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SOME THEOREMS AND WAVE PROPAGATION IN A PIEZOTHERMOELASTIC MEDIUM WITH TWO-TEMPERATURE AND FRACTIONAL ORDER DERIVATIVE

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Abstract. Wave propagation and some basic theorems like variational principle, uniqueness theorem, and theorem of reciprocity are studied for an anisotropic piezothermoelastic solid with two-temperature and fractional order derivative. The basic governing equations are used to study the interesting problem. Also, we characterize an alternative formulation of the mixed initial boundary value problem. These theorems are also summarised for a special case of orthotropic piezothermoelastic solid with the consideration of two-temperature theory and fractional order derivative. The non-trivial solution of the system is insured by a quartic equation whose roots represent the complex velocities of four attenuating waves in the medium. The different characteristics of the waves like phase velocity and attenuation quality factor are plotted three-dimensionally with the change in direction for two different models. Some special cases are also deduced from the present investigation.

Keywords: piezothermoelastic, orthotropic, variational principle, uniqueness, plane waves, phase velocity

1. Introduction

The two-temperature theory of thermoelasticity with two distinct temperatures (conductive temperature φ and the thermodynamic temperature T) was introduced by Chen and Gurtin [1], and Chen et al. [2,3]. Said et al. [4] investigated a problem of rotating-micropolar thermoelastic medium with two-temperature under influence of the magnetic field. Kumar et al. [5] studied the propagation of plane waves in an anisotropic thermoelastic medium with void and two-temperature in the context of three phase lag theory of thermoelasticity.

The theory of thermopiezoelectric material was first proposed by Mindlin [6] and derived governing equations of a thermopiezoelectric plate. The physical laws for the thermopiezoelectric material have been explored by Nowacki [7,8]. Sharma [9] investigated the piezoelectric effect on the velocities of waves in an anisotropic piezo-poroelastic medium. Vashishth and Sukhija [10] studied the inhomogeneous waves at the boundary of an anisotropic piezothermoelastic solid. Kumar and Sharma [11] established basic theorems and discussed wave propagation in a piezothermoelastic medium with the consideration of dual phase lag.

Fractional Calculus is a field of mathematic study that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way, fractional exponents is an outgrowth of exponents with an integer value. Meral and Royston [12] investigated the response of the fractional order on viscoelastic half-space to surface and subsurface sources. Bassiouny and Sabry [13] discussed the two-temperature thermo-elastic behaviour of piezoelectric materials with fractional order derivative. Kumar and Sharma [14]

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discussed the effect of fractional order derivative on energy ratios at the boundary surface of the elastic-piezothermoelastic medium. Lata [15] discussed the fractional order thermoelastic thick circular plate with two temperatures in the frequency domain.

Youssef and Bassiouny [16] proposed the generalised two-temperature theory of thermoelasticity to solve the boundary value problems of one dimensional piezothermoelastic half-space with heating its boundary with different types of heating. Ezzat et al. [17] formulated the theory of two-temperature theory of thermoelasticity for piezoelectric/piezomagnetic materials. Bassiouny and Sabry [18] investigated the propagation of a thermal wave through a semi-infinite slab subjected to thermal loading of the fractional order of exponential type applied for a finite period of time.

Comprehensive work has been done on uniqueness, reciprocity theorems and variational principle by different authors in different media notable among them are Nickell and Sackman [19], Iesan [20], Karamany and Ezzat [21], Othman [22], Ezzat, Kumar et al. [23], Kuang [24], Vashishth and Gupta [25], and Kumar and Sharma [26,27].

In the present investigation, the variational principle, reciprocity theorem, and the uniqueness theorem have been proved. The mixed initial boundary value problem and its alternative approach are also discussed. Further, wave propagation in an orthotropic piezothermoelastic medium with the effect of the two-temperature and fractional order parameter is studied and characteristics like phase velocity and attenuation quality factor of waves are demonstrated graphically depicting the effect of fractional order and two-temperature parameter. The established results will be helpful for further investigation of the various problems.

2. Basic Equations

Following Kumar et al. [5] and Kumar and Sharma [11], the governing equations in a homogeneous, anisotropic piezothermoelastic medium with two-temperature and fractional order derivative in the absence of thermal sources and independent of free charge density are:

Constitutive equations:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} - e_{ijk}E_k - \alpha_{ij}T\,, \tag{1}$$

$$-q_{i,i} = \rho T_0 \dot{S}, \tag{2}$$

$$\rho S = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + rT\,, \tag{3}$$

$$D_i = \xi_{ij}E_j + e_{ijk}\varepsilon_{jk} + \tau_i T\,, \tag{4}$$

$$E_i = -\Phi_{,i}\,,\ (i, j, k, l = 1, 2, 3). \tag{5}$$

Equations of motion:

$$\sigma_{ij,j} + \rho(F_i - \ddot{u}_i) = 0. \tag{6}$$

Equation of heat conduction:

$$-K_{ij}\ \varphi_{,j} = \left(1 + \tau_q \frac{\partial^\alpha}{\partial t^\alpha}\right) q_i\,, \tag{7}$$

such that $\varphi - T = a_{ij}\varphi_{,ij}$.

Gauss equation:

$$D_{i,i} = 0. \tag{8}$$

In the equations (1)-(8), the Cartesian reference frame system is used and repeated subscripts imply summation. The subscripts preceded by comma notations are used to represent the partial derivatives with respect to the space variables and the superposed dots denote the order of time differentiation.

$c_{ijkl}\,(= c_{klij} = c_{jikl} = c_{ijlk})$ – Elastic constants,

ρ – Mass density,

q_i – Components of heat flux vector $\boldsymbol{q}$,

F_i – Components of the external forces per unit mass,

u_i – Components of the displacement vector $\mathbf{u}$,

σ_{ij} – Stress tensor,

ε_{ij} – Strain tensor,

K_{ij} – Thermal conductivity tensor,

S – Entropy per unit mass,

E_i – Electric field intensity,

D_i – Electric displacement,

Φ – Electric potential,

T, φ – Absolute and conductive temperature of the medium,

$a_{ij}(>0)$ – Two-Temperature parameters,

T_0 – Reference temperature of the body,

$\alpha_{ij}, \tau_i, \xi_{ij}, e_{ijk}, r$ – Piezothermal moduli, respectively,

τ_q – Thermal relaxation time,

α – Fractional order derivative such that $0 \le \alpha < 1$.

3. Variational Principle

The principle of virtual work with a variation of displacements for the elastic deformable body of volume V and surface A is written as

$$\int_V \rho(F_i - \ddot{\mathrm{u}}_i)\delta \mathrm{u}_i \, dV + \int_A h_i \delta \mathrm{u}_i \, dA + \int_A c_0 \delta\Phi \, dA = \int_A (\sigma_{ij} n_j)\delta \mathrm{u}_i \, dA + \int_A (D_i n_i)\delta\Phi \, dA, \tag{9}$$

where $h_i = \sigma_{ij} n_j$ and $c_0 = D_i n_i$.

On the left hand side, we have the virtual work of body forces F_i, inertial forces $\rho \ddot{\mathrm{u}}_i$, surface forces h_i, whereas, on the right hand side, we have the virtual work of internal forces and n_i denotes the outward normal of ∂V, c_0 is the electric charge density. Using divergence theorem and the symmetry of the stress tensor, equation (9) can be written in the alternative form as

$$\int_V \rho(F_i - \ddot{\mathrm{u}}_i)\delta \mathrm{u}_i \, dV + \int_A h_i \delta \mathrm{u}_i \, dA + \int_A c_0 \delta\Phi \, dA = \int_V (\sigma_{ij} \delta \mathrm{u}_{i,j}) \, dV + \int_V (D_i \delta\Phi_{,i}) \, dV. \tag{10}$$

Substituting the value of σ_{ij} from the relation (1) in the equation (10), we obtain

$$\begin{aligned} \int_V \rho(F_i - \ddot{\mathrm{u}}_i)\delta \mathrm{u}_i \, dV + \int_A h_i \delta \mathrm{u}_i \, dA + \int_A c_0 \delta\Phi \, dA &= \int_V (c_{ijkl}\, \varepsilon_{kl} - e_{ijk} E_k - \alpha_{ij} T)\delta\varepsilon_{ij} \, dV - \int_V D_i \delta E_i dV \\ &= \delta W - \int_V e_{ijk}\, E_k \delta\varepsilon_{ij} dV - \int_V \alpha_{ij}\left(\varphi - a_{ij}\varphi_{,ij}\right)\delta\varepsilon_{ij} dV - \int_V D_i \delta E_i dV, \end{aligned} \tag{11}$$

where $W = \frac{1}{2}\int_V c_{ijkl}\, \varepsilon_{kl} \varepsilon_{ij} \, dV$, $\delta \mathrm{u}_{i,j} = \delta\varepsilon_{ij}$, $\delta\Phi_{,i} = -\delta E_i$, $T = \varphi - a_{ij}\varphi_{,ij}$.

The equation (11) formulated the uncoupled problem of anisotropic piezothermoelastic with two temperature and fractional order derivative where φ and Φ are known functions. In this case, when we take into account the coupling of the deformation field with the temperature, there arises the necessity of considering an additional relation characterizing the phenomenon of the thermal conductivity. Following Biot [28] we define a vector $\boldsymbol{J}$ connected with the entropy through the relation

$$\rho S = -J_{i,i}. \tag{12}$$

Equations (2) and (12) implies

$$q_i = \dot{J}_i T_0 .$$

Combination of equations (2), (3), (7), and (12) yield

$$T_0 L_{ij} \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) J_i + \varphi_{,j} = 0, \tag{13}$$

$$-J_{i,i} = \alpha_{ij}\varepsilon_{ij} + \tau_i E_i + rT, \tag{14}$$

where L_{ij} the resistivity matrix, is the inverse of the thermal conductivity K_{ij}. Multiplying both sides of the equation (13) by δJ_j and integrating over the region of the body, gives

$$\int_V \left[\varphi_{,j} + T_0 L_{ij} \left(\frac{\partial J_i}{\partial t} + \tau_q \frac{\partial^{\alpha+1} J_i}{\partial t^{\alpha+1}} \right) \right] \delta J_j \, dV = 0. \tag{15}$$

Also, we have,

$$\int_V \varphi_{,j} \delta J_j dV = \int_V \left(\varphi \delta J_j \right)_{,j} dV - \int_V \varphi \delta J_{j,j} dV. \tag{16}$$

Applying the divergence theorem defined by

$$\int_V \left(\varphi \delta J_j \right)_{,j} dV = \int_A \left(\varphi \delta J_j \right) n_j dA, \tag{17}$$

in the equation (16), yield

$$\int_V \varphi_{,j} \delta J_j dV = \int_A \left(\varphi \delta J_j \right) n_j dA - \int_V \varphi \delta J_{j,j} dV. \tag{18}$$

Substituting equation (18) in equation (15), we obtain

$$\int_A \left(\varphi \delta J_j \right) n_j dA - \int_V \varphi \delta J_{j,j} dV + T_0 \int_V L_{ij} \left(\frac{\partial J_i}{\partial t} + \tau_q \frac{\partial^{\alpha+1} J_i}{\partial t^{\alpha+1}} \right) \delta J_j dV = 0. \tag{19}$$

Making use of equation (14) in equation (19), yield the second variational equation

$$\int_A \left(\varphi \delta J_j \right) n_j dA + \int_V \varphi \alpha_{ij} \delta \varepsilon_{ij} dV + \int_V \varphi \tau_j \delta E_j dV - r \int_V a_{ij} \varphi_{,ij} \varphi dV + \delta(M+H) = 0, \tag{20}$$

where δM is defined by

$$\delta M = r \int_V \varphi \delta \varphi dV, \tag{21}$$

and δH is

$$\delta H = T_0 \int_V L_{ij} \left(\frac{\partial J_i}{\partial t} + \tau_q \frac{\partial^{\alpha+1} J_i}{\partial t^{\alpha+1}} \right) \delta J_j dV. \tag{22}$$

