22}}{R}-\frac{\partial N_{13}}{\partial x}-\frac{1}{R}\frac{\partial N_{23}}{\partial \theta}=q_3,\\ \frac{\partial L_{11}}{\partial x}+\frac{1}{R}\frac{\partial L_{21}}{\partial \theta}+N_{23}=-m_1,\quad \frac{1}{R}\frac{\partial L_{22}}{\partial \theta}+\frac{\partial L_{12}}{\partial x}+\frac{L_{23}}{R}-N_{13}=-m_2,\\ \frac{L_{22}}{R}-\frac{\partial L_{13}}{\partial x}-\frac{1}{R}\frac{\partial L_{23}}{\partial \theta}-(S_{12}-S_{21})=m_3,\end{gathered}\tag{11}$$

where $q_k(x,\phi)$, $m_k(x,\phi)$, $k=1,2,3$ are surface density components of externally distributed forces and moments.

Geometric relations (3), elasticity relations (9) and equilibrium equations (11) are the basic equations of the moment-membrane theory of elastic thin cylindrical shells, at the same time, the basic equations of the continual model of deformations of a single-layer carbon nanotube.

By substituting geometric relations (3) into elasticity relations (9) and into the obtained equilibrium equations (11) (at $m_k=0,\ k=1,2,3$), we reduce the problem to a resolving system of six differential equations with respect to generalized displacements $u_i, w, \Omega_i, \Omega_3, i=1,2,$, the order of which is twelve:

$$\begin{gathered}\tilde{L}_{11}u_1+\tilde{L}_{12}u_2+\tilde{L}_{13}w+\tilde{L}_{14}\Omega_1+\tilde{L}_{15}\Omega_2+\tilde{L}_{16}\Omega_3=-\frac{q_1}{\tilde{E}_*},\\ \tilde{L}_{21}u_1+\tilde{L}_{22}u_2+\tilde{L}_{23}w+\tilde{L}_{24}\Omega_1+\tilde{L}_{25}\Omega_2+\tilde{L}_{26}\Omega_3=-\frac{q_2}{\tilde{E}_*},\\ \tilde{L}_{31}u_1+\tilde{L}_{32}u_2+\tilde{L}_{33}w+\tilde{L}_{34}\Omega_1+\tilde{L}_{35}\Omega_2+\tilde{L}_{36}\Omega_3=\frac{q_3}{\tilde{E}_*},\\ \tilde{L}_{41}u_1+\tilde{L}_{42}u_2+\tilde{L}_{43}w+\tilde{L}_{44}\Omega_1+\tilde{L}_{45}\Omega_2+\tilde{L}_{46}\Omega_3=0,\\ \tilde{L}_{51}u_1+\tilde{L}_{52}u_2+\tilde{L}_{53}w+\tilde{L}_{54}\Omega_1+\tilde{L}_{55}\Omega_2+\tilde{L}_{56}\Omega_3=0,\\ \tilde{L}_{61}u_1+\tilde{L}_{62}u_2+\tilde{L}_{63}w+\tilde{L}_{64}\Omega_1+\tilde{L}_{65}\Omega_2+\tilde{L}_{66}\Omega_3=0,\end{gathered}\tag{12}$$

where the coefficients $\tilde{L}_{km}$ are partial differential operators in the following form:

$$\begin{aligned}
&\tilde{L}_{11} = \frac{\partial^2}{\partial x^2} + \frac{D_2}{\tilde{E}_*}\frac{1}{R^2}\frac{\partial^2}{\partial \theta^2},\ \tilde{L}_{12} = \tilde{L}_{21} = \left(\nu + \frac{D_2\eta_1}{\tilde{E}_*}\right)\frac{1}{R}\frac{\partial^2}{\partial x\partial\theta},\ \tilde{L}_{13} = \tilde{L}_{31} = \frac{\nu}{R}\frac{\partial}{\partial x},\\
&\tilde{L}_{14} = \tilde{L}_{41} = 0,\ \tilde{L}_{15} = \tilde{L}_{51} = 0,\ \tilde{L}_{16} = \tilde{L}_{61} = \frac{D_2}{\tilde{E}_*}(1-\eta_1)\frac{1}{R}\frac{\partial}{\partial\theta},\\
&\tilde{L}_{22} = \frac{D_2}{\tilde{E}_*}\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{D_2}{\tilde{E}_*\cdot R^2},\ \tilde{L}_{23} = \tilde{L}_{32} = \frac{1}{R}\left(1 + \frac{D_1}{\tilde{E}_*}\right)\frac{1}{R}\frac{\partial}{\partial\theta},\\
&\tilde{L}_{24} = \tilde{L}_{42} = -\frac{D_1}{\tilde{E}_*\cdot R},\ \tilde{L}_{25} = \tilde{L}_{52} = 0,\ \tilde{L}_{26} = \tilde{L}_{62} = -\frac{D_2}{\tilde{E}_*}(1-\eta_1)\frac{\partial}{\partial x},\\
&\tilde{L}_{33} = -\frac{D_1}{\tilde{E}_*}\left(\frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\right),\ \tilde{L}_{34} = \tilde{L}_{43} = \frac{D_1}{\tilde{E}_*}\cdot\frac{1}{R}\frac{\partial}{\partial\theta},\\
&\tilde{L}_{35} = \tilde{L}_{53} = -\frac{D_1}{\tilde{E}_*}\frac{\partial}{\partial x},\ \tilde{L}_{36} = \tilde{L}_{63} = 0,\ \tilde{L}_{44} = \frac{D'(1+2\eta_2)}{\tilde{E}_*}\frac{\partial^2}{\partial x^2} + \frac{D'}{\tilde{E}_*}\cdot\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - \frac{D_1}{\tilde{E}_*},\\
&\tilde{L}_{45} = \tilde{L}_{54} = \frac{2D'\eta_2}{\tilde{E}_*}\cdot\frac{1}{R}\frac{\partial^2}{\partial x\partial\theta},\ \tilde{L}_{46} = \tilde{L}_{64} = \frac{D'\cdot\eta_2}{\tilde{E}_*\cdot R}\frac{\partial}{\partial x},\\
&\tilde{L}_{55} = \frac{D'}{\tilde{E}_*}\frac{\partial^2}{\partial x^2} + \frac{D'(1+2\eta_2)}{\tilde{E}_*}\cdot\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - \left(\frac{B_*}{\tilde{E}_* R^2} + \frac{D_1}{\tilde{E}_*}\right),\\
&\tilde{L}_{56} = \tilde{L}_{65} = \left[\frac{D'(1+2\eta_2)}{\tilde{E}_*\cdot R} + \frac{B_*}{\tilde{E}_*\cdot R}\right]\cdot\frac{1}{R}\frac{\partial}{\partial\theta},\\
&\tilde{L}_{66} = -\frac{B_*}{\tilde{E}_*}\cdot\frac{\partial^2}{\partial x^2} - \frac{B_*}{\tilde{E}_*}\cdot\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} + \left[\frac{2D_2(1-\eta_1)}{\tilde{E}_*} + \frac{D'(1+2\eta_2)}{\tilde{E}_*\cdot R^2}\right].
\end{aligned} \tag{13}$$

