

ASYMPTOTIC ANALYSIS OF THE EQUATIONS OF HYDROELASTIC OSCILLATIONS IN THIN-WALLED ELASTIC PIPELINE

O.P. Tkachenko✉, A.S. Ryabokon

Computing Center of the Far Eastern Branch of the Russian Academy of Sciences, Kim Yu Chen Str., 65,
Khabarovsk, 680000, Russia
✉ olegt1964@gmail.com

Abstract. A mathematical model of the dynamics of a curved pipeline as a membrane shell filled with an ideal compressible fluid is constructed. A form of an approximate solution of the model equations is found in which the dimension of the problem reduces by one. An asymptotic analysis of the obtained differential equations is performed for various small parameters. Two systems of model equations with different characteristics are obtained. These systems are transformed into systems of first-order equations in canonical form with explicitly defined Riemann invariants. The characteristics of the obtained systems of equations are found and the main types of waves propagating in a curved pipe are identified. Numerical experiments have been performed. The results obtained correspond to the results from the literature sources.

Keywords: curved pipeline, shell, hydroelastic vibrations, asymptotic series, dimension reduction, characteristics method

Acknowledgements. *The work has been supported by the Russian Science Foundation grant № 21-11-00039, <https://rscf.ru/en/project/21-11-00039/>.*

Citation: Tkachenko O.P., Ryabokon A.S. Asymptotic analysis of the equations of hydroelastic oscillations in thin-walled elastic pipeline // Materials Physics and Mechanics. 2021, V. 47. N. 5. P. 747-766. DOI: 10.18149/MPM.4752021_9.

1. Introduction

The problem of the pressure wave propagation in a fluid-filled long curved elastic pipe and the effect of the pipe bending on this process are considered. The theoretical propositions of the study are described in [1,2].

The engineering background of the problem consists in the fact that pipelines can shift from their initial position during operation [3]. It is assumed that there is a possibility to find the coordinates of the position of the pipeline profile based on the results of the analysis of the propagation of hydroelastic vibrations in it [4]. To solve this inverse problem, it is necessary to study the direct problem, which is the subject of our study. In addition, the analysis of wave processes in complex pipeline systems is a separate, practically important task [5].

In the fundamental work [6], water hammer in rectilinear cylindrical pipes is investigated. This research is continued in numerous works of mechanics, among which we highlight the articles [7-9].

In the work of G.T. Aldoshin [7], a review of studies from L. Euler to the present day is performed. The deep interrelation of various problem statements, from hemodynamics to

waves in metal pipes, is noted. The importance of studying composite pipes and nonlinear phenomena is noted. In this context, the article [10] is of great importance, which solves the problem of the propagation of water hammer in a system of two coaxial cylinders, with a filled with liquid space between them and a filled with gas inner cylinder. For a long time this problem resisted theoretical analysis.

A review of the foreign literature up to 2015 related to the dynamic analysis of filled with liquid pipeline systems, taking into account the fluid-structure interaction, is carried out in [11]. Various types of models and modeling algorithms of various levels of complexity are compared, and the scope of their application is discussed.

The modern literature on vibrations in pipelines is very extensive.

Nonclassical theories of water hammer are developed in [12,13]. In [12], a one-dimensional mathematical model is presented that describes the acoustic vibrations of thick-walled pipes filled with liquid. The model has correction conditions and coefficients that take into account the wall thickness. In [13], a water hammer accompanied by cavitation is numerically investigated.

The phenomena of resonance in connection with the vibrations of underwater pipelines induced by vortices are investigated in [14-16]. In [14], focused on modelling of transverse vibrations of an underwater pipeline in a vortex fluid flow, dynamic analysis, and prediction of alternating stress in the pipe. In [15], the free vibrations of two cylinders in a laminar fluid flow are numerically investigated. In [16], a mathematical model of reduced order is constructed that describes the interaction of a cylinder and a liquid under vortex-induced vibrations. The presented model contains an explicitly distinguished singularity at the resonance point.

The theoretical foundations and numerous important examples of the statics and dynamics of curvilinear pipes are described in the works of V.A. Svetlitsky, for example, in [17]. The monograph [17] is collected extensive research material on the vibrations of flexible rods and hoses.

Let's refer to the question of history. N.E. Zhukowski [6] determined that the wave velocity of the water hammer in the pipeline is related to the relative circumferential stiffness of the pipe. He assumed that in any cross-section, the pressure is distributed uniformly, and neglected the inertia of the fluid in the radial direction, the mass of the pipe wall, and the axial and bending stresses in the pipe wall. In 1956, R. Skalak [8] extended the Zhukowski formula, taking into account the Poisson effect. He presented the pipe wall in the form of an elastic membrane, which allowed to take into account the axial stresses and inertia of the pipe. A. Thorlay [18] performed a similar study but also conducted an experimental test of the theory. This has given rise to many publications related to the study of new effects of water hammer in pipes.

Vibrations of curved pipelines and pipes with knees, taking into account the Poisson effect and various ways of fixing the knee, are studied in [8,18,19]. In [9,20], the effect of fixing the pipeline knee on the propagation of pressure waves in a stopped flow of an ideal liquid is studied. Developments [21,22] in this direction followed the path of increasing the number of branches and knees of the pipeline, leaving the main parameters and characteristics of mathematical models unchanged. These studies are carried out within the framework of the theory of rods.

Taking into account the Poisson effect and the connection between the dynamics of the fluid and the pipe through deformation led to the discovery of fundamentally new phenomena in the theory of water hammer. It is advisable to continue studying the relationship of the geometric parameters of the pipeline with the Poisson ratio in the framework of the theory of shells.

The question of mathematical modeling of pipes as shells is considered in [23]. The limits of the applicability of the rod and shell models for one class of problems about pipelines, the problem of tunneling, when the pipe is directly laid in the ground, are established. It is pointed out the limitations of the rod model for the considered range of tasks since the pipeline is a three-dimensional structure, surrounded on all sides by the external environment.

The fundamental problem of interaction between a cylindrical shell and a liquid filling it is studied in the work of A.S. Volmir [24]. The focus is on questions about the flow around the shells and their vibrations. Shells of regular cylindrical or conical shape are considered, the liquid is considered incompressible. An extensive bibliography on hydroelasticity problems is given.

Much attention is given to stationary problems of curvilinear pipes as fragments of toroidal shells in the classical work of V.V. Novozhilov [25]. Numerous examples of calculating stresses and deformations under various methods of applying a load to a segment of a toroidal shell are considered.

The discussion about the advisability of applying the theory of shells to pipeline systems continued in the articles [26,27]. In the publication [26], the dynamics of the pipe under the influence of the reverse shift of the ground during an earthquake are investigated. The problems that cannot be solved by the rod model are investigated. The article [27] is devoted to the topic of modeling the dynamics of underground pipelines under the influence of discontinuous ground shifts based on the theory of shells.

Thus, there is an extensive class of problems about the dynamics of pipelines, to which it is necessary to apply the methods of shell theory.

We applied this approach in the article [28], where a mathematical model of vibrations in a curved underground pipeline is constructed and partially investigated. This approach was then developed in [1]. The mathematical model is verified using various examples from the literature in the article [2].

In the 30s-40s of the XX century, V.Z. Vlasov [29] found that in their general formulation, the equations of the theory of shells contain terms that do not make a significant contribution to stresses, deformations, and resultant forces. This conclusion was confirmed later in the works of V.V. Novozhilov, for example, in [25]. This means that there is a class of problems in which these terms can be neglected. From this conclusion, the problem of simplifying the model arose [1,2].

Based on the classical works on water hammer [6,8,10,18] and the main assumptions made in them (see [30]), the model of the pipe in this work is based on the model of a membrane shell. The model of liquid motion is based on the model of an ideal compressible fluid.

The aim of the work is to study the influence of pipe bending on the propagation of vibrations in the pipe-liquid system, on the one hand, within the framework of the mechanical models used, and on the other hand, within the methods of simplifying these models proposed in [1,31].