Thus, we obtain the variational principle in the following form

$$\begin{aligned} \delta\left(W + M + H\right) = & \int_V \rho (F_i - \ddot{u}_i) \delta u_i \, dV + \int_A h_i \delta u_i \, dA + \int_A c_0 \delta \Phi \, dA + \int_V D_i \delta E_i dV + \int_V e_{ijk} E_k \delta \varepsilon_{ij} dV \\ & - \int_A \left(\varphi \delta J_j \right) n_j dA - \int_V \varphi \tau_j \delta E_j dV - \int_V a_{ij} \alpha_{ij} \varphi_{,ij} \delta \varepsilon_{ij} dV + r \int_V a_{ij} \varphi \delta \varphi_{,ij} dV. \end{aligned} \tag{23}$$

On the right-hand side of equation (23), we find all the causes, the mass forces, inertial forces, the surface forces, the heating, the electric potential on the surface A bounding the body**.**

Mixed initial boundary value problem. For the mixed initial boundary value problem we assume, that $\bar{V}$ denotes the closure of an open, bounded, and connected set characterizing anisotropic piezothermoelastic solid with two-temperature such that the constitutive and field equations are defined on $\bar{V} = V \times [0, \infty)$. Let ∂V denotes the boundary of $\bar{V}$. Let ∂V_i $(i = 1, 2, 3, 4, 5, 6)$ denotes the subsets of ∂V such that $\partial V = \partial V_1 \bigcup \partial V_2 = \partial V_3 \bigcup \partial V_4 = \partial V_5 \bigcup \partial V_6$ and $\partial V_1 \bigcap \partial V_2 = \partial V_3 \bigcap \partial V_4 = \partial V_5 \bigcap \partial V_6 = \varnothing$ with initial conditions on the surface at $t = 0$,

$$\mathrm{u}_i = \mathrm{u}_i^{\,0}\ , \dot{\mathrm{u}}_i = \dot{\mathrm{u}}_i^{\,0}\ , \varphi = \varphi^0, \Phi = \Phi^0, q_i = q_i^{\,0}, \dot{q}_i = \dot{q}_i^{\,0} \ \text{ on } V, \tag{24}$$

and boundary conditions on the surface are

$$\begin{gathered}\mathrm{u}_i = \mathrm{u}_{i1} \text{ on } \partial V_1 \times [0, \infty), h_i = \sigma_{ij} n_j = h_{i1} \text{ on } \partial V_2 \times [0, \infty), q = q_i n_i = q_{i1} \text{ on } \partial V_3 \times [0, \infty), \\ \varphi = \varphi_1 \text{ on } \partial V_4 \times [0, \infty), c_0 = D_i n_i = c_{01} \text{ on } \partial V_5 \times [0, \infty),\ \Phi = \Phi_1 \text{ on } \partial V_6 \times [0, \infty).\end{gathered} \tag{25}$$

where $\mathrm{u}_i^{\,0}, \dot{\mathrm{u}}_i^{\,0}, \varphi^0, \Phi^0, q_i^{\,0}, \dot{q}_i^{\,0}$ are the known initial displacements, temperature, electric potential, heat flux, and heat flux rate, respectively and $\mathrm{u}_{i1}, h_{i1}, \varphi_1, q_{i1}, c_{01}, \Phi_1$ denotes the surface displacement, tractions, temperature, heat flux, electric charge density, and electric potential. In order to meet the smoothness requirements and the other regularity assumptions these functions are introduced as the hypothesis on the data

(i) $\mathrm{u}_i^{\,0}, \varphi^0, \Phi^0, q_i^{\,0}$ are continuous on $\bar{V} = V \times [0, \infty)$.

(ii) $\dot{\mathrm{u}}_i^{\,0}, \dot{q}_i^{\,0}$ are continuously differentiable on $\bar{V} = V \times [0, \infty)$.

(iii) $\mathrm{u}_{i1}, \varphi_1, \Phi_1$ are continuous on $\partial V_1 \times [0, \infty), \partial V_4 \times [0, \infty), \partial V_6 \times [0, \infty)$, respectively.

(iv) h_{i1}, q_{i1}, c_{01} are piecewise continuous on $\partial V_2 \times [0, \infty), \partial V_3 \times [0, \infty), \partial V_5 \times [0, \infty)$, respectively.

Further, we assume that the material constants satisfy the following inequalities

$$C_e > 0, T_0 > 0, \tau_q > 0, \rho > 0, \tag{26a}$$

and $c_{ijkl}, \alpha_{ij}, L_{ij}$ are smooth on V such that

$$C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} > 0 \text{ for all tensors } \varepsilon_{ij} \text{ and } L_{ij} \vartheta_i \vartheta_j > 0 \text{ for any real } \vartheta_i \text{ defined on } V. \tag{26b}$$

A solution of the mixed initial boundary value problem is defined as an admissible state $R = [\mathrm{u}_i, \varepsilon_{ij}, \sigma_{ij}, T, q_i, \Phi, D_i, E_i, S]$, an ordered array of functions with properties $\mathrm{u}_i \in \mathrm{C}^{2,2}$, $\sigma_{ij} \in \mathrm{C}^{1,0}, \varphi \in \mathrm{C}^{2,2}, q_i \in \mathrm{C}^{1,2}, D_i \in \mathrm{C}^{1,0}, \Phi \in \mathrm{C}^{1,0}, S \in \mathrm{C}^{0,1}$ on $V \times [0, \infty)$. The set of all admissible states is a linear space as it satisfies the addition of admissible states and scalar multiplication of an admissible state. R satisfies the equations (1)-(8), initial conditions (24), and boundary conditions (25). Now, we assume that the virtual displacements $\delta\,\mathrm{u}_i$, the virtual increment of the temperature $\delta\varphi$, etc. correspond to the increments occurring in the body. Then

$$\delta\,\mathrm{u}_i = \frac{\partial\,\mathrm{u}_i}{\partial t} dt = \dot{\mathrm{u}}_i\ dt\ \ , \delta\varphi = \frac{\partial \varphi}{\partial t} dt = \dot{\varphi} dt, \text{ etc.} \tag{27}$$

and equation (23) reduces to the following relation

$$\begin{aligned}\frac{d}{dt}\left(W + M + H\right) = & \int_V \rho F_i\ \dot{\mathrm{u}}_i\ dV - \int_V \rho\,\ddot{\mathrm{u}}_i\ \dot{\mathrm{u}}_i\ dV + \int_A h_i\ \dot{\mathrm{u}}_i\ dA + \int_A c_0 \dot{\Phi}\ dA + \int_V D_i \dot{E}_i dV \\ & + \int_V e_{ijk}\ E_k \dot{\varepsilon}_{ij} dV - \int_A \left(\varphi \dot{J}_j\right) n_j dA - \int_V \varphi \tau_j \dot{E}_j dV - \int_V a_{ij} \alpha_{ij}\ \varphi_{,ij} \dot{\varepsilon}_{ij} dV \\ & + r \int_V a_{ij} \varphi \dot{\varphi}_{,ij} dV.\end{aligned} \tag{28}$$

Now,

$$\int_V \rho \ddot{u}_i \dot{u}_i \, dV = \frac{\partial K}{\partial t}, \tag{29}$$

where $K = \frac{1}{2}\int_V \rho \dot{u}_i \dot{u}_i \, dV$, is the kinetic energy of the body enclosed by the volume V.

Using equation (29) in the equation (28), we obtain

$$\begin{aligned} \frac{d}{dt}\left(W + H + K + \frac{1}{2}\int_V r\varphi^2 \, dV\right) = \int_V \rho F_i \, \dot{u}_i \, dV + \int_A h_i \dot{u}_i \, dA + \int_A c_0 \dot{\Phi} \, dA + \int_V e_{ijk} \, E_k \dot{\varepsilon}_{ij} dV \\ + \int_V D_i \dot{E}_i dV - \int_A \left(\varphi \dot{J}_j\right) n_j dA - \int_V \varphi \tau_j \dot{E}_j dV - \int_V a_{ij}\alpha_{ij} \, \varphi_{,ij} \dot{\varepsilon}_{ij} dV + r\int_V a_{ij}\varphi\dot{\varphi}_{,ij} dV. \end{aligned} \tag{30}$$

The above equation is the basis for the proof of the following uniqueness theorem.

Theorem1: The mixed initial boundary value problem with two-temperature theory has only one solution of the equations (6)-(8), subject to the initial conditions (24) and boundary conditions (25).

Proof: Let $u_i^{(1)}, \varphi^{(1)}, \Phi^{(1)}$ and $u_i^{(2)}, \varphi^{(2)}, \Phi^{(2)}$ be two solutions sets of equations (1)-(8). Let us take

$$u_i = u_i^{(1)} - u_i^{(2)}, \; \varphi = \varphi^{(1)} - \varphi^{(2)}, \; \Phi = \Phi^{(1)} - \Phi^{(2)}. \tag{31}$$

The functions u_i, φ and Φ satisfy the governing equations with zero body forces and homogeneous initial and boundary conditions. Thus, these functions satisfy an equation similar to the equation (30) with zero right hand side, that is,

$$\frac{d}{dt}\left(W + H + K + \frac{1}{2}\int_V r\varphi^2 dV\right) = 0. \tag{32}$$

Since we have

$L_{ij} = L_{ji}$,

therefore, from equation (22) and with the aid of the definition of fractional order derivative given by Riemann Liouville i.e.

${}_aD_t^{\alpha}(f(t)) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \; (n-1) \le \alpha < n,$ where n is an integer and α is a real number, we obtain

$$\frac{dH}{dt} = T_0 \int_V L_{ij} \dot{J}_i \dot{J}_j dV + \frac{T_0 \tau_q}{\Gamma(1-\alpha)} \frac{d}{dt}\left[\int_V L_{ij}\left(\int_0^t \frac{\dot{J}_i(\tau)}{(t-\tau)^{\alpha}} d\tau\right)\dot{J}_j dV\right]. \tag{33}$$

Substitution of equation (33) in the equation (32), yields

$$\frac{d}{dt}\left(W + K + \frac{1}{2}\int_V r\varphi^2 \, dV + \frac{T_0\tau_q}{\Gamma(1-\alpha)}\left[\int_V L_{ij}\left(\int_0^t \frac{\dot{J}_i(\tau)}{(t-\tau)^{\alpha}} d\tau\right)\dot{J}_j dV\right]\right) + T_0\int_V L_{ij}\dot{J}_i \dot{J}_j dV = 0. \tag{34}$$

Integrating equation (34) twice with respect to time variable over the interval $(0,t)$ and using homogeneous initial conditions we thus, see that

$$\begin{aligned} \int_0^t \left(W + K + \frac{1}{2}\int_V r\varphi^2 dV + \frac{T_0\tau_q}{\Gamma(1-\alpha)}\left[\int_V L_{ij}\left(\int_0^t \frac{\dot{J}_i(\tau)}{(t-\tau)^{\alpha}} d\tau\right)\dot{J}_j dV\right]\right) d\zeta + \\ + T_0 \int_0^t \int_0^t \int_V L_{ij}\dot{J}_i \dot{J}_j dV d\zeta d\zeta = 0. \end{aligned} \tag{35}$$

We also note that the expression $\int_V r\varphi^2 dV$ occurring in the equation (35) is always positive, since by the laws of thermodynamics Nowacki [7], $0 < a^2 < rT$. Following Kothari and Mukhopadhyay [29], the inequalities (26) implies,

$$\int_0^t \left(W + K + \frac{1}{2}\int_V r\varphi^2 dV + \frac{T_0\tau_q}{\Gamma(1-\alpha)}\left[\int_V L_{ij}\left(\int_0^t \frac{\dot{J}_i(\tau)}{(t-\tau)^\alpha} d\tau \right)\dot{J}_j dV \right]\right) d\zeta + T_0 \int_0^t\int_0^t\int_V L_{ij}\dot{J}_i\dot{J}_j dV d\zeta d\zeta. \quad (36)$$

The component in each integrand of expression (36) is non-negative. Thus, we conclude that each term in the expression (36) must be zero, which implies that $\mathrm{u}_i = \Phi = \varphi = \varepsilon_{ij} = \sigma_{ij} = 0$ on $V \times [0,\infty)$. This proves the uniqueness of the solution to the complete system of field equations subjected to the displacement- electric potential-temperature, initial and boundary conditions.