When solving specific problems, this system should be attached by boundary conditions, the total number of which is equal to six at each of the ends $x = 0$, and $x = l$of the shell (l is the length of the shell), as well as periodicity conditions (due to a closed shell that is considered).

Potential energy of deformation of the middle surface. The principle of possible displacements in the moment-membrane theory of elastic thin cylindrical shells. Boundary conditions

In accordance with the general relations of the three-dimensional moment theory of elasticity [15], by applying the expression for the potential energy of the deformation of a shell which we consider a three-dimensional body and, further, by applying Eq. (2) for the components of the deformations and bending-torsion tensors, as well as Eq. (8) and hypothesis (3), the potential energy of deformation in the moment-membrane theory of elastic thin cylindrical shells will be expressed as follows:

$$\begin{aligned}
U_0 = \frac{1}{2}\iint_{(s)}(&T_{11}\cdot\Gamma_{11} + T_{22}\cdot\Gamma_{22} + S_{12}\cdot\Gamma_{12} + S_{21}\cdot\Gamma_{21} + N_{13}\cdot\Gamma_{13} + N_{23}\cdot\Gamma_{23} + L_{11}\cdot k_{11} +\\
&+L_{22}\cdot k_{22} + L_{12}\cdot k_{12} + L_{21}\cdot k_{21} + L_{13}\cdot k_{13} + L_{23}\cdot k_{23})R\,dx\,d\theta,
\end{aligned} \tag{14}$$

where (s) -is an area of the middle surface.

If the expressions for forces and moments are used with the help of elasticity relations (9), the surface density of the potential energy of the deformation of the moment-membrane theory of elastic thin cylindrical shells will be expressed as follows:

$$\begin{aligned}
W_0 = \frac{1}{2}\big\langle &\tilde{E}_*(\Gamma_{11}^2 + \Gamma_{22}^2 + 2\nu\cdot\Gamma_{11}\cdot\Gamma_{22}) + D_2(\Gamma_{12}^2 + \Gamma_{21}^2 + 2\eta_1\cdot\Gamma_{12}\cdot\Gamma_{21}) + D_1(\Gamma_{13}^2 + \Gamma_{23}^2)\\
&+D'\{[(1+2\eta_2)(k_{11}^2 + k_{22}^2) + 2\eta_2 k_{11}k_{22}] + (k_{12}^2 + k_{21}^2 + 2\eta_2 k_{12}k_{21})]\} + B_*(k_{13}^2 + k_{23}^2)\big\rangle.
\end{aligned} \tag{15}$$

Using the law of elasticity (9), it is easy to verify that:

$$T_{11} = \frac{\partial W_0}{\partial \Gamma_{11}},\ldots,S_{21} = \frac{\partial W_0}{\partial \Gamma_{21}},\ldots,L_{12} = \frac{\partial W_0}{\partial k_{12}},\ldots. \tag{16}$$

Thus, the function W_0 can be considered as an elastic potential of forces and moments.

In all cases, when there is an elastic potential, the principle of possible displacements is valid. According to the principle, the actual state of equilibrium of an elastic body differs from adjacent geometrically possible states in that for any infinitely small possible displacements from the equilibrium position, the variation of the total energy of the system is equal to zero.