The work has the following structure. In section 2, the geometry of the problem is described, small parameters are introduced and a general description of the mechanical process is provided. In section 3, from the equations of motion of a membrane shell and an ideal compressible fluid are derived reduced equations of the mathematical model (5), (9), (10), from which the angular coordinates are excluded. The expansion in the second small parameter is performed and equations (11), (16) are obtained. In section 4, the model equations are analyzed with additional assumptions about small parameters. As a result, two different systems of equations (20), (24) are obtained. In section 5, changes of variables are made in these systems of equations, as a result of which they are reduced to the canonical

form. The characteristics of the obtained systems of first-order equations are found. In section 6, numerical experiments have been performed. In sections 4, 5, 6 it is established that the constructed mathematical models describe the types of waves known in the scientific literature, that is, these models are consistent with the known results.

2. Problem geometry and initial assumptions

We consider a long metal pipe (see Fig. 1), filled with a compressible ideal liquid. Let the centerline of the pipe be a plane curve Γ with natural parametrization: $\Gamma = \{x, y : x = x(s), y = y(s), 0 < s < L\}$. Let all the physical parameters of the pipe material and the liquid filling it is known. At the point O , a longitudinal pressure wave in the liquid is excited. It is required to find the dynamics of the system.

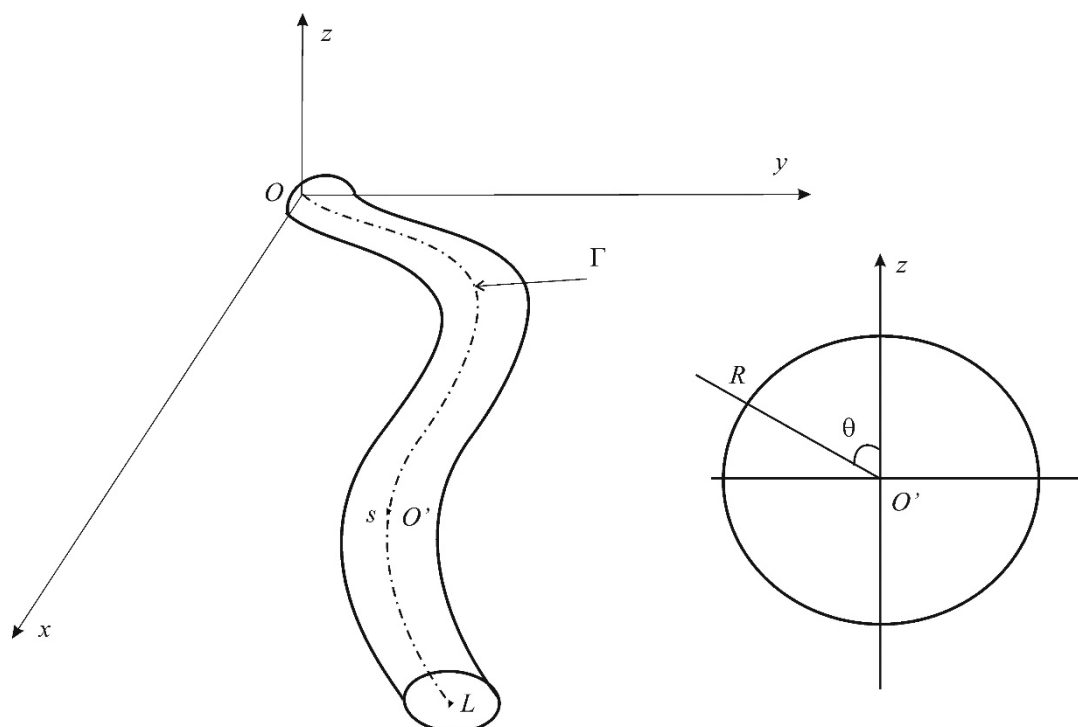


Fig. 1. Pipeline and coordinate systems

To find the dynamics of the pipe-fluid system, we assume the following:

- 1) A linear approximation is acceptable.
- 2) An approximation of the membrane theory of shells is acceptable.

- 3) $\frac{R_0}{\min_{0 \leq s \leq L} |\rho_0(s)|} = \varepsilon$, $\varepsilon \ll 1$, where R_0 is the radius of the pipe, $\rho_0(s)$ is the radius of

curvature of the curve Γ .

- 4) $\mu = \frac{R_0}{\lambda}$, $\mu \ll 1$, where λ is the characteristic wavelength of the process under study.

For what follows, we introduce the curvature function: $f(s) = \frac{\min_{0 \leq s \leq L} |\rho_0(s)|}{\rho_0(s)}$.

3. Equations of motion

The purpose of this section is to derive the quasi-one-dimensional equations of fluid and pipe motion of the corresponding system of equations describing the three-dimensional case.

The curvilinear coordinate system used (s, θ, R) is shown in Fig. 1. Its detailed description is given in [1]. Here (θ, R) are the polar coordinates in the cross-section corresponding to the arc length s .

The linearized system of equations of fluid motion in an arbitrary curvilinear coordinate system has the form:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= -\frac{1}{\rho_f} \text{grad}(p); \\ \frac{1}{c_f^2} \frac{\partial p}{\partial t} + \rho_f \text{div}(\mathbf{v}) &= 0. \end{aligned} \quad (1)$$

The system of equations for the shell of the studied shape in the forces (see [25,32]) are:

$$\begin{aligned} \left(1 + \frac{R_0}{\rho_0(s)} \sin \theta\right)^{-1} \frac{\partial N_s}{\partial s} + \frac{1}{R_0} \frac{\partial S}{\partial \theta} + 2S \frac{\cos \theta}{\rho_0(s) + R_0 \sin \theta} + X_v &= 0; \\ \left(1 + \frac{R_0}{\rho_0(s)} \sin \theta\right)^{-1} \frac{\partial S}{\partial s} + \frac{1}{R_0} \frac{\partial N_\theta}{\partial \theta} + \frac{(N_\theta - N_s) \cos \theta}{\rho_0(s) + R_0 \sin \theta} + Y_v &= 0; \\ \frac{N_s \sin \theta}{\rho_0(s) + R_0 \sin \theta} + \frac{N_\theta}{R_0} - Z_v &= 0. \end{aligned} \quad (2)$$

In the systems of equations (1), (2), it is denoted: N_s – longitudinal stresses of the shell along the axis s , N_θ – the same along the axis θ , S – shearing forces, X , Y , Z – inertia forces along the s , θ and the normal, respectively, c_f – the velocity of sound in liquid, ρ_f – fluid density under normal conditions, \mathbf{v} – fluid velocity vector, p – fluid pressure.

The deformations of the shell of this shape we express in terms of displacements by the formulas:

$$\begin{aligned} \varepsilon_s &= \left(1 + \frac{R_0}{\rho_0(s)} \sin \theta\right)^{-1} \left[\frac{\partial u}{\partial s} + \frac{\cos \theta}{\rho_0(s)} v + \frac{\sin \theta}{\rho_0(s)} w \right], \\ \varepsilon_\theta &= \frac{1}{R_0} \frac{\partial v}{\partial \theta} + \frac{w}{R_0}, \\ \gamma &= \left(1 + \frac{R_0}{\rho_0(s)} \sin \theta\right)^{-1} \left(\frac{\partial v}{\partial s} - \frac{\cos \theta}{\rho_0(s)} u \right) + \frac{1}{R_0} \frac{\partial u}{\partial \theta}. \end{aligned}$$

Here u , v , w are displacements of the pipe points in the direction of the axes s , θ and R , respectively.

Further, expressing N_s , N_θ , S according to Hooke law in terms of deformations (see [25,32]), and deformations – in terms of displacements, we substitute the result in (2). We take into account that, according to [33], the inertial forces are equal to:

$$X_v = -\rho_t h \frac{\partial^2 u}{\partial t^2}, \quad Y_v = -\rho_t h \frac{\partial^2 v}{\partial t^2}, \quad Z_v = p_w - \rho_t h \frac{\partial^2 w}{\partial t^2}.$$

Here and below is indicated by: ρ_t – pipe material density, p_w – the pressure on the wall, h – pipe wall thickness, p_a – atmospheric pressure.