Alternative formulation: Following Nickel and Sackman [19] and Iesan [20], an alternative approach to solving the mixed initial boundary problem is formulated by incorporating the initial conditions explicitly into the field equations. Let χ, ψ be the two functions defined on $\overline{V} = V \times [0,\infty)$, and their convolution is defined as

$$[\chi * \psi] = \int_0^t \chi(x, t-s)\psi(x,s)\, ds, \quad (x,t) \in V \times [0,\infty), \quad (37)$$

and satisfy the following properties

(i) $\chi * \psi = \psi * \chi$.

(ii) $\chi * (\psi + \tau) = (\chi * \psi) + (\chi * \tau)$.

(iii) $\chi * (\psi * \tau) = (\chi * \psi) * \tau$.

(iv) $\chi * \psi = 0 \Rightarrow \chi = 0$ or $\psi = 0$.

Consider Laplace transform of equations (2), (6). Using initial conditions (24), we obtain

$$\overline{q}_{i,i} - \rho C_e\left(\varphi^0 - a_{ij}\varphi^0_{,ij}\right) - T_0\alpha_{ij}\,\mathrm{u}^0_{i,j} + T_0\tau_i\Phi^0_{,i} + \rho T_0\overline{S} = 0, \quad (38a)$$

$$\overline{\sigma}_{ij,j} + \overline{F}_i + \rho s\,\mathrm{u}_i^0 + \rho\,\dot{\mathrm{u}}_i^0 = \rho s^2\,\overline{\mathrm{u}}_i, \quad (38b)$$

$$\overline{D}_{i,i} + \xi_{i,j}\overline{\Phi}_{,ji} - e_{ijk}\,\overline{\mathrm{u}}_{j,ki} - \tau_i\overline{T}_{,i} = 0, \quad (38c)$$

where "s" is the transformation parameter and a superimposed bar indicates the transformed function. Applying the inverse transformation, yields

$$\rho T_0 S = h - g' * q_{i,i}, \quad (39a)$$

$$\rho\,\mathrm{u}_i = g * \sigma_{ij,j} + f_i, \quad (39b)$$

$$D_{i,i} + \xi_{i,j}\Phi_{,ji} - e_{ijk}\,\mathrm{u}_{j,ki} - \tau_i T_{,i} = 0, \quad (39c)$$

where g, g', f_i and h are

$$g(t) = t,\ g'(t) = 1, \quad t \in [0,\infty), \quad (40a)$$

$$f_i(x,t) = g * F_i(x,t) + \rho t\,\mathrm{u}_i^0(x) + \rho\,\dot{\mathrm{u}}_i^0(x), \quad (40b)$$

$$h(x) = \rho C_e\left(\varphi^0 - a_{ij}\varphi^0_{,ij}\right)(x) + \alpha_{ij}T_0\mathrm{u}^0_{i,j}(x) - \tau_i T_0\Phi^0_{,i}(x), \quad (40c)$$

with the aid of equations (38)-(40), alternative formulations of the problem can be made.

Theorem 2: Let $\mathrm{u}_i \in C^{0,2}$, $\sigma_{ij} \in C^{0,1}$, and suppose $\sigma_{ij} = \sigma_{ji}$. Then, $\mathrm{u}_i, \sigma_{ji}$ satisfy the equations of motion (6) as well as the initial conditions (24) on u_i iff

$$g * \sigma_{ij,j} + f_i = \rho\,\mathrm{u}_i, \text{ on } V \times [0,\infty). \quad (41)$$

Following Gurtin [30], the proof of this theorem is trivial.

Theorem 3: Let $S \in C^{0,2}$, $q_i \in C^{0,1}$, suppose the equation (3) holds for $t=0$. Then, S, q_i satisfy the energy equation (2) as well as the initial conditions (24) iff
$$h - g' * q_{i,i} = \rho T_0 S \text{ on } V \times [0,\infty). \tag{42}$$

Proof: Suppose equations (2), (3) and initial conditions (24) hold for $t=0$. Then, equations (2) and (40a) implies
$$-g' * q_{i,i} = \rho T_0 \int_0^t \dot{S}(x,\zeta) d\zeta = \rho T_0 S(x,t) - \rho T_0 S(x,0).$$

Since at $t=0$, $S(x,0) = \rho C_e \left(\varphi^0 - a_{ij}\varphi^0_{,ij}\right)(x) - \alpha_{ij} T_0 \mathrm{u}^0_{i,j}(x) + \tau_i T_0 \Phi^0_{,i}(x)$. Therefore,
$$-g' * q_{i,i} = \rho T_0 S(x,t) - \rho C_e \left(\varphi^0 - a_{ij}\varphi^0_{,ij}\right)(x) - \alpha_{ij} T_0 \mathrm{u}^0_{i,j}(x) + \tau_i T_0 \Phi^0_{,i}(x). \tag{43}$$

Then, by equation (40c), $h - g' * q_{i,i} = \rho T_0 S(x,t)$.

Hence, equation (42) is proved on $V \times [0,\infty)$. Conversely, suppose equation (42) holds. Then, by reversing the argument, and utilizing the equations (40), it is directly verified that S, q_i meet energy equation (2). Since equations (3), (40a), (40b), (40c), (42) imply initial conditions (24) on φ, and therefore the proof of the theorem is complete.

Theorem 4: Let $R = [\mathrm{u}_i, \varepsilon_{ij}, \sigma_{ij}, \varphi, q_i, S, D_i, E_i, \Phi]$ be an admissible state. Then R is a solution to the mixed initial boundary value problem of piezothermoelasticity with two-temperature iff it meets the equations (1), (2), (4), (39a), (39b), (39c) and the boundary conditions (25).

The result of this theorem is the trivial consequence of Theorem 2 and Theorem 3.

This provides an alternative formulation of the solution of the mixed initial boundary value problem by incorporating initial conditions explicitly into field equations.

4. Reciprocity Theorem

We shall consider a homogeneous anisotropic piezothermoelastic body with two-temperature occupying the region V and bounded by the surface A. We assume that the stresses σ_{ij} and the strains ε_{ij} are continuous together with their first derivatives whereas the displacements u_i, temperature φ and the electrical potential Φ are continuous and have continuous derivatives up to second order, for $x \in V + A,\ t > 0$. The components of surface traction, the normal component of the heat flux, the normal component of the electric displacement at regular points of ∂V, are given by
$$h_i = \sigma_{ij} n_j,\ q = q_i n_i,\ c_0 = D_i n_i,\ i = 1,2,3, \tag{44}$$
respectively. To the system of field equations, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:
$$\mathrm{u}_i(x,t) = \mathrm{u}_{i1}(x,t),\ \varphi(x,t) = \varphi_1(x,t),\ \Phi(x,t) = \Phi_1(x,t), \tag{45}$$
for all $x \in A,\ t > 0$ and the homogeneous initial conditions
$$\left.\begin{aligned} &\mathrm{u}_i(x,0) = \dot{\mathrm{u}}_i(x,0) = 0,\ \varphi(x,0) = \dot{\varphi}(x,0) = 0, \\ &\text{and } \Phi(x,0) = \dot{\Phi}(x,0) = 0, \text{ for all } x \in V, t = 0. \end{aligned}\right\} \tag{46}$$

We derive the dynamic reciprocity relationship for a generalised piezothermoelastic bounded body V with two- temperature, which satisfies equations (1)-(8), the boundary conditions (45) and the homogeneous initial conditions (46), and are subjected to the action of body forces $F_i(x,t)$, surface traction $h_i(x,t)$, the heat flux $q(x,t)$, and the surface charge density $c_0(x,t)$. We define the Laplace transform as

$$\overline{f}(x,s)=L(f(x,t))=\int_0^\infty f(x,t)e^{-st}dt. \tag{47}$$

Applying the Laplace transform defined by the equation (47) on the equations (1)-(8) and omitting the bars for simplicity, we obtain

$$\sigma_{ij}=c_{ijkl}\,\varepsilon_{kl}-e_{ijk}E_k-\alpha_{ij}T, \tag{48}$$

$$-q_{i,i}=\rho T_0 sS, \tag{49}$$

$$\rho S=\alpha_{ij}\varepsilon_{ij}+\tau_i E_i+rT, \tag{50}$$

$$\sigma_{ij,j}+\rho\,\mathrm{F}_i=\rho s^2\mathrm{u}_i, \tag{51}$$

$$-K_{ij}\,\varphi_{,j}=(1+\tau_q s^{\alpha})\,q_i, \tag{52}$$

$$D_{i,i}=0, \tag{53}$$

$$D_i=\xi_{ij}E_j+e_{ijk}\varepsilon_{jk}+\tau_i T, \tag{54}$$

$$E_i=-\Phi_{,i},(i,j,k,l=1,2,3). \tag{55}$$

We now consider two problems where applied body forces, surface temperature, and the electric potential are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $\mathrm{u}_i^{(1)},\varepsilon_{ij}^{(1)},\sigma_{ij}^{(1)},\varphi^{(1)},\Phi^{(1)}$ for the first problem and $\mathrm{u}_i^{(2)},\varepsilon_{ij}^{(2)},\sigma_{ij}^{(2)},\varphi^{(2)},\Phi^{(2)}$ for the second problem. Each set of variables satisfies the equations (48) - (55). Using the assumption $\sigma_{ij}=\sigma_{ji}$, we obtain

$$\int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}dV=\int_V \sigma_{ij}^{(1)}\mathrm{u}_{i,j}^{(2)}dV=\int_V\left(\sigma_{ij}^{(1)}\mathrm{u}_i^{(2)}\right)_{,j}dV-\int_V \sigma_{ij,j}^{(1)}\mathrm{u}_i^{(2)}dV. \tag{56}$$

Using the divergence theorem in the first term of the right hand side of equation (56) yields

$$\int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}dV=\int_A \sigma_{ij}^{(1)}\mathrm{u}_i^{(2)}n_j dA-\int_V \sigma_{ij,j}^{(1)}\mathrm{u}_i^{(2)}dV. \tag{57}$$

Equation (57) with the aid of equations (44) and (51) gives

$$\int_V \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}dV=\int_A h_i^{(1)}\mathrm{u}_i^{(2)}dA-\rho\int_V s^2\mathrm{u}_i^{(1)}\mathrm{u}_i^{(2)}dV+\rho\int_V F_i^{(1)}\mathrm{u}_i^{(2)}dV. \tag{58}$$

A similar expression is obtained for the integral $\int_V \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}dV$, from which together with the equation (58), it follows that

$$\int_V (\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})dV=\int_A (h_i^{(1)}\mathrm{u}_i^{(2)}-h_i^{(2)}\mathrm{u}_i^{(1)})dA+\rho\int_V (F_i^{(1)}\mathrm{u}_i^{(2)}-F_i^{(2)}\mathrm{u}_i^{(1)})dV. \tag{59}$$

Now multiplying equation (48) by $\varepsilon_{ij}^{(2)}$ and $\varepsilon_{ij}^{(1)}$ for the first and second problems respectively, subtracting and integrating over the region V, we obtain

$$\int_V (\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})dV=\int_V c_{ijkl}\,(\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)}-\varepsilon_{kl}^{(2)}\varepsilon_{ij}^{(1)})dV-\int_V e_{ijk}(\Phi_{,k}^{(2)}\varepsilon_{ij}^{(1)}-\Phi_{,k}^{(1)}\varepsilon_{ij}^{(2)})dV$$
$$-\int_V \alpha_{ij}(\varphi^{(1)}\varepsilon_{ij}^{(2)}-\varphi^{(2)}\varepsilon_{ij}^{(1)})dV+\int_V a_{ij}\alpha_{ij}(\varphi_{,ij}^{(1)}\varepsilon_{ij}^{(2)}-\varphi_{,ij}^{(2)}\varepsilon_{ij}^{(1)})dV.$$

Using the symmetry properties of c_{ijkl}, we obtain

$$\int_V (\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}-\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)})dV=-\int_V e_{ijk}(\Phi_{,k}^{(2)}\varepsilon_{ij}^{(1)}-\Phi_{,k}^{(1)}\varepsilon_{ij}^{(2)})dV-\int_V \alpha_{ij}(\varphi^{(1)}\varepsilon_{ij}^{(2)}-\varphi^{(2)}\varepsilon_{ij}^{(1)})dV$$
$$+\int_V a_{ij}\alpha_{ij}(\varphi_{,ij}^{(1)}\varepsilon_{ij}^{(2)}-\varphi_{,ij}^{(2)}\varepsilon_{ij}^{(1)})dV. \tag{60}$$

Equating equations (59) and (60), we get the first part of the reciprocity theorem

$$\int_A (h_i^{(1)}u_i^{(2)} - h_i^{(2)}u_i^{(1)})dA + \rho\int_V (F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)})dV = -\int_V e_{ijk}(\Phi_{,k}{}^{(2)}\varepsilon_{ij}{}^{(1)} - \Phi_{,k}{}^{(1)}\varepsilon_{ij}{}^{(2)})dV$$
$$-\int_V \alpha_{ij}(\varphi^{(1)}\varepsilon_{ij}{}^{(2)} - \varphi^{(2)}\varepsilon_{ij}{}^{(1)})dV + \int_V a_{ij}\alpha_{ij}(\varphi_{,ij}{}^{(1)}\varepsilon_{ij}{}^{(2)} - \varphi_{,ij}{}^{(2)}\varepsilon_{ij}{}^{(1)})dV. \quad (61)$$

This contains the mechanical causes of motion F_i and h_i.