If we write the principle of possible displacements of the shell, considering it as a three-dimensional body [15], applying formulas for displacements and rotations (1), components of bending deformation and bending-torsion tensors (2), stresses and moment stresses (6), the equation of the principle of possible displacements of the moment-membrane theory of elastic thin cylindrical shells will be obtained:

$$\iint_{(s)}(T_{11}\cdot\delta\Gamma_{11}+T_{22}\cdot\delta\Gamma_{22}+S_{12}\cdot\delta\Gamma_{12}+S_{21}\cdot\delta\Gamma_{21}+N_{13}\cdot\delta\Gamma_{13}+N_{23}\cdot\delta\Gamma_{23}+ +L_{11}\cdot\delta k_{11}+L_{22}\cdot\delta k_{22}+L_{12}\cdot\delta k_{12}+L_{21}\cdot\delta k_{21}+L_{13}\cdot\delta k_{13}+L_{23}\cdot\delta k_{23})Rdxd\theta-\delta A_0=0, \quad (17)$$

where δA_0 is the work of external forces and moments on possible displacements and rotations $\delta u_1,\delta u_2,\delta w,\delta\Omega_1,\delta\Omega_2,\delta\Omega_3$:

$$\delta A_0=\iint_{(s)}(q_1\cdot\delta u_1+q_2\cdot\delta u_2+q_3\cdot\delta w+m_1\cdot\delta\Omega_1+m_2\cdot\delta\Omega_2+m_3\cdot\delta\Omega_3)Rdxd\theta- -\int_{\theta_0}^{\theta_0+2\pi}(\bar{T}_{11}\cdot\delta u_1+\bar{S}_{12}\cdot\delta u_2+\bar{N}_{13}\cdot\delta w+\bar{L}_{11}\cdot\delta\Omega_1+\bar{L}_{12}\cdot\delta\Omega_2+\bar{L}_{13}\cdot\delta\Omega_3)\Big|_{x=0}Rd\theta+ +\int_{\theta_0}^{\theta_0+2\pi}(\bar{T}_{11}\cdot\delta u_1+\bar{S}_{12}\cdot\delta u_2+\bar{N}_{13}\cdot\delta w+\bar{L}_{11}\cdot\delta\Omega_1+\bar{L}_{12}\cdot\delta\Omega_2+\bar{L}_{13}\cdot\delta\Omega_3)\Big|_{x=l}Rd\theta. \quad (18)$$

Due to the independence of the variations$\delta u_1,\delta u_2,\delta w,\delta\Omega_1,\delta\Omega_2,\delta\Omega_3$, from the variational equation (17), the equilibrium equations (11) follow as Euler equations, and static boundary conditions of the moment-membrane theory of elastic thin closed cylindrical shells follow as natural boundary conditions:

$$T_{11}=\bar{T}_{11},\ S_{12}=\bar{S}_{12},\ N_{13}=\bar{N}_{13},\ L_{11}=\bar{L}_{11},\ L_{12}=\bar{L}_{12},\ L_{13}=\bar{L}_{13},\text{ at } x=0, x=l. \quad (19)$$

The boundary conditions in generalized displacements will be as follows:

$$u_1(x,\theta)=\bar{u}_1(x,\theta),\ u_2(x,\theta)=\bar{u}_2(x,\theta),\ w(x,\theta)=\bar{w}(x,\phi), \Omega_1(x,\theta)=\bar{\Omega}_1(x,\theta),\ \Omega_2(x,\theta)=\bar{\Omega}_2(x,\theta),\ \Omega_3(x,\theta)=\bar{\Omega}_3(x,\theta),\text{ at } x=0, x=l. \quad (20)$$

As noted, the order of the resolving system of Eq. (12) of the moment-membrane theory of cylindrical shells is equal to twelve, therefore six boundary conditions of the type (19) or (20) must be set on each edge of a closed cylindrical shell. There may also be mixed boundary conditions (for example, hinge support conditions).

Axisymmetric deformation of a cylindrical shell

Due to symmetry, we accept $u_2=0$, $\Omega_1=0$, $q_2=0$. Quantities u_1, w, Ω_2, Ω_3 will depend only on the coordinate x.

The resolving system for axisymmetric deformation with respect to u_1,w,Ω_2 takes the following form:

$$\frac{d^2u_1}{dx^2}+\frac{\nu}{R}\frac{dw}{dx}=-\frac{q_1}{\tilde{E}_*},\ \frac{D_1}{\tilde{E}_*}\frac{d^2w}{dx^2}-\frac{w}{R^2}-\frac{\nu}{R}\frac{du_1}{dx}+\frac{D_1}{\tilde{E}_*}\cdot\frac{d\Omega_2}{dx}=-\frac{q_3}{\tilde{E}_*},\ \frac{d^2\Omega_2}{dx^2}-\left(\frac{D_1}{D'}+\frac{B_*}{D'\cdot R^2}\right)\Omega_2-\frac{D_1}{D'}\cdot\frac{dw}{dx}=0. \quad (21)$$

We note that for the axisymmetric problem we also obtain $\Omega_3\equiv 0$. Further, a case is considered when $q_1=0$. In that case, from the first equation of system (21) we obtain:

$$\frac{du_1}{dx}+\frac{\nu}{R}w=C_0=const. \quad (22)$$

This means that $T_{11}=const$. To be specific, we take $T_{11}=0$, i.e.

$$\frac{du_1}{dx}=-\frac{\nu}{R}w. \quad (23)$$

Instead of system (21) we obtain:

$$\frac{d^2w}{dx^2}-\frac{E_*}{D_1R^2}w+\frac{d\Omega_2}{dx}=-\frac{q_3}{D_1},D'\cdot\frac{d^2\Omega_2}{dx^2}-\left(\frac{B_*}{R^2}+D_1\right)\Omega_2-D_1\frac{dw}{dx}=0. \quad (24)$$

From the equations of this system the following formula is obtained for Ω_2:

$$\left(\frac{B_*}{R^2}+D_1\right)\Omega_2=D'\left(-\frac{d^3w}{dx^3}+\frac{E_*}{D_1R^2}\cdot\frac{dw}{dx}-\frac{1}{D_1}\frac{dq_3}{dx}\right)-D_1\frac{dw}{dx}. \quad (25)$$