We transform the unknowns and variables to a dimensionless form. To do this, we introduce the variables $\zeta = s/\lambda$, $\tau = \omega t$, where ω , λ is the characteristic frequency and wavelength of the vibrations processes associated with the relation $\lambda \omega = c_f$. We make the

following change of unknowns: $u' = u/R_0$, $v' = v/R_0$, $w' = w/R_0$, $p'_w = P_w/p_a$. Here u' , v' , w' are dimensionless displacements of the pipe points in the direction of the axes s , θ and R respectively.

Leaving the first-order terms by ε , we get:

$$\begin{aligned}
& (1 - \varepsilon f \sin \theta) \frac{R_0}{\lambda} \frac{\partial}{\partial \zeta} \left[\frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} + v' \varepsilon f \cos \theta + w' \varepsilon f \sin \theta - \varepsilon f \frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} \sin \theta + \right. \\
& \left. + v' \left(\frac{\partial v'}{\partial \theta} + w' \right) \right] + \frac{1 - \nu}{2} \frac{\partial}{\partial \theta} \left[(1 - \varepsilon f \sin \theta) \frac{R_0}{\lambda} \frac{\partial v'}{\partial \zeta} - u' \varepsilon f \cos \theta + \frac{\partial u'}{\partial \theta} \right] + \\
& + \varepsilon f (1 - \nu) \cos \theta \left[\frac{\partial u'}{\partial \theta} + \frac{R_0}{\lambda} \frac{\partial v'}{\partial \zeta} \right] - \rho_t \frac{\omega^2 R_0^2}{E^*} \frac{\partial^2 u'}{\partial \tau^2} = 0; \\
& (1 - \varepsilon f \sin \theta) \frac{R_0 (1 - \nu)}{2\lambda} \frac{\partial}{\partial \zeta} \left[\frac{\partial u'}{\partial \theta} + (1 - \varepsilon f \sin \theta) \frac{R_0}{\lambda} \frac{\partial v'}{\partial \zeta} - u' \varepsilon f \cos \theta \right] + \\
& + \frac{\partial}{\partial \theta} \left[\frac{\partial v'}{\partial \theta} + w' + \nu (1 - \varepsilon f \sin \theta) \frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} + \nu v' \varepsilon f \cos \theta + \nu w' \varepsilon f \sin \theta \right] + \\
& + \varepsilon f (1 - \nu) \cos \theta \left(\frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} - \frac{\partial v'}{\partial \theta} - w' \right) - \rho_t \frac{\omega^2 R_0^2}{E^*} \frac{\partial^2 v'}{\partial \tau^2} = 0; \\
& \varepsilon f \sin \theta \left[\frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} + v' \left(\frac{\partial v'}{\partial \theta} + w' \right) \right] + \frac{\partial v'}{\partial \theta} + w' + \nu (1 - \varepsilon f \sin \theta) \frac{R_0}{\lambda} \frac{\partial u'}{\partial \zeta} + \\
& + \nu \varepsilon f (v' \cos \theta + w' \sin \theta) - \frac{R_0}{h} \frac{p'_w p_a}{E^*} + \rho_t \frac{\omega^2 R_0^2}{E^*} \frac{\partial^2 w'}{\partial \tau^2} = 0.
\end{aligned} \tag{3}$$

Here denote: ν – Poisson ratio; E – Young modulus; $E^* = E/(1 - \nu^2)$, $L' = L/\lambda$. In

accordance with the monograph by V.V. Novozhilov [25], we impose boundary conditions at the ends of the tubular membrane shell:

$$u' = 0, v' = 0 \text{ at } \zeta = 0, \zeta = L'.$$

We make two assumptions about the displacement v :

1) $v = v(\zeta, \tau)$ – no longitudinal waves propagating along the circumference;

2) $|v| \leq |\varepsilon|$ – the smallness of the amplitude of torsional vibrations in problems of propagation of vibrations in a pipe.

Under these assumptions, the displacement v disappears from the first and third equations of the system (3), and then there is no need to solve the second equation.

We represent u' , w' , p'_w in the form [34]:

$$\begin{aligned}
u' &= u_0(\zeta, \tau) + \varepsilon u_1(\zeta, \tau) \sin \theta, \\
w' &= w_0(\zeta, \tau) + \varepsilon w_1(\zeta, \tau) \sin \theta, \\
p'_w &= p_{w0}(\zeta, \tau) + \varepsilon p_{w1}(\zeta, \tau) \sin \theta.
\end{aligned} \tag{4}$$

The correctness of (4) is proved by the sequential expansion of the solution of the system of equations (3) into series, first by powers ε , then by the trigonometric functions $\sin \theta$, $\cos \theta$. In this case, the terms with the factor $\cos \theta$ turn to zero. Substituting (4) in (3) and equating the terms for the same powers of the parameter ε in the left and right parts of the equations, we get:

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left(\frac{\partial u_0}{\partial \zeta} + \frac{\nu}{\mu} w_0 \right) - \frac{c_f^2}{c_t^2} \frac{\partial^2 u_0}{\partial \tau^2} &= 0, \\ w_0 + \nu \mu \frac{\partial u_0}{\partial \zeta} - \frac{R_0}{h} \frac{p_{w0} p_a}{E^*} + \mu^2 \frac{c_f^2}{c_t^2} \frac{\partial^2 w_0}{\partial \tau^2} &= 0, \\ \frac{\partial}{\partial \zeta} \left(\frac{\partial u_1}{\partial \zeta} + \frac{\nu}{\mu} w_1 \right) - \frac{1-\nu}{2\mu^2} (u_1 - f u_0) - \frac{c_f^2}{c_t^2} \frac{\partial^2 u_1}{\partial \tau^2} - f \frac{\partial}{\partial \zeta} \left(\frac{\partial u_0}{\partial \zeta} + \frac{\nu-1}{\mu} w_0 \right) &= 0, \\ w_1 + \nu \mu \frac{\partial u_1}{\partial \zeta} - \frac{R_0}{h} \frac{p_{w1} p_a}{E^*} + \mu^2 \frac{c_f^2}{c_t^2} \frac{\partial^2 w_1}{\partial \tau^2} + f \mu (1-\nu) \frac{\partial u_0}{\partial \zeta} + 2\nu f w_0 &= 0. \end{aligned} \quad (5)$$

Here $c_t = \sqrt{E^*/\rho_t}$ is the longitudinal wave velocity in a pipe material plate.

From the boundary conditions attached to equations (3), we obtain the conditions on (5):

$$u_0(0, \tau) = u_0(L', \tau) = 0, \quad u_1(0, \tau) = u_1(L', \tau) = 0 \quad \text{at } \zeta = 0, \quad \zeta = L'.$$

Further, using the technique given in [1], for a liquid, one can obtain equations with respect to physical variables:

$$\begin{aligned} \rho_f \frac{\partial V_R}{\partial t} + \frac{\partial p}{\partial R} &= 0; \quad \rho_f \frac{\partial V_s}{\partial t} + \left(1 - \frac{R}{R_0} \varepsilon f \sin \theta \right) \frac{\partial p}{\partial s} = 0, \\ \frac{1}{c_f^2} \frac{\partial p}{\partial t} + \rho_f \left[\left(1 - \frac{R}{R_0} \varepsilon f \sin \theta \right) \left(\frac{\partial V_s}{\partial s} + \frac{\sin \theta}{\rho_0(s)} V_R \right) + \frac{\partial V_R}{\partial R} + \frac{V_R}{R} \right] &= 0, \\ V_R \Big|_{R=R_0} &= \frac{\partial w}{\partial t}. \end{aligned} \quad (6)$$

Here R is radial coordinate, V_s , V_R are fluid velocities along the axes s , R respectively. The system of equations of motion is supplemented by a boundary condition on the pipe wall. The boundary conditions at the ends of the pipe have the form [2]:

$$\left(\mu_1 V_s + \mu_2 p \right) \Big|_{s=0} = F_1(t), \quad \left(\mu_3 V_s + \mu_4 p \right) \Big|_{s=L} = F_2(t),$$

where μ_i takes the values 0 or 1, depending on the statement of the problem.