Now, taking the divergence of both sides of equation (52) and using equations (49), (50), we arrive at the equation of heat conduction, namely

$$\frac{\partial}{\partial x_i}\left(K_{ij}\varphi_{,j}\right) = \left(s + \tau_q s^{\alpha+1}\right)T_0(\alpha_{ij}\varepsilon_{ij} + \tau_i E_i + rT). \quad (62)$$

To derive the second part, multiplying equation (62) by $\varphi^{(2)}$ and $\varphi^{(1)}$ for the first and the second problems respectively, subtracting and integrating over V, we get

$$\int_V \left(\left(K_{ij}\varphi_{,j}{}^{(1)}\right)_{,i}\varphi^{(2)} - \left(K_{ij}\varphi_{,j}{}^{(2)}\right)_{,i}\varphi^{(1)}\right)dV = \Omega_1 T_0\int_V \alpha_{ij}\left(\varepsilon_{ij}{}^{(1)}\varphi^{(2)} - \varepsilon_{ij}{}^{(2)}\varphi^{(1)}\right)dV$$
$$-\Omega_1 T_0\int_V \alpha_{ij}a_{ij}\left(\varepsilon_{ij}{}^{(1)}\varphi_{,ij}{}^{(2)} - \varepsilon_{ij}{}^{(2)}\varphi_{,ij}{}^{(1)}\right)dV + \Omega_1 T_0\int_V \tau_i\left(E_i^{(1)}\varphi^{(2)} - E_i^{(2)}\varphi^{(1)}\right)dV$$
$$-\Omega_1 T_0\int_V r a_{ij}\left(E_i^{(1)}\varphi_{,ij}{}^{(2)} - E_i^{(2)}\varphi_{,ij}{}^{(1)}\right)dV, \quad (63)$$

where, $\Omega_1 = s + \tau_q s^{\alpha+1}$. Now,

$$\left(K_{ij}\varphi_{,j}{}^{(1)}\right)_{,i}\varphi^{(2)} = \left(K_{ij}\varphi_{,j}{}^{(1)}\varphi^{(2)}\right)_{,i} - K_{ij}\varphi_{,j}{}^{(1)}\varphi_{,i}{}^{(2)} \text{ and}$$
$$\left(K_{ij}\varphi_{,j}{}^{(2)}\right)_{,i}\varphi^{(1)} = \left(K_{ij}\varphi_{,j}{}^{(2)}\varphi^{(1)}\right)_{,i} - K_{ij}\varphi_{,j}{}^{(2)}\varphi_{,i}{}^{(1)}. \quad (64)$$

Equation (63) with the help of equations (44), (45), (64) and the divergence theorem can be written as

$$\int_A \Omega_1\left(q^{(1)}\varphi^{(2)} - q^{(2)}\varphi^{(1)}\right)dA = -\Omega_1 T_0\int_V \alpha_{ij}\left(\varepsilon_{ij}{}^{(1)}\varphi^{(2)} - \varepsilon_{ij}{}^{(2)}\varphi^{(1)}\right)dV$$
$$+\Omega_1 T_0\int_V \alpha_{ij}a_{ij}\left(\varepsilon_{ij}{}^{(1)}\varphi_{,ij}{}^{(2)} - \varepsilon_{ij}{}^{(2)}\varphi_{,ij}{}^{(1)}\right)dV + \Omega_1 T_0\int_V \tau_i\left(\Phi_{,i}{}^{(1)}\varphi^{(2)} - \Phi_{,i}{}^{(2)}\varphi^{(1)}\right)dV$$
$$-\Omega_1 T_0\int_V r a_{ij}\left(\Phi_{,i}{}^{(1)}\varphi_{,ij}{}^{(2)} - \Phi_{,i}{}^{(2)}\varphi_{,ij}{}^{(1)}\right)dV. \quad (65)$$

This constitutes the second part of the reciprocity theorem which contains the thermal causes of motion φ_1 and q. To derive the third part, multiplying equation (54) by $E_i^{(2)}$ and $E_i^{(1)}$ for the first and the second problems respectively, subtracting and integrating over V, we get

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = \int_V \xi_{ij}(E_j{}^{(1)}E_i^{(2)} - E_j{}^{(2)}E_i^{(1)})dV + \int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}E_i^{(2)} - \varepsilon_{jk}{}^{(2)}E_i^{(1)})dV +$$
$$\int_V \tau_i(\varphi^{(1)}E_i^{(2)} - \varphi^{(2)}E_i^{(1)})dV - \int_V \tau_i a_{ij}(\varphi_{,ij}{}^{(1)}E_i^{(2)} - \varphi_{,ij}{}^{(2)}E_i^{(1)})dV.$$

Since $\xi_{ij} = \xi_{ji}$, therefore, we have

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = \int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}E_i^{(2)} - \varepsilon_{jk}{}^{(2)}E_i^{(1)})dV + \int_V \tau_i(\varphi^{(1)}E_i^{(2)} - \varphi^{(2)}E_i^{(1)})dV$$
$$-\int_V \tau_i a_{ij}(\varphi_{,ij}{}^{(1)}E_i^{(2)} - \varphi_{,ij}{}^{(2)}E_i^{(1)})dV. \quad (66)$$

Equation (66) with the aid of equation (55) yields

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = -\int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}\Phi_{,i}{}^{(2)} - \varepsilon_{jk}{}^{(2)}\Phi_{,i}{}^{(1)})dV - \int_V \tau_i(\varphi^{(1)}\Phi_{,i}{}^{(2)} - \varphi^{(2)}\Phi_{,i}{}^{(1)})dV$$
$$+\int_V \tau_i a_{ij}(\varphi_{,ij}{}^{(1)}\Phi_{,i}{}^{(2)} - \varphi_{,ij}{}^{(2)}\Phi_{,i}{}^{(1)})dV. \tag{67}$$

Also, using (55) with equation (67), we have

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = \int_V \left(D_i^{(2)}\Phi_{,i}{}^{(1)} - D_i^{(1)}\Phi_{,i}{}^{(2)}\right)dV. \tag{68}$$

Now,

$$\left.\begin{aligned} D_i^{(2)}\Phi_{,i}{}^{(1)} &= \left(D_i^{(2)}\Phi^{(1)}\right)_{,i} - D_{i,i}{}^{(2)}\Phi^{(1)}, \\ D_i^{(1)}\Phi_{,i}{}^{(2)} &= \left(D_i^{(1)}\Phi^{(2)}\right)_{,i} - D_{i,i}{}^{(1)}\Phi^{(2)}. \end{aligned}\right\} \tag{69}$$

Using equation (69), (54) and divergence theorem in equation (68), we obtain

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = \int_V \left(\left(D_i^{(2)}\Phi^{(1)}\right)_{,i} - \left(D_i^{(1)}\Phi^{(2)}\right)_{,i}\right)dV + \int_V \left(D_{i,i}{}^{(1)}\Phi^{(2)} - D_{i,i}{}^{(2)}\Phi^{(1)}\right)dV$$
$$= \int_A \left(\left(D_i^{(2)}\Phi^{(1)}n_i\right) - \left(D_i^{(1)}\Phi^{(2)}n_i\right)\right)dA. \tag{70}$$

With the aid of equation (44), we obtain

$$\int_V \left(D_i^{(1)}E_i^{(2)} - D_i^{(2)}E_i^{(1)}\right)dV = \int_A \left(\left(c_0{}^{(2)}\Phi^{(1)}\right) - \left(c_0{}^{(1)}\Phi^{(2)}\right)\right)dA. \tag{71}$$

From equations (67) and (71), we have

$$\int_A \left(\left(c_0{}^{(1)}\Phi^{(2)}\right) - \left(c_0{}^{(2)}\Phi^{(1)}\right)\right)dA = \int_V e_{ijk}(\varepsilon_{jk}{}^{(1)}\Phi_{,i}{}^{(2)} - \varepsilon_{jk}{}^{(2)}\Phi_{,i}{}^{(1)})dV + \int_V \tau_i(\varphi^{(1)}\Phi_{,i}{}^{(2)} - \varphi^{(2)}\Phi_{,i}{}^{(1)})dV$$
$$-\int_V \tau_i a_{ij}(\varphi_{,ij}{}^{(1)}\Phi_{,i}{}^{(2)} - \varphi_{,ij}{}^{(2)}\Phi_{,i}{}^{(1)})dV. \tag{72}$$

This constitutes the third part of the reciprocity theorem which contains the electric potential Φ and surface charge density c_0. Combining equations (61), (65), and (72) we obtain

$$\Omega_1 T_0\left[\int_A (h_i^{(1)}\mathrm{u}_i^{(2)} - h_i^{(2)}\mathrm{u}_i^{(1)})dA + \rho\int_V (F_i^{(1)}\mathrm{u}_i^{(2)} - F_i^{(2)}\mathrm{u}_i^{(1)})dV + \int_A \left(c_0{}^{(1)}\Phi^{(2)} - c_0{}^{(2)}\Phi^{(1)}\right)dA\right.$$
$$\left.+\int_V (\tau_i - r)a_{ij}(\varphi_{,ij}{}^{(1)}\Phi_{,i}{}^{(2)} - \varphi_{,ij}{}^{(1)}\Phi_{,i}{}^{(2)})dV\right] + \int_A \Omega_1\left(q^{(1)}\varphi_1^{(2)} - q^{(2)}\varphi_1^{(1)}\right)dA = 0. \tag{73}$$

This is the general reciprocity theorem in the Laplace transform domain. For applying inverse Laplace transform on the equations (61), (65), (72), and (73), we shall use the convolution theorem

$$L^{-1}(F(s)G(s)) = \int_0^t f(t-\zeta)g(\zeta)d\zeta = \int_0^t g(t-\zeta)f(\zeta)d\zeta, \tag{74}$$

and the symbolic notation

$$\wedge(f) = 1 + \tau_q\frac{\partial^\alpha f(x,\zeta)}{\partial\zeta^\alpha}. \tag{75}$$