Using Eqs. (24) and (25), we arrive at the solution of the following inhomogeneous fourth-order differential equation for w:

$$\frac{d^4w}{dx^4} - \left(\frac{E_*}{D_1R^2} + \frac{B_*}{D'R^2}\right)\frac{d^2w}{dx^2} + \frac{E_*}{D'\cdot D_1R^2}\left(\frac{B_*}{R^2} + D_1\right)w = \frac{1}{D'D_1}\left(\frac{B_*}{R^2} + D_1\right)q_3 - \frac{1}{D_1}\frac{d^2q_3}{dx^2}. \quad (26)$$

Further, a case is considered when $q_3 = const$. In that case, instead of Eq. (26) we obtain:

$$\frac{d^4w}{dx^4} - \left(\frac{E_*}{D_1R^2} + \frac{B_*}{D'R^2}\right)\frac{d^2w}{dx^2} + \frac{E_*}{D'\cdot D_1R^2}\left(\frac{B_*}{R^2} + D_1\right)w = \frac{1}{D'D_1}\left(\frac{B_*}{R^2} + D_1\right)q_3. \quad (27)$$

It is easy to see that a particular solution of Eq. (27) is:

$$\widetilde{w} = \frac{q_3R^2}{E_*}. \quad (28)$$

Let us consider a specific example, where a carbon nanotube, when the ends $x = 0,\ x = l$ are hinge supported, is loaded with a uniformly distributed normally applied surface load q_3. The boundary conditions in this case will be:

$$w|_{\substack{x=0\\x=l}} = 0, \frac{\partial\Omega_2}{\partial x}\bigg|_{\substack{x=0\\x=l}} = 0. \quad (29)$$

The initial data has the following meanings: q_3 = 10 nN/nm^2, R = 10 nm, l = 100 nm. The stiffness characteristics of a nanotube are expressed numerically as follows [10]: $E_* = 287\ \frac{\text{nN}}{\text{nm}};\ \ B_* = 0.505\ \text{nN}\cdot\text{nm},\ D_1 = 86\ \frac{\text{nN}}{\text{nm}},\ \ D' = 0.415\ \text{nN}\cdot\text{nm}.$

The general solution of differential Eq. (27) (considering Eq. (28)) has the form: $w(x) = 3.484 + e^{-1.15x}(C_1 \cos 1.14x + C_2 \sin 1.14x) + e^{1.15x}(C_3 \cos 1.14x + C_4 \sin 1.14x)$, where $C_1,\ C_2,\ C_3,\ C_4$ are constants.

With the help of boundary conditions (29), by determining the constants of integration, the solution of the stated problem will finally be expressed as follows: $w(x) = 3.484 + e^{-1.15x}(-3.484 \cos 1.14x - 0.03 \sin 1.14x)$ (nm).

The maximum deflection of the nanotube is obtained at $x = 50$ nm, w_{max}.

Technical theory of cylindrical shells

The integration of the system of differential equations of a cylindrical shell into generalized displacements based the moment-membrane theory (12) is a complex task, therefore, the resolving toolkit of the theory of cylindrical shells (similar to the classical theory of cylindrical shells [18,19]) should involve equations that are simpler in structure and provide sufficient accuracy in applied calculations.

Such a simplified (technical) version of the theory of a cylindrical shell (the shell in this case has zero Gaussian curvature) is built on the basis of the following assumptions:
a) following members in equilibrium equations (11) can be neglected: in the second one (as in the classical case) $\frac{N_{23}}{R}$, in the fifth one - $\frac{L_{23}}{R}$ and in the sixth one - $\frac{L_{22}}{R}$;
b) following members in the geometrical relations (3) can be neglected: for $\Gamma_{23} - \frac{u_2}{R}$ (as in the classical case); $k_{23} - \frac{\Omega_2}{R}$; $k_{22} - \frac{\Omega_3}{R}$.

In view of the above facts, the main relations of the theory of a cylindrical shell (3), Eq. (11) can be written as follows:

Equilibrium equations:

$$\frac{\partial T_{11}}{\partial x} + \frac{1}{R}\frac{\partial S_{21}}{\partial\theta} = -q_1,\ \ \frac{1}{R}\frac{\partial T_{22}}{\partial\theta} + \frac{\partial S_{22}}{\partial x} = -q_2, \frac{T_{22}}{R} - \frac{\partial N_{13}}{\partial x} - \frac{1}{R}\frac{\partial N_{23}}{\partial\theta} = q_3,$$
$$\frac{\partial L_{11}}{\partial x} + \frac{1}{R}\frac{\partial L_{21}}{\partial\theta} + N_{23} = -m_1,\ \ \frac{1}{R}\frac{\partial L_{22}}{\partial\theta} + \frac{\partial L_{12}}{\partial x} - N_{13} = -m_2, \quad (30)$$
$$\frac{\partial L_{13}}{\partial x} + \frac{1}{R}\frac{\partial L_{23}}{\partial\theta} + (S_{12} - S_{21}) = -m_3.$$