We change of variables:

$$\begin{aligned} \zeta &= s/\lambda, \quad \tau = \omega t, \quad \xi = R/\lambda; \\ V_\xi &= V_R/c_f, \quad V_\zeta = V_s/c_f, \quad p' = p/p_a, \end{aligned}$$

where p_a is atmospheric pressure. Enter the value

$$a^2 = \frac{p_a}{\rho_f c_f^2}.$$

After calculations from (6), we get:

$$\begin{aligned} \frac{\partial V_\xi}{\partial \tau} + a^2 \frac{\partial p'}{\partial \xi} &= 0; \quad \frac{\partial V_\zeta}{\partial \tau} + \left(1 - \xi \frac{\varepsilon}{\mu} f \sin \theta \right) a^2 \frac{\partial p'}{\partial \zeta} = 0; \\ a^2 \frac{\partial p'}{\partial \tau} + \left(1 - \xi \frac{\varepsilon}{\mu} f \sin \theta \right) \left(\frac{\partial V_\zeta}{\partial \zeta} + \frac{\varepsilon}{\mu} f V_\xi \sin \theta \right) + \frac{\partial V_\xi}{\partial \xi} + \frac{V_\xi}{\xi} &= 0; \\ V_\xi \Big|_{\xi=\mu} &= \frac{\partial w'}{\partial \tau}. \end{aligned} \quad (7)$$

Representing the velocities and pressure in (7) in a form similar to (4):

$$\begin{aligned} V_{\xi} &= V_{\xi 0} + \varepsilon V_{\xi 1} \sin \theta + O(\varepsilon^2); \\ V_{\zeta} &= V_{\zeta 0} + \varepsilon V_{\zeta 1} \sin \theta + O(\varepsilon^2); \\ p' &= p_0 + \varepsilon p_1 \sin \theta + O(\varepsilon^2); \end{aligned} \quad (8)$$

and equating the terms for the same powers of ε , from (7), (8) we get:

$$\begin{aligned} \frac{\partial V_{\xi 0}}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \xi} = 0; \quad \frac{\partial V_{\zeta 0}}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} = 0; \\ a^2 \frac{\partial p_0}{\partial \tau} + \frac{\partial V_{\zeta 0}}{\partial \zeta} + \frac{\partial V_{\xi 0}}{\partial \xi} + \frac{V_{\xi 0}}{\xi} = 0; \end{aligned} \quad (9)$$

$$\begin{aligned} V_{\xi 0} \Big|_{\xi=\mu} &= \mu \frac{\partial w_0}{\partial \tau}. \\ \frac{\partial V_{\xi 1}}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \xi} &= 0; \\ \frac{\partial V_{\zeta 1}}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} - \frac{\xi}{\mu} a^2 \frac{\partial p_0}{\partial \zeta} &= 0; \\ a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_{\zeta 1}}{\partial \zeta} + \frac{\partial V_{\xi 1}}{\partial \xi} + \frac{V_{\xi 1}}{\xi} - \frac{\xi f}{\mu} \frac{\partial V_{\zeta 0}}{\partial \zeta} + f \frac{V_{\xi 0}}{\mu} &= 0; \end{aligned} \quad (10)$$

$$V_{\xi 1} \Big|_{\xi=\mu} = \mu \frac{\partial w_1}{\partial \tau}.$$

Using the expansions of the unknowns into a series by the powers of the parameter μ , the system of equations (9) can be reduced (see [1,28]) to the form:

$$\begin{aligned} a^2 \frac{\partial p_0}{\partial \tau} + \frac{\partial V_{\zeta 0}}{\partial \zeta} + 2 \frac{\partial w_0}{\partial \tau} &= 0, \\ \frac{\partial V_{\zeta 0}}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} &= 0, \\ p_{w0} &= p_0 - \frac{\mu^2}{2a^2} \frac{\partial^2 w_0}{\partial \tau^2}, \\ V_{\xi 0} &= \xi \frac{\partial w_0}{\partial \tau}; \quad \frac{\partial p_0}{\partial \xi} = 0. \end{aligned} \quad (11)$$

$$\mu_1 V_{\zeta 0}(0, \tau) + \mu_2 p_0(0, \tau) = F_{01}(\tau), \quad \mu_3 V_{\zeta 0}(L', \tau) + \mu_4 p_0(L', \tau) = F_{02}(\tau).$$

Equations (11) describe the propagation of hydroelastic vibrations in a straight cylindrical pipe. The second-order correction to p_{w0} in μ is known and found in another way in [8]. Thus, the mathematical model of fluid motion (9), (10) is in agreement with the known scientific results.

The right side of the system of equations (10) depends on the given function $f(\zeta)$. For simplicity, we assume that the curvature of the pipeline profile is constant. Then $f \equiv 1$.

Further, the entire analysis is performed under this assumption.

We change of variable $\xi = \mu r$ and represent the solutions (10) in the form:

$$V_{\xi 1} = V_{\xi 1}^{(0)} + \mu V_{\xi 1}^{(1)} + \mu^2 V_{\xi 1}^{(2)} + O(\mu^3),$$

$$\begin{aligned} V_{\zeta_1} &= V_{\zeta_1}^{(0)} + \mu V_{\zeta_1}^{(1)} + \mu^2 V_{\zeta_1}^{(2)} + O(\mu^3), \\ p_1 &= p_1^{(0)} + \mu p_1^{(1)} + \mu^2 p_1^{(2)} + O(\mu^3). \end{aligned} \quad (12)$$

We substitute (12) in (10) and equate the terms at the same powers μ in the left and right parts of the equations. We denote

$$p_1 = p_1^{(0)} + \mu p_1^{(1)}; V_{\zeta_1} = V_{\zeta_1}^{(0)} + \mu V_{\zeta_1}^{(1)}; V_{\xi_1} = V_{\xi_1}^{(1)} + \mu V_{\xi_1}^{(2)}.$$

We get the relations:

$$\begin{aligned} \frac{\partial V_{\zeta_1}}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} - r a^2 \frac{\partial p_0}{\partial \zeta} &= 0, \quad \frac{\partial p_1}{\partial r} = 0, \\ a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_{\zeta_1}}{\partial \zeta} + \frac{\partial V_{\xi_1}}{\partial r} + \frac{V_{\xi_1}}{r} - r \frac{\partial V_{\zeta_0}}{\partial \zeta} + r \frac{\partial w_0}{\partial \tau} &= 0; \\ V_{\xi_1} \Big|_{r=1} = \frac{\partial w_1}{\partial \tau}, \quad V_{\xi_1} \Big|_{r=0} &= 0. \end{aligned} \quad (13)$$

Differentiating the third equation (13) sequentially with respect to r and τ , and using (11), we obtain for V_{ξ_1} :

$$\begin{aligned} \frac{\partial^2 V_{\xi_1}}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{V_{\xi_1}}{r} \right) - 2 \frac{\partial V_{\zeta_0}}{\partial \zeta} + \frac{\partial w_0}{\partial \tau} &= 0; \\ V_{\xi_1} \Big|_{r=1} = \frac{\partial w_1}{\partial \tau}; \quad V_{\xi_1} \Big|_{r=0} &= 0. \end{aligned} \quad (14)$$

The solution of the problem (14) is:

$$V_{\xi_1} = -\frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_{\zeta_0}}{\partial \zeta} \right) r^2 + \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_{\zeta_0}}{\partial \zeta} \right) \right] r. \quad (15)$$

Substituting (15) into (13), and representing V_{ζ_1} as:

$$V_{\zeta_1} = V(r, \zeta, \tau) + V_1(\zeta, \tau),$$

from (13) we get:

$$\begin{aligned} a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_1}{\partial \zeta} + 2 \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_{\zeta_0}}{\partial \zeta} \right) \right] &= 0, \\ \frac{\partial V_1}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} &= 0. \end{aligned} \quad (16)$$

$$\mu_1 V_1(0, \tau) + \mu_2 p_1(0, \tau) = F_{11}(\tau), \quad \mu_3 V_1(L, \tau) + \mu_4 p_1(L, \tau) = F_{12}(\tau).$$

With an accuracy of terms of the order of μ^2 , the pressure on the pipe wall is equal to

$$p_{w1} = p_1.$$

Equations and boundary conditions (5), (11), (15), (16) form a closed system for determining the first two terms of the decompositions (4), (8) of unknown functions over a small parameter ε .