Equations (61), (65), (72) and (73) with the aid of equation (75) yield the first, second, third, and fourth parts of the reciprocity theorem in the final form

$$\int_A\int_0^t (h_i^{(1)}(x,t-\zeta)\mathrm{u}_i^{(2)}(x,\zeta)d\zeta dA+\rho\int_V\int_0^t (F_i^{(1)}(x,t-\zeta)\mathrm{u}_i^{(2)}(x,\zeta)d\zeta dV$$

$$-\int_V\int_0^t e_{ijk}(\Phi_{,k}^{(1)}(x,t-\zeta)\varepsilon_{ij}^{(2)}(x,\zeta))d\zeta dV-\int_V \alpha_{ij}\varphi^{(1)}(x,t-\zeta)\varepsilon_{ij}^{(2)}(x,\zeta)dV$$

$$+\int_V a_{ij}\alpha_{ij}\varphi_{,ij}^{(1)}(x,t-\zeta)\varepsilon_{ij}^{(2)}(x,\zeta)dV=S_{21}^{12}, \qquad (76)$$

$$\int_A\int_0^t\left(q^{(1)}(x,t-\zeta)\frac{\partial\wedge\varphi^{(2)}(x,\zeta)}{\partial\zeta}\right)d\zeta dA+T_0\int_V\int_0^t\alpha_{ij}\left(\varepsilon_{ij}^{(1)}(x,t-\zeta)\frac{\partial\wedge(\varphi^{(2)}(x,\zeta))}{\partial\zeta}\right)d\zeta dV$$

$$-T_0\int_V\int_0^t\alpha_{ij}a_{ij}\left(\varepsilon_{ij}^{(1)}(x,t-\zeta)\frac{\partial\wedge(\varphi_{,ij}^{(2)}(x,\zeta))}{\partial\zeta}\right)d\zeta dV-T_0\int_V\int_0^t\tau_i\left(\Phi_{,i}^{(1)}(x,t-\zeta)\frac{\partial\wedge(\varphi^{(2)}(x,\zeta))}{\partial\zeta}\right)d\zeta dV$$

$$+T_0\int_V\int_0^t\tau_i a_{ij}\left(\Phi_{,i}^{(1)}(x,t-\zeta)\frac{\partial\wedge(\varphi_{,ij}^{(2)}(x,\zeta))}{\partial\zeta}\right)d\zeta dV=S_{21}^{12}, \qquad (77)$$

and,

$$\int_A\int_0^t c_0^{(1)}(x,t-\zeta)\Phi^{(2)}(x,\zeta)\,d\zeta dA+\int_V\int_0^t e_{ijk}\Phi_{,i}^{(1)}(x,t-\zeta)\varepsilon_{jk}^{(2)}(x,\zeta)d\zeta dV$$

$$+\int_V\int_0^t\tau_i\,\Phi_{,i}^{(1)}(x,t-\zeta)\varphi^{(2)}(x,\zeta)d\zeta dV+\int_V\int_0^t\tau_i\,a_{ij}\Phi_{,i}^{(1)}(x,t-\zeta)\varphi_{,ij}^{(2)}(x,\zeta)d\zeta dV=S_{21}^{12}. \qquad (78)$$

Here, S_{21}^{12} indicates the same expression as on the left-hand side except that the superscripts (1) and (2) are interchanged. Finally, equation (73) with the aid of equation (74) gives the general reciprocity theorem in the final form

$$\int_A\int_0^t h_i^{(1)}(x,t-\zeta)\frac{\partial\wedge(\mathrm{u}_i^{(2)}(x,\zeta))}{\partial\zeta}d\zeta dA+\rho\int_V\int_0^t F_i^{(1)}(x,t-\zeta)\frac{\partial\wedge(\mathrm{u}_i^{(2)}(x,\zeta))}{\partial\zeta}d\zeta dV$$

$$+\int_A\int_0^t c_0^{(1)}(x,t-\zeta)\frac{\partial\wedge(\Phi^{(2)}(x,\zeta))}{\partial\zeta}d\zeta dA+\int_V\int_0^t(\tau_i-r)a_{ij}\varphi_{,ij}^{(1)}(x,t-\zeta)\frac{\partial\wedge(\Phi_{,i}^{(2)}(x,\zeta))}{\partial\zeta}d\zeta dV$$

$$+\frac{1}{T_0}\int_A\int_0^t\left(q^{(1)}(x,t-\zeta)\frac{\partial\wedge\varphi^{(2)}(x,\zeta)}{\partial\zeta}\right)d\zeta dA=S_{21}^{12}. \qquad (79)$$

Special Cases. If we restrict our work to the following sub-cases with two-temperature, the constitutive relations change according to the following independent constants. The variational principle, uniqueness, and reciprocity theorems can be established by following similar steps.

Case 1: Monoclinic medium

$$\left.\begin{aligned}
\sigma_{11} &= c_{11}u_{1,1}+c_{12}u_{2,2}+c_{13}u_{3,3}+c_{14}(u_{2,3}+u_{3,2})+e_{11}\Phi_{,1}-\alpha_{11}T,\\
\sigma_{22} &= c_{12}u_{1,1}+c_{22}u_{2,2}+c_{23}u_{3,3}+c_{24}(u_{2,3}+u_{3,2})+e_{12}\Phi_{,1}-\alpha_{22}T,\\
\sigma_{33} &= c_{13}u_{1,1}+c_{23}u_{2,2}+c_{33}u_{3,3}+c_{34}(u_{2,3}+u_{3,2})+e_{13}\Phi_{,1}-\alpha_{33}T,\\
\sigma_{23} &= c_{41}u_{1,1}+c_{42}u_{2,2}+c_{43}u_{3,3}+c_{44}(u_{2,3}+u_{3,2})+e_{14}\Phi_{,1},
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\sigma_{13} &= c_{55}(u_{1,3}+u_{3,1})+e_{25}\Phi_{,2}+e_{35}\Phi_{,3},\\
\sigma_{12} &= c_{66}(u_{1,2}+u_{2,1})+e_{26}\Phi_{,2}+e_{36}\Phi_{,3},\\
D_1 &= e_{11}u_{1,1}+e_{12}u_{2,2}+e_{13}u_{3,3}+e_{14}(u_{2,3}+u_{3,2})-\xi_{11}\Phi_{,1}+\tau_1T,\\
D_2 &= e_{25}(u_{1,3}+u_{3,1})+e_{26}(u_{1,2}+u_{2,1})-\xi_{22}\Phi_{,2}-\xi_{23}\Phi_{,3}+\tau_2T,\\
D_3 &= e_{35}(u_{1,3}+u_{3,1})+e_{36}(u_{1,2}+u_{2,1})-\xi_{23}\Phi_{,2}-\xi_{33}\Phi_{,3}+\tau_3T,
\end{aligned}\right\} \tag{80}$$

Case 2: Orthotropic medium

$$\left.\begin{aligned}
\sigma_{11} &= c_{11}u_{1,1}+c_{12}u_{2,2}+c_{13}u_{3,3}+e_{31}\Phi_{,3}-\alpha_{11}T,\\
\sigma_{22} &= c_{12}u_{1,1}+c_{22}u_{2,2}+c_{23}u_{3,3}+e_{32}\Phi_{,3}-\alpha_{22}T,\\
\sigma_{33} &= c_{13}u_{1,1}+c_{23}u_{2,2}+c_{33}u_{3,3}+e_{33}\Phi_{,3}-\alpha_{33}T,\\
\sigma_{23} &= c_{44}(u_{2,3}+u_{3,2})+e_{24}\Phi_{,2},\\
\sigma_{13} &= c_{55}(u_{1,3}+u_{3,1})+e_{15}\Phi_{,1},\\
\sigma_{12} &= c_{66}(u_{1,2}+u_{2,1}),\\
D_1 &= -\xi_{11}\Phi_{,1}+e_{15}(u_{1,3}+u_{3,1}),\\
D_2 &= -\xi_{22}\Phi_{,2}+e_{24}(u_{2,3}+u_{3,2}),\\
D_3 &= -\xi_{33}\Phi_{,3}+e_{31}u_{1,1}+e_{32}u_{2,2}+e_{33}u_{3,3}+\tau_3T,
\end{aligned}\right\} \tag{81}$$

Case 3: Transversely Isotropic medium

$$\left.\begin{aligned}
\sigma_{11} &= c_{11}u_{1,1}+c_{12}u_{2,2}+c_{13}u_{3,3}+e_{13}\Phi_{,3}-\alpha_{11}T,\\
\sigma_{22} &= c_{12}u_{1,1}+c_{11}u_{2,2}+c_{13}u_{3,3}+e_{13}\Phi_{,3}-\alpha_{11}T,\\
\sigma_{33} &= c_{13}u_{1,1}+c_{13}u_{2,2}+c_{33}u_{3,3}+e_{33}\Phi_{,3}-\alpha_{33}T,\\
\sigma_{23} &= c_{44}(u_{2,3}+u_{3,2})+e_{15}\Phi_{,2},\\
\sigma_{13} &= c_{44}(u_{1,3}+u_{3,1})+e_{15}\Phi_{,1},\\
\sigma_{12} &= c_{66}(u_{1,2}+u_{2,1}),\\
D_1 &= e_{15}(u_{1,3}+u_{3,1})-\xi_{11}\Phi_{,1},\\
D_2 &= e_{15}(u_{1,2}+u_{2,1})-\xi_{11}\Phi_{,2},\\
D_3 &= e_{13}u_{1,1}+e_{13}u_{2,2}+e_{33}u_{3,3}-\xi_{33}\Phi_{,3}+\tau_3T,
\end{aligned}\right\} \tag{82}$$

5. Plane wave propagation

Formulation and solution of the problem. Substituting the constitutive relations (81) (Following Tzou and Bao [31]) into the field equations (6)-(8) without body forces, heat sources, yield

$$c_{11}u_{1,11}+c_{12}u_{2,21}+c_{13}u_{3,13}+e_{31}\Phi_{,31}+c_{66}(u_{1,22}+u_{2,12})+c_{55}(u_{1,33}+u_{3,13})-\alpha_{11}(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,1}$$
$$+e_{15}\Phi_{,13}-\rho\ddot{u}_1=0, \tag{83a}$$

$$c_{66}(u_{1,21}+u_{2,11})+c_{12}u_{1,12}+c_{22}u_{2,22}+c_{23}u_{3,32}+e_{32}\Phi_{,32}-\alpha_{22}(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,2}+c_{44}(u_{2,33}+u_{3,23})$$

$$+e_{24}\Phi_{,23}-\rho\,\ddot{u}_2=0, \tag{83b}$$

$$c_{55}(u_{1,31}+u_{3,11})+c_{44}(u_{2,32}+u_{3,22})+c_{13}\,u_{1,13}+c_{23}\,u_{2,23}+c_{33}\,u_{3,33}-\alpha_{33}(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,3}$$
$$+e_{15}\Phi_{,11}+e_{24}\Phi_{,22}\ +e_{33}\Phi_{,33}-\rho\,\ddot{u}_3=0, \tag{83c}$$

$$-\xi_{11}\Phi_{,11}+e_{15}(u_{1,31}+u_{3,11})-\xi_{22}\Phi_{,22}+e_{24}(u_{2,32}+u_{3,22})-\xi_{33}\Phi_{,33}+e_{31}\,u_{1,31}+e_{32}\,u_{2,23}+e_{33}\,u_{3,33}$$
$$+\tau_3(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,3}=0, \tag{83d}$$

$$(K_{11}\varphi_{,11}+K_{22}\varphi_{,22}+K_{33}\varphi_{,33})-\left(1+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}\right)T_0(\alpha_{11}\,\dot{u}_{1,1}+\alpha_{22}\,\dot{u}_{2,2}+\alpha_{33}\,\dot{u}_{3,3}-\tau_3\dot{\Phi}_{,3}+r(\dot{\varphi}-a_{11}\dot{\varphi}_{,11}-a_{22}\dot{\varphi}_{,22}$$
$$-a_{33}\dot{\varphi}_{,33}))=0. \tag{83e}$$

To facilitate a solution, we introduce the following dimensionless quantities

$$\left(x_i',u_i'\right)=\frac{\omega_1}{c_1}\left(x_i,u_i\right),\ \left(t',\tau_q'\right)=\omega_1\left(t,\tau_q\right),\ \left(T',\varphi'\right)=\frac{\alpha_{11}}{\rho c_1^2}\left(T,\varphi\right),\ \ \Phi'=\frac{\omega_1 e_{31}\Phi}{c_1\alpha_{11}T_0},\ a_{ij}'=\frac{\omega_1^2}{c_1^2}a_{ij},$$

where $c_1=\sqrt{\dfrac{c_{11}}{\rho}}$ and $\omega_1=\dfrac{\rho c_e c_1^2}{K_{11}}$, $i=1,2,3$.