Geometrical relations:

$$\Gamma_{11}=\frac{\partial u_1}{\partial x},\ \ \Gamma_{22}=\frac{1}{R}\frac{\partial u_2}{\partial\theta}+\frac{w}{R},\ \ \Gamma_{12}=\frac{\partial u_2}{\partial x}-\Omega_3,$$
$$\Gamma_{21}=\frac{1}{R}\frac{\partial u_1}{\partial\theta}+\Omega_3,\ \ \Gamma_{13}=\frac{\partial w}{\partial x}+\Omega_2,\ \ \Gamma_{23}=\frac{1}{R}\frac{\partial w}{\partial\theta}-\Omega_1, \tag{31}$$
$$k_{11}=\frac{\partial\Omega_1}{\partial x},\ \ k_{22}=\frac{1}{R}\frac{\partial\Omega_2}{\partial\theta},\ \ k_{12}=\frac{\partial\Omega_2}{\partial x},k_{21}=\frac{1}{R}\frac{\partial\Omega_1}{\partial\theta},\ \ k_{13}=\frac{\partial\Omega_3}{\partial x},\ \ k_{23}=\frac{1}{R}\frac{\partial\Omega_3}{\partial\theta}.$$

We add the elasticity relations (9) to the equilibrium equations (30) and geometric relations (31).

From the geometric relations (31), by eliminating displacements u_1,u_2, we arrive at the following equations for the continuity of deformations:

$$\frac{\partial\Gamma_{21}}{\partial x}-\frac{1}{R}\frac{\partial\Gamma_{11}}{\partial\theta}-k_{13}=0,\frac{1}{R}\frac{\partial\Gamma_{12}}{\partial\theta}-\frac{\partial\Gamma_{22}}{\partial x}+k_{23}=-\frac{1}{R}\frac{\partial w}{\partial x}. \tag{32}$$

It should be mentioned that equations $\frac{1}{R}\frac{\partial k_{11}}{\partial\theta}=\frac{\partial k_{21}}{\partial x},\ \ \frac{\partial k_{22}}{\partial x}=\frac{1}{R}\frac{\partial k_{12}}{\partial\theta}$ and $\frac{1}{R}\frac{\partial k_{13}}{\partial\theta}=\frac{\partial k_{23}}{\partial x}$ are automatically satisfied.

Let us consider a more essential case of loading the cylindrical shell, when $q_3=q,q_1=q_2=m_1=m_2=m_3=0$.

Introducing auxiliary functions $\Phi_1$and Φ_2 by formulas:

$$T_{11}=\frac{1}{R}\frac{\partial\Phi_1}{\partial\theta},S_{21}=-\frac{\partial\Phi_1}{\partial x},T_{22}=\frac{\partial\Phi_2}{\partial x},S_{12}=-\frac{1}{R}\frac{\partial\Phi_2}{\partial\theta}, \tag{33}$$

the first two equilibrium equations (30) are identically satisfied.

Let us consider the sixth equilibrium equation from the system (30):

$$\frac{\partial L_{13}}{\partial x}+\frac{1}{R}\frac{\partial L_{23}}{\partial\theta}+(S_{12}-S_{21})=0. \tag{34}$$

Function ψ is introduced by formulas:

$$L_{13}=\frac{\partial\psi}{\partial x},L_{23}=\frac{1}{R}\frac{\partial\psi}{\partial\theta}. \tag{35}$$

By considering Eq. (33) for S_{12} and S_{21}, as well as by satisfying Eq. (34), we arrive at the equality:

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}+\Phi_1\right)=\frac{1}{R}\frac{\partial}{\partial\theta}\left(\Phi_2-\frac{1}{R}\frac{\partial\psi}{\partial\theta}\right). \tag{36}$$

Now, if we introduce the function ϕ according to the formulas:

$$\Phi_2-\frac{1}{R}\frac{\partial\psi}{\partial\theta}=\frac{\partial\phi}{\partial x},\ \ \frac{\partial\psi}{\partial x}+\Phi_1=\frac{1}{R}\frac{\partial\phi}{\partial\theta}. \tag{37}$$

Eq. (34) will be identically satisfied.

For efforts $T_{11},T_{22},S_{12},S_{21}$ and moments L_{13},L_{23}, we finally have:

$$T_{11}=\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta},\ \ T_{22}=\frac{\partial^2\phi}{\partial x^2}+\frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta},$$
$$S_{21}=-\frac{1}{R}\frac{\partial^2\phi}{\partial x\partial\theta}+\frac{\partial^2\psi}{\partial x^2},\ \ S_{12}=-\frac{1}{R}\frac{\partial^2\phi}{\partial x\partial\theta}-\frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}, \tag{38}$$
$$L_{13}=\frac{\partial\psi}{\partial x},\ \ L_{23}=\frac{1}{R}\frac{\partial\psi}{\partial\theta}.$$

For the indicated forces and moments, by applying Eq. (38), as noted above, the first two and the sixth equilibrium equations from the system (30) are identically satisfied. As a result of the replacement of forces and moments through generalized displacements w,Ω_1,Ω_2 and functions ϕ and ψ, the other three equations from the system (30), with the help of Eq. (38), take the form:

$$\frac{1}{R}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta}\right)-D_1\Delta w-D_1\left(\frac{\partial\Omega_2}{\partial x}-\frac{1}{R^2}\frac{\partial\Omega_1}{\partial\phi}\right)=q,$$
$$\Delta\Omega_1+2\eta_2\frac{\partial}{\partial x}\left(\frac{\partial\Omega_1}{\partial x}+\frac{1}{R}\frac{\partial\Omega_2}{\partial\theta}\right)+\frac{D_1}{D'}\left(\frac{1}{R}\frac{\partial w}{\partial\theta}-\Omega_1\right)=0, \tag{39}$$
$$\Delta\Omega_2+2\eta_2\frac{1}{R}\frac{\partial}{\partial\theta}\left(\frac{1}{R}\frac{\partial\Omega_2}{\partial\theta}+\frac{\partial\Omega_1}{\partial x}\right)-\frac{D_1}{D'}\left(\frac{\partial w}{\partial x}+\Omega_2\right)=0,$$

where

$$\Delta(\cdot)=\frac{\partial^2(\cdot)}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2(\cdot)}{\partial\theta^2}. \tag{40}$$