4. Particular asymptotic expansions

The important role of the Poisson effect in the propagation of hydroelastic waves in pipes is established in [8,9,22] and others. Therefore, the relation of small parameters to the Poisson ratio is given below, and it is shown how the introduction of various relations between them makes it possible to simplify the equations (5), (11), (16).

We consider two cases:

- 1) $\varepsilon \sim \mu^{3/2}$;
- 2) $\varepsilon \sim \mu$.

Here we adhere to the restriction $\nu \sim \mu^{1/2}$, emphasizing that we are talking about orders of magnitude. The base for simplifying the equations is that the order terms ε^2 were discarded during the derivation, so we also neglect them in the model.

In the first case, we present the solutions of the fourth equation of system (5) in the form:

$$\begin{aligned} w_1 &= w_1^{(0)} + \mu w_1^{(1)} + \mu^2 w_1^{(2)} + O(\mu^3), \\ u_1 &= u_1^{(0)} + \mu u_1^{(1)} + \mu^2 u_1^{(2)} + O(\mu^3). \end{aligned} \quad (17)$$

We substitute (17) in the last equation of the system (5) and equate the terms at the same powers μ in the left and right parts of the equation, and replace the term $\nu\mu$ with $\mu^{3/2}$. We get:

$$\begin{aligned} \mu^0: \quad w_1^{(0)} - \frac{R_0 p_a}{h E^*} p_1 + 2\nu w_0 &= 0; \\ \mu^1: \quad w_1^{(1)} + \frac{\partial u_0}{\partial \zeta} &= 0; \\ \mu^{3/2}: \quad \frac{\partial}{\partial \zeta} (u_1 - u_0) &= 0. \end{aligned} \quad (18)$$

Multiplying the second equation (18) by the parameter μ , adding it to the first equation, and denoting $w_1 = w_1^{(0)} + \mu w_1^{(1)}$, we get:

$$w_1 - \frac{R_0 p_a}{h E^*} p_1 + \mu \frac{\partial u_0}{\partial \zeta} + 2\nu w_0 = 0. \quad (19)$$

Since the first equation of the system (16) is composed accurately to the first orders in μ , the values of w_1 in (16) and (19) can be considered equal. Therefore, the complete system of model equations for case 1 has the form:

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left(\mu \frac{\partial u_0}{\partial \zeta} + \nu w_0 \right) - \mu \frac{c_f^2}{c_t^2} \frac{\partial^2 u_0}{\partial \tau^2} &= 0; \\ w_0 + \nu \mu \frac{\partial u_0}{\partial \zeta} - \frac{R_0 p_a}{h E^*} p_0 + \mu^2 \left(\frac{1}{2a^2} \frac{R_0 p_a}{h E^*} + \frac{c_f^2}{c_t^2} \right) \frac{\partial^2 w_0}{\partial \tau^2} &= 0; \\ \frac{\partial V_0}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} &= 0; \\ a^2 \frac{\partial p_0}{\partial \tau} + \frac{\partial V_0}{\partial \zeta} + 2 \frac{\partial w_0}{\partial \tau} &= 0; \\ w_1 - \frac{R_0 p_a}{h E^*} p_1 + \mu \frac{\partial u_0}{\partial \zeta} + 2\nu w_0 &= 0; \\ a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_1}{\partial \zeta} + 2 \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_0}{\partial \zeta} \right) \right] &= 0; \\ \frac{\partial V_1}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} &= 0. \end{aligned} \quad (20)$$

The equation for u_1 is excluded, since this value does not affect the pressure variations, which corresponds to the conclusion [8,9].

Analyzing system (5) in case 2, when $\varepsilon \sim \mu$, we represent w_0 and w_1 as:

$$\begin{aligned} w_0 &= w_0^{(0)} + \mu w_0^{(1)} + O(\mu^2), \\ w_1 &= w_1^{(0)} + \mu w_1^{(1)} + O(\mu^2). \end{aligned} \quad (21)$$

Substituting (21) in (5) and equating the terms for the same powers of μ in the left and right parts of the equations, we get:

$$\begin{aligned} \text{at } \mu^0: & w_0^{(0)} - \frac{R_0 p_a}{h E^*} p_0 = 0, \quad w_1^{(0)} - \frac{R_0 p_a}{h E^*} p_1 + 2\nu w_0 = 0; \\ \text{at } \mu^1: & w_0^{(1)} + \nu \frac{\partial u_0}{\partial \zeta} = 0, \quad w_1^{(1)} + \nu \frac{\partial u_1}{\partial \zeta} + (1-\nu) \frac{\partial u_0}{\partial \zeta} = 0. \end{aligned} \quad (22)$$

In this case, $w_1^{(1)}$ is not calculated, since substituting (21) in (4), we get:

$$w = w_0(\zeta, \tau) + \varepsilon(w_1^{(0)} + \mu w_1^{(1)}) \sin \theta,$$

and the product $\varepsilon\mu$ has the order ε^2 . From (22), denoting $w_0 = w_0^{(0)} + \mu w_0^{(1)}$, $w_1 = w_1^{(0)}$, we get:

$$\begin{aligned} w_0 - \frac{R_0 p_a}{h E^*} p_0 + \nu \mu \frac{\partial u_0}{\partial \zeta} &= 0, \\ w_1 - \frac{R_0 p_a}{h E^*} p_1 + 2\nu w_0 &= 0. \end{aligned} \quad (23)$$

The function u_1 disappeared from the equations. The equations of the mathematical model for case 2, taking into account (23), take the form:

$$\begin{aligned} \frac{\partial}{\partial \zeta} \left(\mu \frac{\partial u_0}{\partial \zeta} + \nu w_0 \right) - \mu \frac{c_f^2}{c_t^2} \frac{\partial^2 u_0}{\partial \tau^2} &= 0, \\ w_0 + \nu \mu \frac{\partial u_0}{\partial \zeta} - \frac{R_0 p_a}{h E^*} p_0 &= 0, \\ \frac{\partial V_0}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} &= 0, \\ a^2 \frac{\partial p_0}{\partial \tau} + \frac{\partial V_0}{\partial \zeta} + 2 \frac{\partial w_0}{\partial \tau} &= 0, \\ w_1 - \frac{R_0 p_a}{h E^*} p_1 + 2\nu w_0 &= 0, \\ a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_1}{\partial \zeta} + 2 \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_0}{\partial \zeta} \right) \right] &= 0, \\ \frac{\partial V_1}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} &= 0. \end{aligned} \quad (24)$$

Thus, for a fixed wavelength λ , in the membrane shell approximation, it is possible to describe the processes occurring in pipes either with a smaller bend with greater accuracy (case 1), or with a stronger bend, but with less accuracy (case 2). When choosing the method of description, it should be borne in mind that the characteristics of the systems of equations (20) and (24) differ, as will be shown below.

5. Analysis of equations by the method of characteristics

To reduce (20), (24) to systems of first-order equations, we change of variables:

$$\begin{aligned}\sigma_\zeta &= \frac{c_t^2}{c_f^2} \left(\mu \frac{\partial u_0}{\partial \zeta} + \nu w_0 \right); \\ \sigma_\theta &= \frac{c_t^2}{c_f^2} \left(w_0 + \nu \mu \frac{\partial u_0}{\partial \zeta} \right); \\ \dot{w}_0 &= \frac{\partial w_0}{\partial \tau}; \quad \dot{u}_0 = \frac{\partial u_0}{\partial \tau}.\end{aligned}\tag{25}$$

We denote:

$$\alpha = \frac{R_0 \rho_f c_f^2}{h \rho_t c_t^2}.$$

Taking into account (25), equations (20), (24) take the following form.