Incorporating these dimensionless quantities, the system of equations (83), after removal of prime ('), reduces to the following form

$$c_{11}\,u_{1,11}+(c_{12}+c_{66})\,u_{2,12}+(c_{13}+c_{55})\,u_{3,13}+c_{66}\,u_{1,22}+c_{55}\,u_{1,33}+\frac{\alpha_{11}T_0}{e_{31}}(e_{31}+e_{15})\Phi_{,13}-\rho c_1^{\,2}(\varphi-a_{11}\varphi_{,11}$$
$$-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,1}-\rho c_1^{\,2}\,\ddot{u}_1=0, \tag{84a}$$

$$(c_{66}+c_{12})\,u_{1,12}+c_{66}\,u_{2,11}+c_{22}\,u_{2,22}+(c_{23}+c_{44})\,u_{3,23}+c_{44}\,u_{2,33}+\frac{\alpha_{11}T_0}{e_{31}}(e_{32}+e_{24})\Phi_{,32}-\rho c_1^{\,2}\frac{\alpha_{22}}{\alpha_{11}}(\varphi-a_{11}\varphi_{,11}$$
$$-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,2}-\rho c_1^{\,2}\,\ddot{u}_2=0, \tag{84b}$$

$$(c_{55}+c_{13})\,u_{1,31}+c_{55}\,u_{3,11}+(c_{44}+c_{23})\,u_{2,23}+c_{44}\,u_{3,22}+c_{33}\,u_{3,33}-\rho c_1^{\,2}\frac{\alpha_{33}}{\alpha_{11}}(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,3}$$
$$+\frac{\alpha_{11}T_0}{e_{31}}(e_{15}\Phi_{,11}+e_{24}\Phi_{,22}+e_{33}\Phi_{,33})-\rho c_1^{\,2}\,\ddot{u}_3=0, \tag{84c}$$

$$-\frac{\alpha_{11}T_0}{e_{31}}(\xi_{11}\Phi_{,11}+\xi_{22}\Phi_{,22}+\xi_{33}\Phi_{,33})+e_{15}(u_{1,31}+u_{3,11})+e_{24}(u_{2,32}+u_{3,22})+e_{31}\,u_{1,31}+e_{32}\,u_{2,23}+e_{33}\,u_{3,33}$$
$$+\tau_3\frac{\rho c_1^{\,2}}{\alpha_{11}}(\varphi-a_{11}\varphi_{,11}-a_{22}\varphi_{,22}-a_{33}\varphi_{,33})_{,3}=0, \tag{84d}$$

$$\frac{\rho\omega_1}{\alpha_{11}}(K_{11}\varphi_{,11}+K_{22}\varphi_{,22}+K_{33}\varphi_{,33})-\left(1+\tau_q\frac{\partial^\alpha}{\partial t^\alpha}\right)T_0(\alpha_{11}\,\dot{u}_{1,1}+\alpha_{22}\,\dot{u}_{2,2}+\alpha_{33}\,\dot{u}_{3,3}-\tau_3\dot{\Phi}_{,3}+r(\dot{\varphi}-a_{11}\dot{\varphi}_{,11}$$
$$-a_{22}\dot{\varphi}_{,22}-a_{33}\dot{\varphi}_{,33})))=0. \tag{84e}$$

For plane harmonic waves, we assume

$$\left(u_k,\Phi,\varphi\right)=\left(\bar{u}_k,\bar{\Phi},\bar{\varphi}\right)\exp\left[\boldsymbol{i}\omega\left(\frac{n_k x_k}{v}-t\right)\right],k=1,2,3, \tag{85}$$

where, ω – circular frequency, v – phase velocity of the wave propagating along the direction vector $\boldsymbol{n}$, $\bar{u}_1,\bar{u}_2,\bar{u}_3,\bar{\Phi}$ and $\bar{\varphi}$ – the undetermined amplitude vectors.

Upon using equation (85) in the set of equations (84), we obtain a system $PX=0$, where

$$P=\begin{bmatrix} t_{11}\nu^2+x_{11} & x_{12} & x_{13} & x_{14} & t_{12}\nu^2+m_{11} \\ x_{15} & t_{11}\nu^2+x_{16} & x_{17} & x_{18} & t_{14}\nu^2+m_{12} \\ x_{19} & x_{20} & t_{11}\nu^2+x_{21} & x_{22} & t_{15}\nu^2+m_{13} \\ x_{23} & x_{24} & x_{25} & x_{26} & t_{16}\nu^2+m_{14} \\ x_{27}\nu & x_{28}\nu & x_{29}\nu & x_{30}\nu & t_{17}\nu^2+m_{15} \end{bmatrix}$$

and $X=\left[\overline{\mathrm{u}}_1 \quad \overline{\mathrm{u}}_2 \quad \overline{\mathrm{u}}_3 \quad \overline{\Phi} \quad \overline{\varphi}\right]^{tr}$, "$tr$" represents the transpose of the matrix. The symbols used in the matrix P are mentioned in Appendix A. For this system to possess a non-trivial solution, the determinant of the matrix P vanishes which yields a characteristic equation in ν^2. On Solving this characteristic equation, we obtain four roots of ν^2, in which we are interested in those roots whose imaginary parts are positive. The complex phase velocities of the quasi-waves, given by $\nu_i, i=1,2,3,4$ will be varying with the direction of phase propagation. Corresponding to these roots, there exist four waves corresponding to descending order of their velocities, namely quasi longitudinal wave (qP), two quasi transverse waves (qS_1) and (qS_2), and quasi thermal wave (qT). The complex velocity of the quasi-waves, i.e. $\nu=\nu_R+\boldsymbol{i}\nu_I$, defines the phase propagation velocity $V_i=\left(\dfrac{{\nu_R}^2+{\nu_I}^2}{\nu_R}\right)_i$ and attenuation quality factor $Q_i^{-1}=\dfrac{\mathrm{Img}(1/{\nu_i}^2)}{\mathrm{Re}(1/{\nu_i}^2)}$, $i=1,2,3,4$, for the corresponding waves. Therefore, the four waves in such a medium are attenuating.

Special cases.

(a) For propagation of wave along the x_1 axis, $\boldsymbol{n}=(1,0,0)$ Then,

$$t_{11}=-\omega\rho c_1^2, t_{12}=\boldsymbol{i}\rho c_1^2,\; t_s=\frac{\alpha_{11}T_0\omega}{e_{31}}, t_{13}=\omega^2 a_{11}, t_{14}=0, t_{15}=0, t_{16}=0, \tau_H=T_0(1+(-\boldsymbol{i}\omega)^{\alpha}\tau_q),$$

$$t_{17}=\boldsymbol{i}r\tau_H\rho c_1^2, x_{11}=\omega c_{11}, x_{12}=x_{13}=x_{14}=x_{15}=0, x_{16}=\omega c_{66}, x_{17}=x_{18}=x_{19}=x_{20}=0,$$

$$x_{21}=\omega c_{55}, x_{22}=t_s e_{15}, x_{23}=x_{24}=0, x_{25}=-\omega e_{15}, x_{26}=t_s\xi_{11}, x_{27}=-\tau_H\omega\alpha_{11}, x_{28}=x_{29}=x_{30}=0,$$

$$x_{31}=-\frac{\rho\omega_1}{\alpha_{11}}\omega K_{11}, m_{11}=t_{12}t_{13}, m_{12}=0, m_{13}=0, m_{14}=0, m_{15}=x_{31}+t_{17}t_{13}.$$

Substituting these expressions, we can solve the determinant i.e. det (P) = 0, and further, the obtained characteristic equation can be solved to find the characteristics of the waves. Similarly, if we consider wave propagating in x_2 axis then its direction is given by $\boldsymbol{n}=(0,1,0)$ and for a wave propagating along x_3 axis then its direction is given by $\boldsymbol{n}=(0,0,1)$. Further, we can solve the det (P) = 0 in order to find the characteristic equation and the characteristics of the waves. In these cases, the generated waves will not be known as quasi since the waves propagate along with the principal directions.

(b) Let's consider the plane wave propagation in x_1-x_2 plane i.e. $\boldsymbol{n}=(\sin\theta,\cos\theta,0)$ such that $n_1^2+n_2^2=1$ and,

$t_{11} = -\omega\rho c_1^2, t_{12} = \boldsymbol{i}\sin\theta\rho c_1^2,\ t_s = \dfrac{\alpha_{11}T_0\omega}{e_{31}}, t_{13} = \omega^2(a_{11}\sin^2\theta + a_{22}\cos^2\theta), t_{14} = \boldsymbol{i}\cos\theta\rho c_1^2\dfrac{\alpha_{22}}{\alpha_{11}},$

$t_{15} = 0, t_{16} = 0, \tau_H = T_0(1+(-\boldsymbol{i}\omega)^{\alpha}\tau_q), t_{17} = \boldsymbol{i}r\tau_H\rho c_1^2, x_{11} = \omega(c_{11}\sin^2\theta + c_{66}\cos^2\theta),$

$x_{12} = \omega(c_{12}+c_{66})\sin\theta\cos\theta, x_{13} = 0, x_{14} = 0, x_{15} = x_{12}, x_{16} = \omega(c_{66}\sin^2\theta + c_{22}\cos^2\theta),$

$x_{17} = x_{18} = x_{19} = x_{20} = 0, x_{21} = \omega(c_{55}\sin^2\theta + c_{44}\cos^2\theta), x_{22} = t_s(e_{15}\sin^2\theta + e_{24}\cos^2\theta),$

$x_{23} = x_{24} = 0, x_{25} = -\omega(e_{15}\sin^2\theta + e_{24}\cos^2\theta), x_{26} = t_s(\xi_{11}\sin^2\theta + \xi_{22}\cos^2\theta),$

$x_{27} = -\tau_H\omega\alpha_{11}\sin\theta, x_{28} = -\tau_H\omega\alpha_{22}\cos\theta, x_{29} = x_{30} = 0, x_{31} = -\dfrac{\rho\omega_1}{\alpha_{11}}\omega(K_{11}\sin^2\theta +$

$K_{22}\cos^2\theta), m_{11} = t_{12}t_{13}, m_{12} = t_{14}t_{13}, m_{13} = 0, m_{14} = 0, m_{15} = x_{31} + t_{17}t_{13}.$

Substituting these expressions in the matrix P and solving the det (P) = 0, we obtain the characteristic equation and further, the characteristics of the waves. Similarly, if we consider wave propagating in x_2- x_3 plane then its direction is given by $\boldsymbol{n} = (0, \sin\theta, \cos\theta)$ and for a wave propagating in x_1- x_3 plane then its direction is given by $\boldsymbol{n} = (\sin\theta, 0, \cos\theta)$. Further, we can solve the det (P) = 0 in order to find the characteristic equation and different characteristics of the waves like phase velocity, attenuation quality factor, specific heat loss, and penetration depth.