Now the angles of free rotation $\Omega_1$and $\Omega_2$will be represented as follows [19-21]:

$$\Omega_1=-\frac{1}{R}\frac{\partial G}{\partial\theta}+\frac{\partial\chi}{\partial x},\ \Omega_2=\frac{\partial G}{\partial x}+\frac{1}{R}\frac{\partial\chi}{\partial\theta}, \tag{41}$$

where G and χ are auxiliary functions to be determined.

By substituting Eq. (41) into the second and third equations from the system (39), it is easy to see that for the function χ we arrive at the solution of the following Helmholtz equation:

$$\Delta\chi-\tilde{k}^2\cdot\chi=0, \tag{42}$$

where

$$\tilde{k}^2=\frac{D_1}{D'(1+2\eta_2)}. \tag{43}$$

To determine the function G, we obtain the following equation:

$$G=-\frac{1}{D_1}(D'\Delta w+D_1w)+\frac{D'}{D_1^2}(\Delta_k'\phi+\Delta_k''\psi)-\frac{D'}{D_1^2}q. \tag{44}$$

In addition, if the second equation from the system (39) is differentiated with respect to the operator $\frac{1}{R}\frac{\partial}{\partial\theta}$, while the third one- with respect to $\frac{\partial}{\partial x}$, the second equation is subtracted from the obtained first equality, by using the first equation from system (39), we arrive at the following equation:

$$D'\Delta\Delta w+\frac{1}{R}\left(1-\frac{D'}{D_1}\Delta\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta}\right)=q-\frac{D'}{D_1}\Delta q. \tag{45}$$

Thus, instead of system (39), we have obtained a system of two Eqs. (45) and (42), which contain four functions $w,\ \phi,\ \psi,\ \chi$. We will turn to two equations of continuity of deformations (32), by substituting in them the inverse relations of elasticity (9), which express $\Gamma_{11},\Gamma_{22},\Gamma_{12},\Gamma_{21}$through forces $T_{11},T_{22},S_{12},S_{21}$ and by using Eq. (38), instead of the equations of compatibility of deformations (32), we obtain the following equations:

$$\begin{gathered}\frac{\partial}{\partial x}(l_*^2\Delta-1)\psi-\frac{B_*}{E_*}\cdot\frac{1}{R}\frac{\partial\Delta\phi}{\partial\theta}=0,\\ \frac{1}{R}\frac{\partial}{\partial\theta}(l_*^2\Delta-1)\psi+\frac{B_*}{E_*}\cdot\frac{\partial\Delta\phi}{\partial x}=B_*\frac{1}{R}\frac{\partial w}{\partial x},\end{gathered} \tag{46}$$

where

$$l_*^2=\frac{B_*(\mu_*+\alpha_*)}{4\mu_*\alpha_*}. \tag{47}$$

Eq. (45), in conjunction with Eqs. (42) and (46), will form a complete system of equations for determining four functions $w,\ \phi,\ \psi,\ \chi$:

$$\begin{gathered}D'\Delta\Delta w+\frac{1}{R}\left(1-\frac{D'}{D_1}\Delta\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{1}{R}\ \frac{\partial^2\psi}{\partial x\partial\theta}\right)=q-\frac{D'}{D_1}\Delta q,\\ \frac{\partial}{\partial x}(l_*^2\Delta-1)\psi-\frac{B_*}{E_*}\cdot\frac{1}{R}\frac{\partial\Delta\phi}{\partial\theta}=0,\quad \frac{1}{R}\frac{\partial}{\partial\theta}(l_*^2\Delta-1)\psi+\frac{B_*}{E_*}\cdot\frac{\partial\Delta\phi}{\partial x}=B_*\frac{1}{R}\frac{\partial w}{\partial x},\\ \Delta\chi-\tilde{k}^2\chi=0.\end{gathered} \tag{48}$$

The system of resolving equations of the moment-membrane technical theory (48) lends itself to further simplifications.

Based on the second and third equations of system (48), it is easy to obtain the following equation:

$$\Delta\Delta\phi=E_*\frac{1}{R}\frac{\partial^2 w}{\partial x^2}. \tag{49}$$

If we consider the first and third equations of system (48), it will be possible to express the mixed derivative of the function ψ through functions w and ϕ:

$$\frac{1}{R}\left(\frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta}\right)=-D'c_0\Delta\Delta w-\frac{c_0}{R}\cdot\frac{\partial^2\phi}{\partial x^2}+\frac{1}{R}\cdot\frac{D'}{D_1}\cdot\frac{c_0}{a_0}\Delta\frac{\partial^2\phi}{\partial x^2}+\frac{d_0}{R^2}B_*\cdot\frac{d^2w}{dx^2}+c_0q-c_0\frac{D'}{D_1}\Delta q, \tag{50}$$

where

$$c_0=\frac{l_*^2}{l_*^2-\frac{D'}{D_1}},\quad d_0=\frac{\frac{D'}{D_1}}{l_*^2-\frac{D'}{D_1}},\quad a_0=\frac{l_*^2}{l_*^2-\frac{B_*}{E_*}}. \tag{51}$$