Case 1:

$$\begin{aligned}\frac{\partial \sigma_\zeta}{\partial \zeta} - \mu \frac{\partial \dot{u}_0}{\partial \tau} &= 0, \\ \left(1 + \frac{1}{2} \frac{R_0 \rho_f}{h \rho_t} \right) \mu^2 \frac{\partial \dot{w}_0}{\partial \tau} + \sigma_\theta - \frac{R_0 p_a c_t^2}{h E^* c_f^2} p_0 &= 0, \\ \frac{\partial \sigma_\zeta}{\partial \tau} - \frac{c_t^2}{c_f^2} \left(\mu \frac{\partial \dot{u}_0}{\partial \zeta} + \nu \dot{w}_0 \right) &= 0, \\ \frac{\partial \sigma_\theta}{\partial \tau} - \frac{c_t^2}{c_f^2} \left(\dot{w}_0 + \nu \mu \frac{\partial \dot{u}_0}{\partial \zeta} \right) &= 0, \\ \frac{\partial V_0}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} &= 0, \\ a^2 \frac{\partial p_0}{\partial \tau} + \frac{\partial V_0}{\partial \zeta} + 2 \frac{\partial w_0}{\partial \tau} &= 0, \\ a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_1}{\partial \zeta} + 2 \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\frac{\partial w_0}{\partial \tau} - 2 \frac{\partial V_0}{\partial \zeta} \right) \right] &= 0, \\ \frac{\partial V_1}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} &= 0, \\ w_1 - \frac{R_0 p_a}{h E^*} p_1 + \mu \frac{\partial u_0}{\partial \zeta} + 2 \nu w_0 &= 0.\end{aligned}\tag{26}$$

Case 2:

$$\begin{aligned}\frac{\partial \sigma_\zeta}{\partial \zeta} - \mu \frac{\partial \dot{u}_0}{\partial \tau} &= 0, \\ \frac{\partial \sigma_\zeta}{\partial \tau} - \frac{c_t^2}{c_f^2} (1 - \nu^2) \mu \frac{\partial \dot{u}_0}{\partial \zeta} - \nu \frac{R_0 \rho_f}{h \rho_t} a^2 \frac{\partial p_0}{\partial \tau} &= 0, \\ \frac{\partial V_0}{\partial \tau} + a^2 \frac{\partial p_0}{\partial \zeta} &= 0, \\ a^2 (1 + 2\alpha) \frac{\partial p_0}{\partial \tau} + \frac{\partial V_0}{\partial \zeta} - 2 \nu \mu \frac{\partial \dot{u}_0}{\partial \zeta} &= 0, \\ \frac{\partial V_1}{\partial \tau} + a^2 \frac{\partial p_1}{\partial \zeta} &= 0,\end{aligned}\tag{27}$$

$$a^2 \frac{\partial p_1}{\partial \tau} + \frac{\partial V_1}{\partial \zeta} + 2 \left[\frac{\partial w_1}{\partial \tau} + \frac{1}{3} \left(\dot{w}_0 - 2 \frac{\partial V_0}{\partial \zeta} \right) \right] = 0,$$

$$w_1 - \frac{R_0}{h} \frac{p_a}{E^*} p_1 + 2\nu \frac{R_0}{h} \frac{p_a}{E^*} p_0 = 0.$$

We divide the analysis of equations of the zeroth and first approximation. In this case, the solutions of the zeroth approximation equations are considered known when analyzing the first approximation equations [35].

The canonical form of the system of equations (26) for the zeroth approximation:

$$\begin{aligned} \frac{\partial I_+}{\partial \tau} + \frac{\partial I_+}{\partial \zeta} + 2\dot{w}_0 &= 0, \\ \frac{\partial I_-}{\partial \tau} - \frac{\partial I_-}{\partial \zeta} + 2\dot{w}_0 &= 0; \\ \frac{\partial J_-}{\partial \tau} + \frac{\partial J_-}{\partial \zeta} - \frac{c_t^2}{c_f^2} \nu \dot{w}_0 &= 0, \\ \frac{\partial J_+}{\partial \tau} - \frac{\partial J_+}{\partial \zeta} - \frac{c_t^2}{c_f^2} \nu \dot{w}_0 &= 0; \\ \frac{\partial \Sigma}{\partial \tau} - \frac{c_t^2}{c_f^2} (1 - \nu^2) \dot{w}_0 &= 0; \\ \left(1 + \frac{1}{2} \frac{R_0}{h} \frac{\rho_f}{\rho_t} \right) \mu^2 \frac{\partial \dot{w}_0}{\partial \tau} + \Sigma + \frac{\nu}{2} (J_+ + J_-) - \frac{1}{2} \frac{R_0}{h} \frac{\rho_f}{\rho_t} (I_+ + I_-) &= 0. \end{aligned} \quad (28)$$

The desired functions of the system of equations (28):

$$I_+ = a^2 p_0 + V_0; \quad I_- = a^2 p_0 - V_0;$$

$$J_+ = \sigma_\zeta + \mu \frac{c_t}{c_f} \dot{u}_0; \quad J_- = \sigma_\zeta - \mu \frac{c_t}{c_f} \dot{u}_0;$$

$$\Sigma = \sigma_\theta - \nu \sigma_\zeta; \quad \dot{w}_0 = \dot{w}_0$$

are Riemann invariants for the system (26). They propagate along with the characteristics:

$$\begin{aligned} \frac{d\tau}{d\zeta} = 1; \quad \frac{d\tau}{d\zeta} = -1; \\ \frac{d\tau}{d\zeta} = -\frac{c_f}{c_t}; \quad \frac{d\tau}{d\zeta} = \frac{c_f}{c_t}; \\ \frac{d\tau}{d\zeta} = 0 \quad (\text{two multiple characteristics}). \end{aligned} \quad (29)$$

We find the characteristics of the system (27) by equating to zero the determinant of the matrix composed of coefficients for derivatives [36]. We get:

$$\delta - (k\delta + 2\nu^2\beta + 1)x^2 + kx^4 = 0. \quad (30)$$

Here

$$\begin{aligned} \delta = 1 + 2\alpha; \quad \alpha = \frac{R_0}{h} \frac{\rho_f c_f^2}{\rho_t c_t^2}; \\ k = \frac{c_t^2}{c_f^2} (1 - \nu^2); \quad \beta = \frac{R_0}{h} \frac{\rho_f}{\rho_t}; \quad x = \frac{d\tau}{d\zeta}. \end{aligned}$$

We represent the solution (30) as a series with respect to the parameter ν^2 :

$$x = x_0 + \nu^2 x_1 + O(\nu^4).$$

For unknown x_0 we get

$$\begin{aligned} x_{01} &= \pm\sqrt{\delta} = \pm\sqrt{1+2\alpha}; \\ x_{02} &= \pm\sqrt{1/k} = \pm\frac{c_f}{c_t}\sqrt{1-\nu^2} \end{aligned} \quad (31)$$

The corrections for the coefficient ν^2 will be equal to

$$\begin{aligned} x_{11} &= \pm\frac{2\beta\sqrt{\delta}}{k\delta-1}, \\ x_{12} &= \pm\frac{2\beta}{\sqrt{k}(1-k\delta)}. \end{aligned} \quad (32)$$

Approximate solutions are made according to the rule:

$$\frac{d\tau}{d\zeta} = x_{0i} + \nu^2 x_{1i}. \quad (33)$$

The search for the exact solution of equation (30) is not difficult, but this solution is very extensive. It can be seen from (31)-(33) that x_{0i} describes the waves that arise in the classical theory of water hammer [6], and x_{1i} – the influence of the Poisson effect on their propagation. Based on the assumptions about the order of the parameters for this case, taking into account the Poisson effect is necessary to keep the declared accuracy of the model, since $\nu^2 \sim \mu$. It follows from (32) that this effect increases the wave velocity in the liquid and decreases it in the pipe wall.

It can be seen from (29), (33) that the zeroth approximations of both cases give the same types of waves that are described in [19,9], respectively.