Numerical results and discussion. The characteristics of the plane wave propagating in the orthotropic piezothermoelastic medium with the consideration of two- temperature and fractional order derivative can be explained through numerical examples. Matlab 9.0 software is used to solve the different characteristics like phase velocity, attenuation quality factor, specific heat loss, and penetration depth. Also, the effect of two-temperature parameter on phase velocity and attenuation quality factor with respect to thermal relaxation time is computed. The numerical values of cadmium selenide (CdSe) have been taken. Elastic constants (in units of GPa) are $c_{11} = 74.1$, $c_{12} = 45.2$, $c_{13} = 39.3$, $c_{22} = 79.4$, $c_{23} = 42.6$, $c_{33} = 83.6$, $c_{44} = 13.2$, $c_{55} = 15.1$, $c_{66} = 14.7$. Thermoelastic coupling constants (in units of $10^5\ NK^{-1}m^{-2}$) are given by $\alpha_{11} = 6.21$, $\alpha_{22} = 5.93$, $\alpha_{33} = 5.51$, electric permittivity constants ($10^{-11}\ C^2N^{-1}m^{-2}$) are $\xi_{11} = 8.26, \xi_{22} = 8.71$, $\xi_{33} = 9.03$, thermal conductivity constants ($Wm^{-1}K^{-1}$) are $K_{11} = 9$, $K_{22} = 7$, $K_{33} = 8$. The pyroelectric constant is $\tau_3 = -2.6\times10^{-6}\ Cm^{-2}K^{-1}$, piezoelectric constants ($10^{-3}\ Cm^{-2}$) are $e_{15} = 3$, $e_{24} = 2$, $e_{31} = 35$, $e_{32} = 32$, $e_{33} = 34$, thermal constants are given by $a_{11} = 5\times10^{-5}$, $a_{22} = 3\times10^{-5}$, $a_{33} = 7\times10^{-5}$. Numerical values for the remaining constants are $\rho = 5500\ Kgm^{-3}, T_0 = 300K$, $\tau_T = 10^{-8}\ s, C_e = 260\ JKg^{-1}K^{-1}$, $\tau_q = 2\times10^{-8}\ s$, $\omega = 2\pi\times10^{-6}$ Hz. A unit vector $\boldsymbol{n} = (\cos\phi_n \sin\theta_n, \sin\phi_n\sin\theta_n, \cos\theta_n)$, where θ denotes the polar angle with x_3-axis and ϕ is the azimuthal angle between x_1-axis and x_2-axis.represents the direction of propagation of the waves such that θ_n is varying from 0 to 200º. Using the above numerical values the variations of phase velocity and attenuation quality factor of four waves are displayed for the fixed different values of $\phi = 30º, 78º, 156º$, respectively but with the variation in the angle of incidence and fractional order parameter. These characteristics are compared for TT (Two-Temperature i.e. $a_{ij} \neq 0$) model and for WTT (Without Two-Temperature i.e. $a_{ij} = 0$) model. The comparison of corresponding plots in these figures signifies the effectiveness of two-temperature and fractional order parameters with the change in direction as exhibited in graphs as follows.

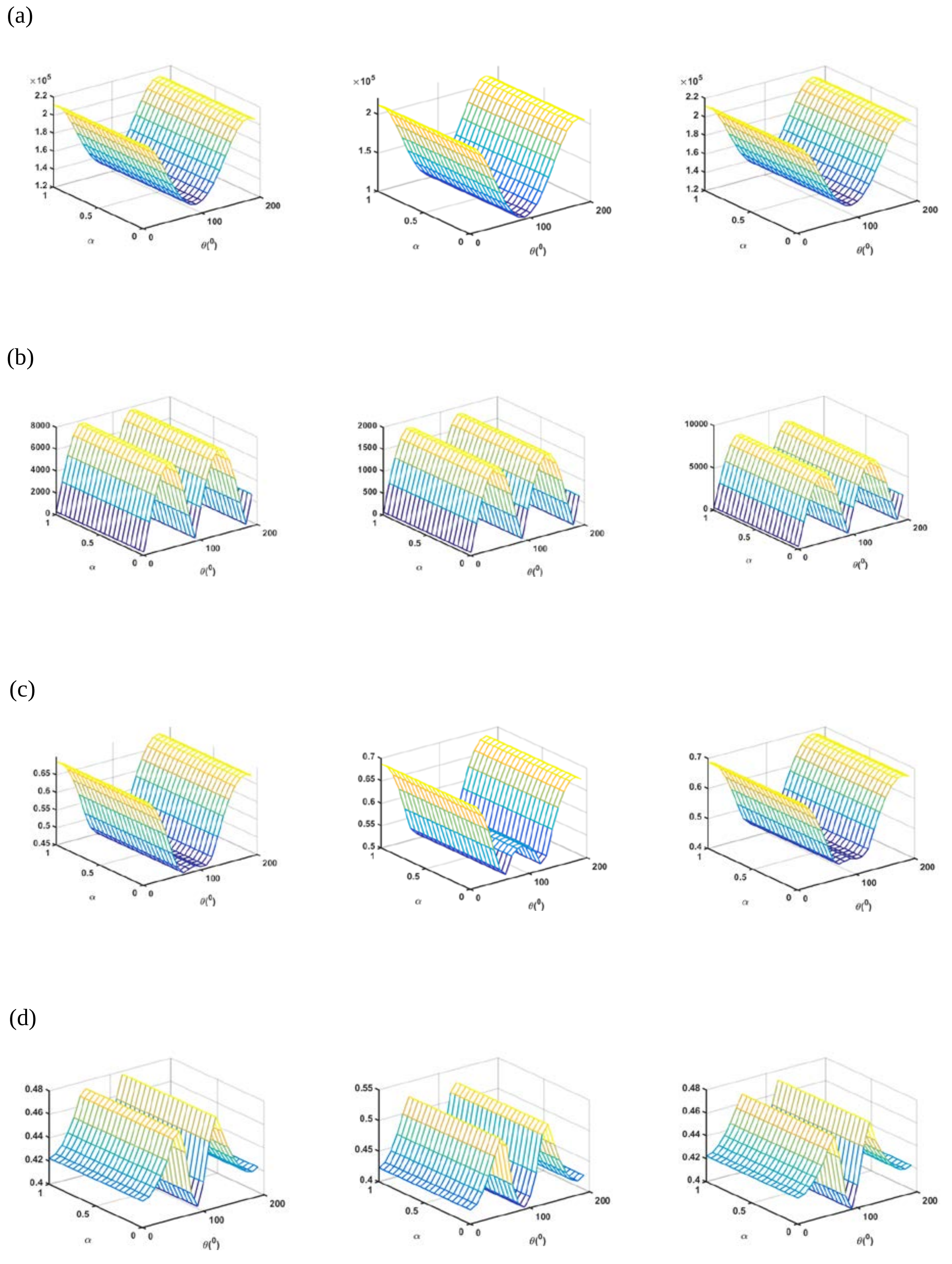

Fig. 1. Profile of (a) V_1 (b) V_2 (c) V_3 (d) V_4 w.r.t. α and θ for $\phi = 30^o, 78^o, 156^o$, respectively for TT model

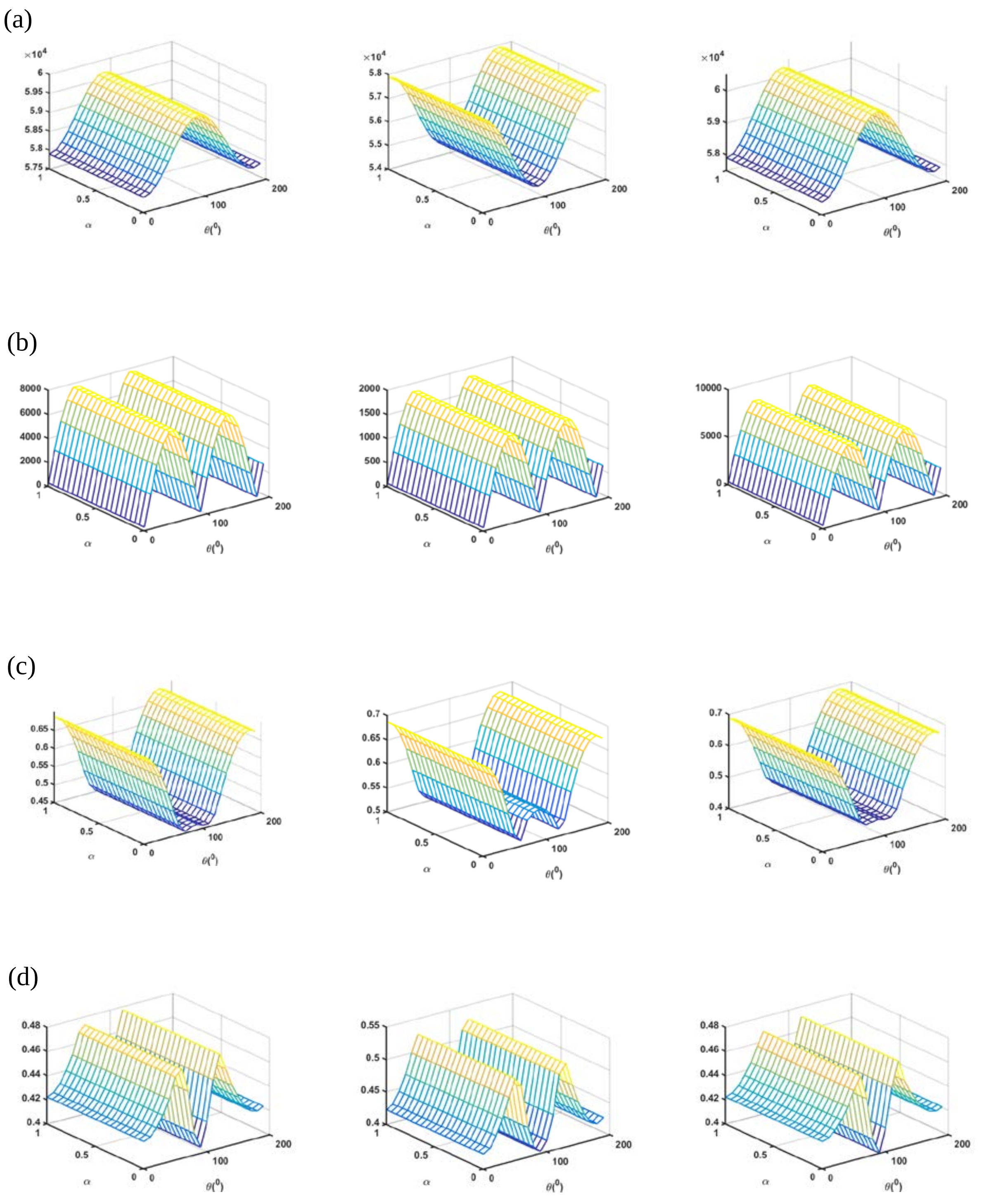

Fig. 2. Profile of (a) V_1 (b) V_2 (c) V_3 (d) V_4 w.r.t. α and θ for $\phi = 30^{\circ}, 78^{\circ}, 156^{\circ}$, respectively for WTT model

Comparing Figures 1 and 2, it is clear that for the different values of ϕ, the phase velocity of qP wave (V_1) depict the significant impact of factional order parameter and two-temperature parameters as the angle of incidence varies. It possesses the highest value which is near to 2.2×10^5 for $\phi = 78^\circ$ and at $\theta = 0, 200^\circ$. For $\phi = 30^\circ, 156^\circ$ and for WTT, V_1 increases when $\theta \leq 100^\circ$ and decreases as θ exceeds 100°. For the different values of α, the phase velocities $V_i (i = 2,3,4)$ of other waves depict the similar behaviour for both models with a slight difference in numerical values as ϕ and θ vary. It is noticed that the phase velocities of the waves for both TT and WTT models exhibit an abrupt change in trend at $\theta = 100°$. The variations of phase velocity of the respective waves clearly signify the impact of ϕ, θ, α and a_{ij}.