The substitution of Eq. (50) into the first equation of system (48), will result in the following system of equations, instead of the system of equations (48):

$$D'\Delta\Delta w - D'c_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta w - \frac{d_0}{R}\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^2\phi}{\partial x^2} + \frac{1}{R}\cdot\frac{D'}{D_1}\cdot\frac{c_0}{a_0}\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^2\phi}{\partial x^2} +$$
$$+\frac{d_0}{R^2}B_*\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^2 w}{\partial x^2} = -d_0\left(1-\frac{D'}{D_1}\Delta\right)q + c_0\frac{D'}{D_1}\left(1-\frac{D'}{D_1}\Delta\right)\Delta q,\ \Delta\Delta\phi = E_*\frac{1}{R}\frac{\partial^2 w}{\partial x^2},\ \ \Delta\chi - \tilde{k}^2\chi = 0. \quad (52)$$

Accepting,

$$w = \Delta\Delta F,\ \ \phi = \frac{E_*}{R}\cdot\frac{\partial^2 F}{\partial x^2}, \quad (53)$$

we identically satisfy the second equation of system (52). From the first equation of system (52), we obtain a tenth-order equation with respect to the new potential function F. As a result, instead of system (52), we arrive at the following twelfth-order system of two equations for the functions F and χ:

$$D'\Delta\Delta\Delta\Delta F - D'c_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\Delta\Delta F - \frac{d_0}{R^2}E_*\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^4 F}{\partial x^4} + \frac{1}{R^2}\cdot\frac{c_0}{a_0}\cdot\frac{D'}{D_1}\cdot E_*\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^4 F}{\partial x^4} +$$
$$\frac{d_0}{R^2}B_*\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\frac{\partial^2 F}{\partial x^2} = -d_0\left(1-\frac{D'}{D_1}\Delta\right)q + c_0\frac{D'}{D_1}\left(1-\frac{D'}{D_1}\Delta\right)\Delta q,\ \ \Delta\chi - \tilde{k}^2\chi = 0. \quad (54)$$

If a closed cylindrical shell is hinged at the ends $x = 0,\ x = l$, it can be shown that in this particular case of boundary conditions, we have:

$$\chi \equiv 0. \quad (55)$$

In that case, the problem is reduced to solving one equation of the tenth order with respect to the function F:

$$D'\Delta\Delta\Delta\Delta F - D'c_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\Delta\Delta F - \frac{d_0}{R^2}E_*\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^4 F}{\partial x^4} + \frac{1}{R^2}\cdot\frac{c_0}{a_0}\cdot\frac{D'}{D_1}\cdot E_*\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^2 F}{\partial x^2} +$$
$$+\frac{d_0}{R^2}B_*\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\frac{\partial^2 F}{\partial x^2} = -d_0\left(1-\frac{D'}{D_1}\Delta\right)q + c_0\frac{D'}{D_1}\left(1-\frac{D'}{D_1}\Delta\right)\Delta q. \quad (56)$$

We note that the simplified version of the moment-membrane technical theory of a cylindrical shell presented here can be used for studying various static problems for a carbon nanotube, in particular, for studying problems of its stability.

Stability of an axially compressed closed cylindrical shell according to the moment-membrane theory (axisymmetric problem)

Let us present the basic equations necessary for studying the stability of a cylindrical shell according to the moment-membrane theory.

We consider a case when a closed cylindrical shell is under the action of a uniform axial compression (the main momentless state). For stability problems, the general equations of the technical theory of a cylindrical shell must be supplemented with terms that consider subcritical forces in the shell.

In this case, in the first equation from the system (48), instead $q$we take a "fictitious" transverse load:

$$q = -p_1\frac{\partial^2 w}{\partial x^2}, \quad (57)$$

where p_1 is the axial force that occurs in the shell before buckling. The specified system will take the form (further we consider the case when the ends $x = 0,\ x = l$ of the cylindrical shell are hinge supported, i.e. $\chi \equiv 0$):

$$D'\Delta\Delta w + \frac{1}{R}\left(1-\frac{D'}{D_1}\Delta\right)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{1}{R}\frac{\partial^2\psi}{\partial x\partial\theta}\right) = -p_1\frac{\partial^2 w}{\partial x^2} + \frac{D'}{D_1}p_1\Delta\frac{\partial^2 w}{\partial x^2},$$
$$\frac{\partial}{\partial x}(l_*^2\Delta - 1)\psi - \frac{B_*}{E_*}\cdot\frac{1}{R}\frac{\partial\Delta\phi}{\partial\theta} = 0,\ \ \frac{1}{R}\frac{\partial}{\partial\theta}(l_*^2\Delta - 1)\psi + \frac{B_*}{E_*}\cdot\frac{\partial\Delta\phi}{\partial x} = B_*\frac{1}{R}\frac{\partial w}{\partial x}. \quad (58)$$

Repeating the steps of the previous paragraph, instead of Eq. (58) we obtain:

$$D'\Delta\Delta w - D'c_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta w - \frac{d_0}{R}\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^2\phi}{\partial x^2} + \frac{1}{R}\cdot\frac{D'}{D_1}\cdot\frac{c_0}{a_0}\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^2\phi}{\partial x^2} +$$
$$+\frac{d_0}{R^2}B_*\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^2 w}{\partial x^2} = -p_1 d_0\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^2 w}{\partial x^2} - p_1 c_0\frac{D'}{D_1}\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^2 w}{\partial x^2},\quad \Delta\Delta\phi = E_*\frac{1}{R}\frac{\partial^2 w}{\partial x^2}. \tag{59}$$