Considering the solutions of the zeroth approximation to be known and comparing the equations of the first approximation in systems (26) and (27), we can conclude that the characteristics of the subsystems of the first approximation in (26) and (27) coincide. They have the form:

$$\frac{d\tau}{d\zeta} = \pm\sqrt{1+2\alpha},$$

and are obtained from the corresponding homogeneous system of equations, which is the same for both cases:

$$\frac{\partial I_{+1}}{\partial \tau} + \frac{1}{\sqrt{1+2\alpha}} \frac{\partial I_{+1}}{\partial \zeta} = 0,$$

$$\frac{\partial I_{-1}}{\partial \tau} - \frac{1}{\sqrt{1+2\alpha}} \frac{\partial I_{-1}}{\partial \zeta} = 0,$$

where

$$I_{+1} = a^2\sqrt{1+2\alpha} p_1 + V_1,$$

$$I_{-1} = V_1 - a^2\sqrt{1+2\alpha} p_1.$$

Thus, in the first approximation of the parameter ε , a wave moving at the velocity $c_f\sqrt{1+2\alpha}$ appears in a curved tube. It can be hypothesized that this wave is the precursor wave mentioned in the article [9].

6. Results of numerical analysis

To verify the mathematical model, a test example was chosen, described in [37,2]. In [37], a mathematical model is developed and the acoustic vibrations of a curved pipe, which is a fragment of a torus, are calculated. In [2], the problem statement is modified to analyze the mutual influence of the pipe and the liquid filling it.

Let the metal pipe be a segment of the torus with a length $1/4$ of the total length of its circumference. The pipe has a centerline radius of the $\rho_0 = 0.4$ m, section radius $R_0 = 0.0015$ m and the length of the axis $L = 0.3$ m. Other parameters of the mechanical system: $h = 0.0006$ m, $\rho_t = 7800$ kg/m³, $E = 2 \cdot 10^{11}$ Pa, $\nu = 0.3$, $\rho_f = 870$ kg/m³, $c_f = 1300$ m/c, $P_0 = 10^5$ Pa, $\omega_0 = 150$ Hz.

The frequency and amplitude of pressure fluctuations are indicated by ω_0 and P_0 .

Constant pressure is maintained at $s = L$.

Comparison with the results of the calculation of the technical shell. The numerical analysis is performed in comparison with the mathematical model [2].

For the consistency of the numerical analysis, the mathematical model [2] is reduced to a membrane form. To do this, the moment components are excluded from the shell equations, as a result, the shell is described by equations (2). This is equivalent to adding to a mathematical model (5), (11), (16) dissipative components responsible for the internal friction of the flow and the resistance of the external medium in the work [2].

The system of equations of the mathematical model is supplemented with initial and boundary conditions:

$$w_0(s, 0) = 0, u_0(s, 0) = \frac{\partial u_0}{\partial t}(s, 0) = \frac{\partial w_0}{\partial t}(s, 0) = p_0(s, 0) = V_0(s, 0) = 0;$$

$$u_0(0, t) = u_0(L, t) = 0, p_0(0, t) = 2P_0 \cos \omega_0 t, \frac{\partial V_{0p}}{\partial t}(L, t) = 0.$$

For the numerical solution, an explicit difference scheme is applied, described in detail in [2].

The graphs of the dimensional functions u_0 are shown in Fig. 2.

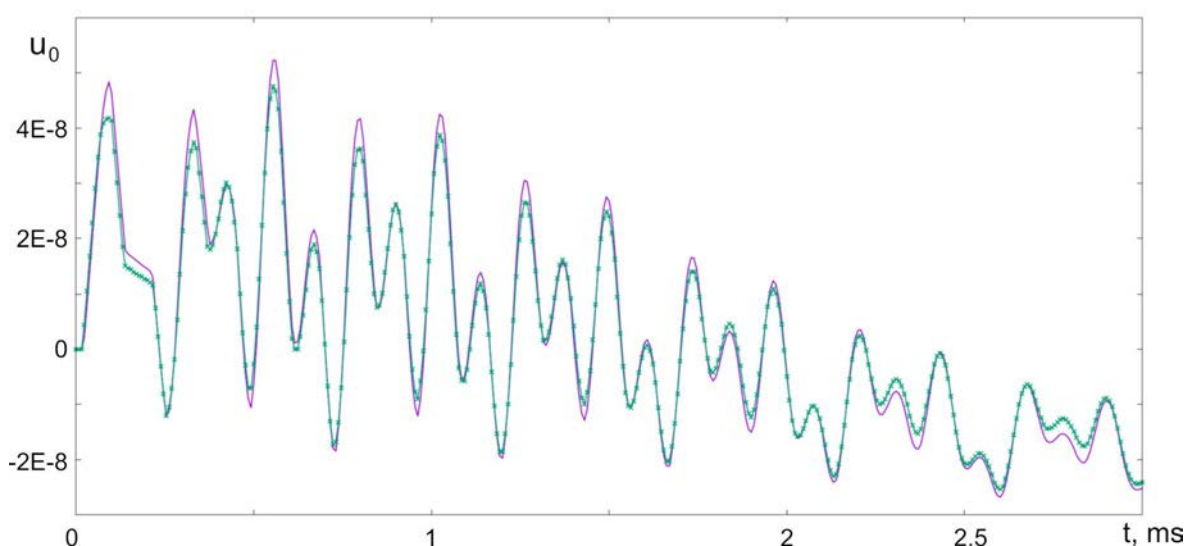


Fig. 2. Displacement of the centerline at $s = 0.1$ m. The continuous line is the displacement of the membrane shell, the line with markers is the semi-momentless shell

The graphs show that taking into account the bending moments in the framework of V.Z. Vlasov shell theory [29] does not lead to a significant change in the solution in the zeroth approximation with respect to the parameter ε (4).

To estimate the relative deviation, the values of magnitude are found:

$$\varepsilon_1 = \frac{|u_{T0} - u_0|}{\max |u_{T0}|},$$

where u_{T0} is the longitudinal displacement of the semi-momentless shell [2]. The graph of ε_1 as a function of time is shown in Fig. 3. The calculation is performed at a time interval of 16 ms.

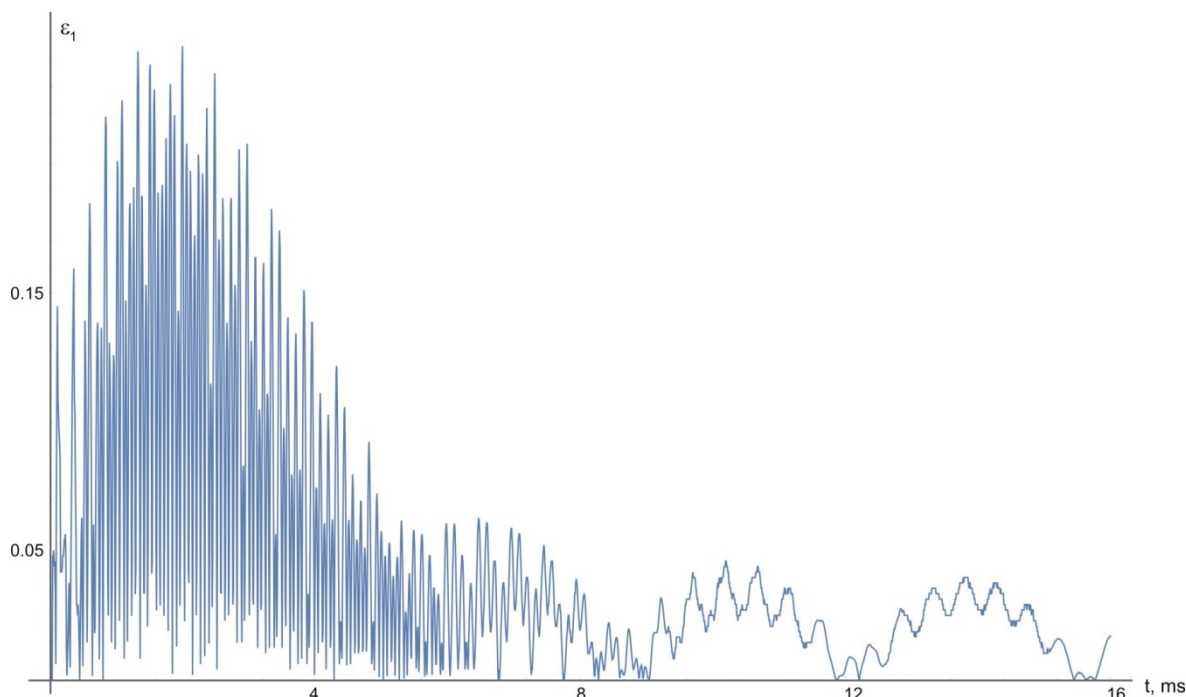


Fig. 3. Relative deviation of displacement ε_1

The graph shows that the greatest relative difference in the displacements of the semi-momentless and membrane shells is observed with a strongly nonstationary motion at the initial moments of time. At the same time, the absolute values of deviations are small even under these conditions.