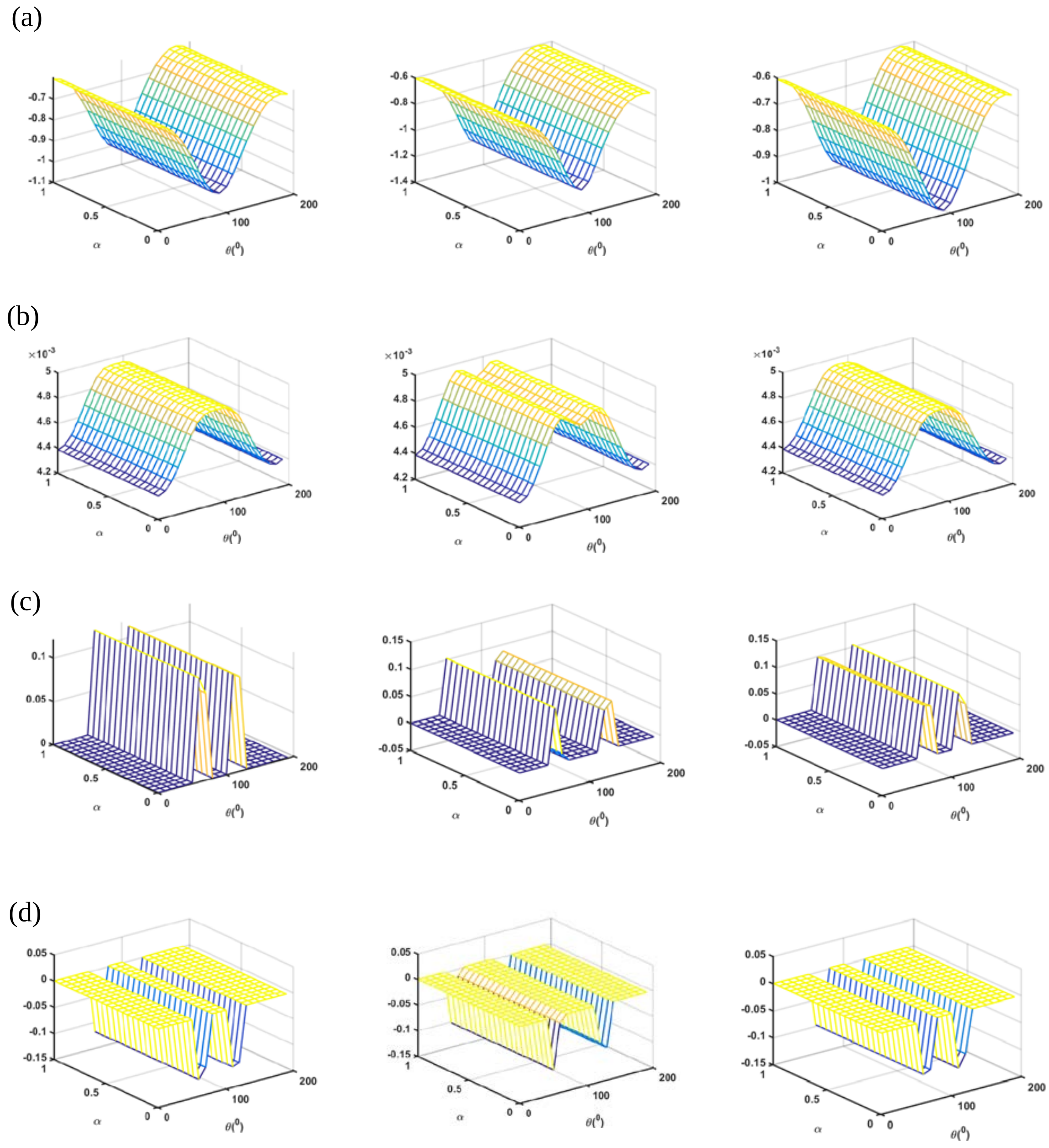

Fig. 3. Profile of (a) Q_1^{-1} (b) Q_2^{-1} (c) Q_3^{-1} (d) Q_4^{-1} w.r.t. α and θ for $\phi = 30°, 78°, 156°$, respectively for TT model

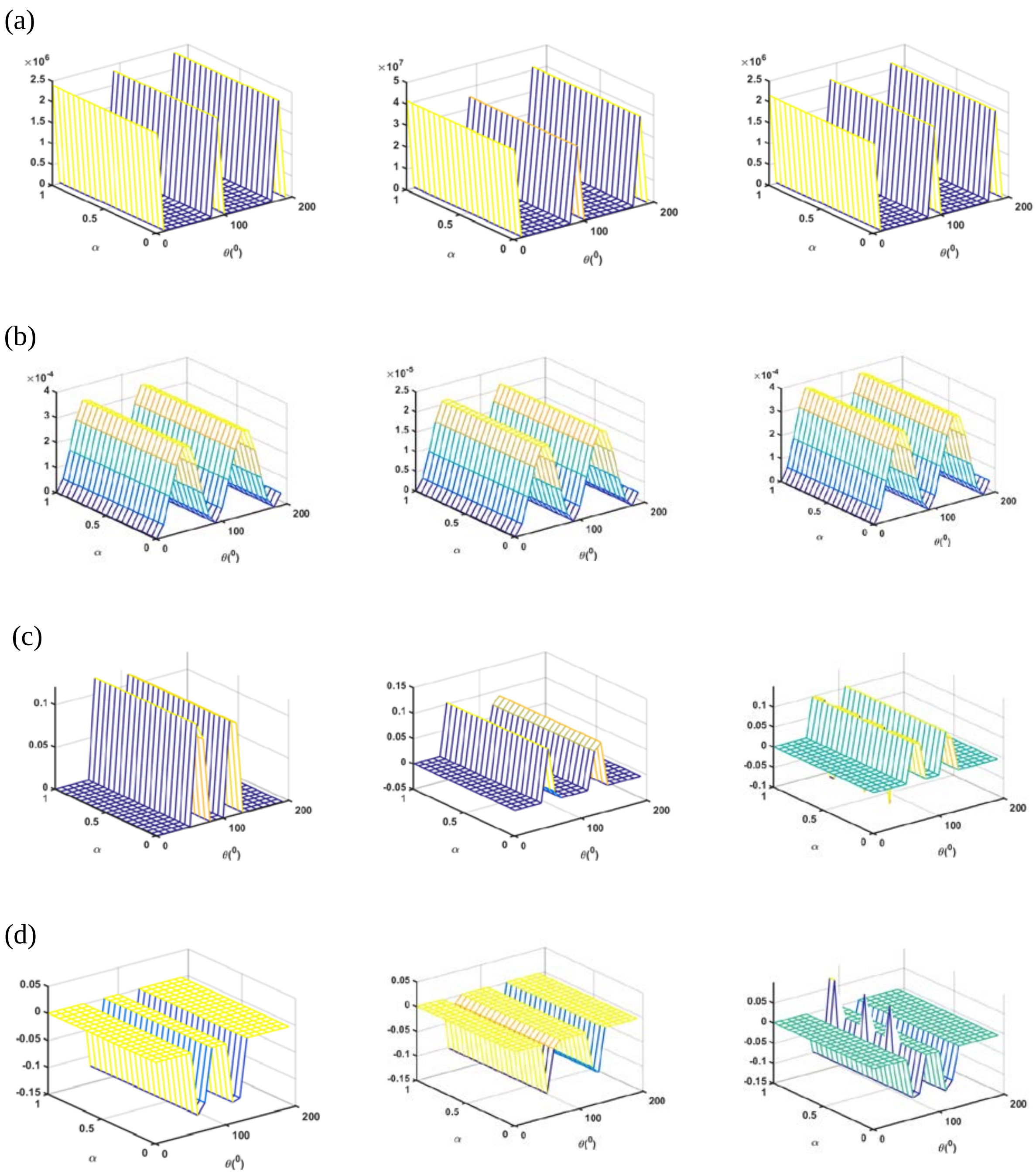

Fig. 4. Profile of (a) Q_1^{-1} (b) Q_2^{-1} (c) Q_3^{-1} (d) Q_4^{-1} w.r.t. α and θ for $\phi = 30°, 78°, 156°$, respectively for WTT model

It is clear from Fig. 3(a) that for TT model the attenuation quality factor of qP wave initially tends to decrease then increases as θ exceeds $100°$ for the different values of ϕ while in the case of WTT, Q_1^{-1} demonstrates an oscillatory behaviour as shown in Fig. 4(a). Evidently, two-temperature parameters and α have a substantial effect. Figure 3(b) depicts that for $\phi = 78°$ and $\theta = 100°$, the attenuation quality factor of qS_1 wave (Q_2^{-1}) is less than the value obtained for $\phi = 30°, 156°$ and $\theta = 100^0$. It gradually decreases for the same value of α as the angle of incidence increases whereas, for WTT, it displays oscillatory behaviour with variations in numerical values for different values of ϕ and θ. The attenuation quality factor of qS_2 wave (Q_3^{-1}) is similar for WTT and TT models as shown in Figs. 3(c) and 4(c). Q_3^{-1} and Q_4^{-1} of qS_2 and qT waves are noticed as unaffected for both mediums attaining smaller values with some variations and may not be of that much significance. Both exhibit constant behaviour as θ increases for different values of ϕ, α.

6. Conclusions

A mathematical model of an anisotropic piezothermoelastic medium with two-temperature and fractional order derivative is presented. Also, the mathematical formulation for orthotropic piezothermoelastic medium with two-temperature and fractional order derivative is presented. Significant theorems like variational principle, uniqueness theorem for the mixed initial boundary value problem, and theorem of reciprocity are established. Some special cases of interest are also given.

Appreciable effects of two-temperature parameters and fractional order parameters are observed on the various characteristics of the waves for the considered model.

Phase velocity and attenuation quality factor are presented graphically for two different models (TT, WTT) to depict the response of the considered model.

It is observed that the qP wave propagates with the highest phase velocity with the change in direction in comparison to the other plots. Phase velocities of qS_1 qS_2 and qT waves are also affected by both parameters as shown by variations in numerical values.

Due to the effects of fractional order and two-temperature parameters, the attenuation quality factor of qP wave increases for different values of ϕ and $\theta \geq 100°$. Attenuation quality factor of qS_1 wave also show the impact of both parameters. It attains the highest value for the intermediate values of θ.

Attenuation quality factor of qS_2 and qT waves are also affected by α, ϕ and θ.

There is no independent wave mode in the electric field, whereas the electric potential wave still can propagate with the elastic wave modes via constitutive relations.

The established results will be helpful for the investigators working on piezothermoelastic models.

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Appendix A

$t_{11} = -\omega\rho c_1^2, \qquad t_{12} = \boldsymbol{i} n_1 \rho c_1^2, \qquad t_s = \frac{\alpha_{11} T_0 \omega}{e_{31}}, \qquad t_{13} = \omega^2 (a_{11} n_1^2 + a_{22} n_2^2 + a_{33} n_3^2),$

$t_{14} = \boldsymbol{i} n_2 \rho c_1^2 \frac{\alpha_{22}}{\alpha_{11}}, \qquad t_{15} = \boldsymbol{i} n_3 \rho c_1^2 \frac{\alpha_{33}}{\alpha_{11}}, \qquad t_{16} = \boldsymbol{i} n_3 \rho c_1^2 \frac{\tau_3}{\alpha_{11}}, \qquad \tau_H = T_0 (1 + (-\boldsymbol{i}\omega)^\alpha \tau_q),$

$t_{17} = \boldsymbol{i} r \tau_H \rho c_1^2.$

$x_{11} = \omega(c_{11} n_1^2 + c_{66} n_2^2 + c_{55} n_3^2), \qquad x_{12} = \omega(c_{12} + c_{66}) n_1 n_2, \qquad x_{13} = \omega(c_{13} + c_{55}) n_1 n_3,$

$x_{14} = t_s (e_{31} + e_{15}) n_1 n_3, \qquad x_{15} = x_{12}, \qquad x_{16} = \omega(c_{66} n_1^2 + c_{22} n_2^2 + c_{44} n_3^2),$

$x_{17} = \omega(c_{23} + c_{44}) n_2 n_3, \qquad x_{18} = t_s (e_{32} + e_{24}) n_2 n_3, \qquad x_{19} = x_{13}, x_{20} = x_{17},$

$x_{21} = \omega(c_{55} n_1^2 + c_{44} n_2^2 + c_{33} n_3^2), \qquad x_{22} = t_s (e_{15} n_1^2 + e_{24} n_2^2 + e_{33} n_3^2), \qquad x_{23} = -\omega(e_{31} + e_{15}) n_1 n_3,$

$x_{24} = -\omega(e_{32} + e_{24}) n_2 n_3, \qquad x_{25} = -\omega(e_{15} n_1^2 + e_{24} n_2^2 + e_{33} n_3^2), \qquad x_{26} = t_s (\xi_{11} n_1^2 + \xi_{22} n_2^2 + \xi_{33} n_3^2),$

$x_{27} = -\tau_H \omega \alpha_{11} n_1, \qquad x_{28} = -\tau_H \omega \alpha_{22} n_2, \qquad x_{29} = -\tau_H \omega \alpha_{33} n_3,$

$x_{30} = \tau_H \omega \alpha_{11} T_0 n_3 \frac{\tau_3}{e_{31}}, \qquad x_{31} = -\frac{\rho \omega_1}{\alpha_{11}} \omega (K_{11} n_1^2 + K_{22} n_2^2 + K_{33} n_3^2).$

$m_{11} = t_{12} t_{13}, \qquad m_{12} = t_{14} t_{13}, \qquad m_{13} = t_{15} t_{13}, \qquad m_{14} = t_{16} t_{13}, \qquad m_{15} = x_{31} + t_{17} t_{13}.$