Based on the second equation from system (59), when studying the stability of a cylindrical closed shell, we arrive at the solution of the following 10th order equation with respect to w:

$$D'\Delta\Delta\Delta\Delta w - D'c_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\Delta\Delta w - \frac{d_0}{R^2}E_*\left(1-\frac{D'}{D_1}\Delta\right)\frac{\partial^4 w}{\partial x^4} + \frac{E_*}{R^2}\cdot\frac{D'}{D_1}\cdot\frac{c_0}{a_0}\left(1-\frac{D'}{D_1}\Delta\right)\Delta\frac{\partial^4 w}{\partial x^4} +$$
$$+\frac{d_0}{R^2}B_*\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\frac{\partial^2 w}{\partial x^2} = p_1 d_0\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\frac{\partial^2 w}{\partial x^2} - p_1 c_0\frac{D'}{D_1}\left(1-\frac{D'}{D_1}\Delta\right)\Delta\Delta\Delta\frac{\partial^2 w}{\partial x^2}. \tag{60}$$

If the curved surface of the shell remains axisymmetric after buckling, the deflection $w$depends only on the coordinate x, and Eq. (60) looks as follows:

$$\left[\frac{D'^2 c_0}{D_1} - p_1 c_0\left(\frac{D'}{D_1}\right)^2\right]\frac{\partial^{10} w}{\partial x^{10}} + \left[D' - D'c_0 - \frac{1}{R^2}\left(\frac{D'}{D_1}\right)^2\frac{c_0}{a_0}E_* - B_*\frac{d_0\cdot D'}{R^2\cdot D_*} + p_1(d_0+c_0)\frac{D'}{D_1}\right]\frac{\partial^8 w}{\partial x^8} +$$
$$+\left[\frac{d_0\cdot E_*}{R^2}\frac{D'}{D_1} + \frac{E_*}{R^2}\frac{D'}{D_1}\frac{c_0}{a_0} + \frac{d_0}{R^2}B_* - p_1 d_0\right]\frac{\partial^6 w}{\partial x^6} - \frac{d_0\cdot E_*}{R^2}\frac{\partial^4 w}{\partial x^4} = 0. \tag{61}$$

Based on the homogeneous differential Eq. (61), we consider the problem of determining the critical force of a carbon nanotube uniformly compressed in the axial direction.

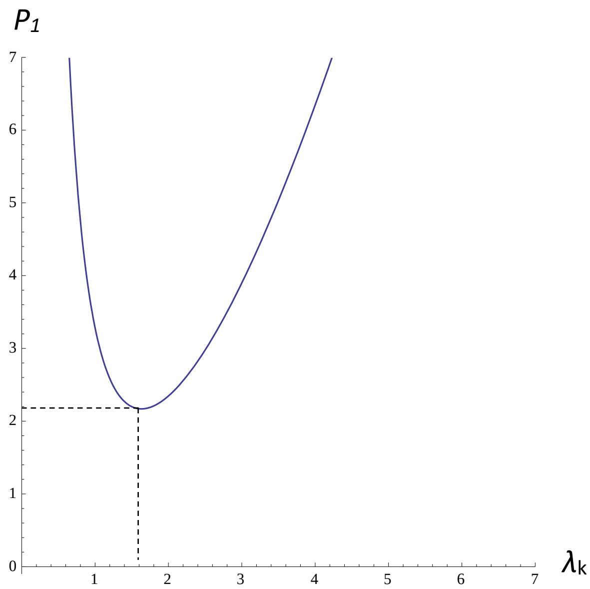

Fig. 2. Graphic definition of the critical force as a result of $p_1(\lambda_k)$ dependency minimization

Assuming deflection $w(x) = f\,sin\,\lambda_k x$, $f = const$, $\lambda_k = \frac{\pi k}{l}$, $k = 1,2,3,\ldots,$ by satisfying the boundary conditions of hinge support at $x = 0, x = l$, we obtain (here the data of paragraph 5 of this paper are used): $p_1 = \frac{140072+1249{,}43\lambda_k{}^2+20257{,}1\lambda_k{}^4+82{,}93\lambda_k{}^6}{(48805+435{,}34\lambda_k{}^2+0{,}964\lambda_k{}^4)\lambda_k{}^2}$

Let us determine the smallest (critical) value of the compressive force. Equating to zero the derivative of p_1 by λ_k: $\frac{dp_1}{d\lambda_k} = 0$, after determining the corresponding value for λ_k, for the critical force we obtain the following numerical value: $p_{cr} = 2.169\,\frac{\mathrm{nN}}{\mathrm{nm}}$. This result is also substantiated graphically, by constructing a dependence graph $p_1(\lambda_k)$ (Fig. 2).

Conclusion

Based on the three-dimensional moment linear theory of elasticity, with the application of the hypotheses' method, the basic equations, relations and boundary conditions of the moment-membrane theory of elastic cylindrical shells are constructed, which is interpreted as a continual model of the deformation behavior of a single-layer carbon nanotube. The resolving equations of the moment-membrane technical theory of elastic cylindrical shells are also constructed. The constructed continual theories of a cylindrical shell open new possibilities for studying various applied problems of the mechanical behavior of a single-walled carbon nanotube.

As particular problems, in an axisymmetric formulation, the problems of the static bending of a single-layer carbon nanotube and the problem of the stability of its initial axial compressed state have been studied.

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