For stationary vibrations, the relative deviation does not exceed 5%, this value is equal to the error of the shell theory. Similar results are obtained for other parameters (pressure and velocity of fluid, radial displacement).

It follows from the performed numerical experiments that in the linear approximation and with stationary vibrations in the mathematical model of a uniformly curved thin-walled pipe, it is possible not to take into account bending moments.

Verification of the mathematical model. The constructed mathematical model of the dynamics of a curved pipeline as a membrane shell is studied in MATLAB. This package requires presenting the equations in a special form:

$$c \left(x, t, u, \frac{\partial u}{\partial x} \right) \frac{\partial u}{\partial t} = x^{-m} \frac{\partial}{\partial x} \left(x^m f \left(x, t, u, \frac{\partial u}{\partial x} \right) \right) + s \left(x, t, u, \frac{\partial u}{\partial x} \right), \quad (34)$$

where c , f , s are vectors, x and t are independent variables in space and time, u is the required vector-function, m is the symmetry constant, which equals to zero in our case.

The system of equations of the model (20) for case 1 in the zeroth approximation, reduced to the form (34):

$$\begin{aligned} \frac{1}{c_t^2} \frac{\partial \dot{u}_{0p}}{\partial t} &= \frac{\partial}{\partial s} \left(\frac{\partial u_{0p}}{\partial s} + \nu \frac{w_{0p}}{R_0} \right); \\ R_0 \left(\frac{R_0 \rho_f}{2h E^*} + \frac{1}{c_t^2} \right) \frac{\partial \dot{w}_{0p}}{\partial t} &= -\nu \frac{\partial u_{0p}}{\partial s} - \frac{w_{0p}}{R_0} + \frac{R_0 p_{0p}}{h E^*}; \\ \frac{\partial V_{0p}}{\partial t} &= -\frac{1}{\rho_f} \frac{\partial p_{0p}}{\partial s}; \\ \frac{1}{\rho_f c_f^2} \frac{\partial p_{0p}}{\partial t} &= -\frac{\partial V_{0p}}{\partial s} - \frac{2}{R_0} \dot{w}_{0p}; \\ \frac{\partial w_{0p}}{\partial t} &= \dot{w}_{0p}; \\ \frac{\partial u_{0p}}{\partial t} &= \dot{u}_{0p}. \end{aligned} \quad (35)$$

Here all the values are dimensional.

To solve the system of equations (35), the *pdepe* function is called with the following initial and boundary conditions:

$$\begin{aligned} w_{0p}(s, 0) &= \frac{R_0^2 p_a}{h E^*}, \quad u_{0p}(s, 0) = \frac{\partial u_{0p}}{\partial t}(s, 0) = \frac{\partial w_{0p}}{\partial t}(s, 0) = p_{0p}(s, 0) = V_{0p}(s, 0) = 0; \\ u_{0p}(0, t) &= u_{0p}(L, t) = 0, \quad p_{0p}(0, t) = P_0 \cos \omega_0 t, \quad \frac{\partial V_{0p}}{\partial t}(L, t) = 0. \end{aligned}$$

The parameters of the mechanical system are listed at the beginning of section 6.

The result of calculating the longitudinal displacement of the pipe wall at $s = 0.2$ according to the formulas (35) is shown in Fig. 4 in dimensionless parameters. The oscillation pattern is qualitatively consistent with the results of the calculation [37] without taking into account friction, where a mathematical model of hydroelastic vibrations of a pipeline loaded with a pulsating fluid flow with similar parameters is numerically analyzed. The time and distance from the beginning of the pipe are agreed with [37]. In both cases, the oscillations are not damped.

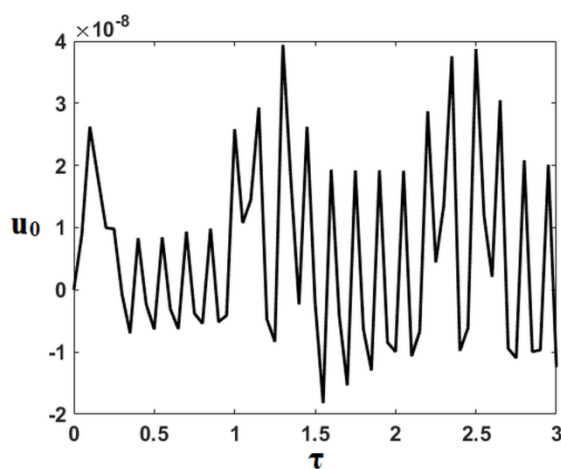


Fig. 4. Dimensionless longitudinal displacement of the pipe wall at a distance of 0.2 m from its beginning

As a result of the fast Fourier transform in MATLAB, two graphs of the amplitude spectrum of longitudinal vibrations of the pipe wall are obtained using the built-in *fft* function. Segments of length 3 (Fig. 5a) and 26 τ units (Fig. 5b) are studied. You can notice a sharp increase in the amplitude of vibrations of the pipe wall at a frequency of 400 Hz, which can be caused by the phenomenon of resonance.

Peaks of 150 and 425 Hz on the amplitude spectrum are obtained in the source [37]. The spectrum in Fig. 5 looks complex, but the maxima near the above frequencies are well highlighted.

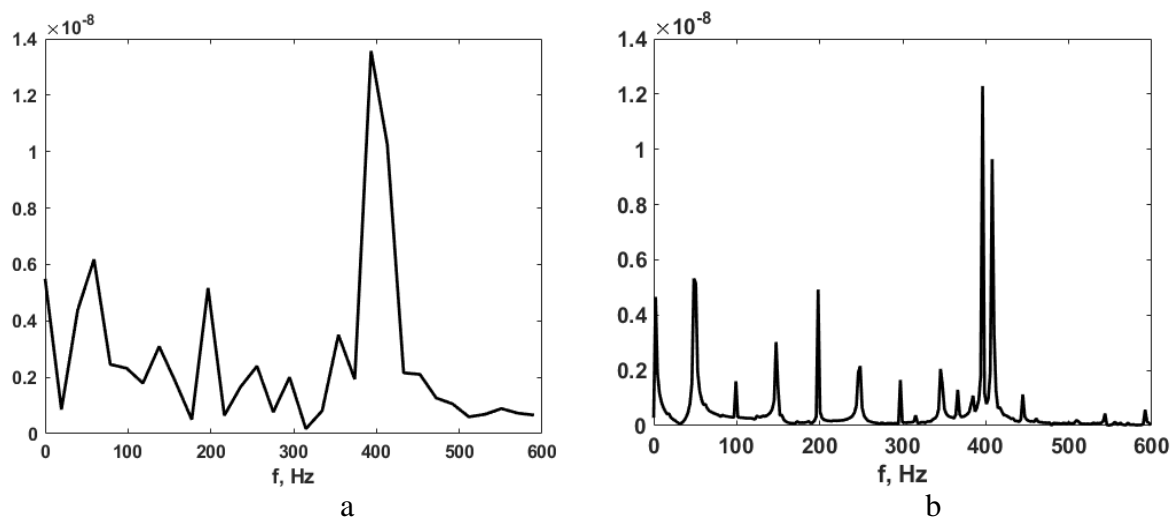


Fig. 5. The amplitude spectrum of the longitudinal vibrations of the pipe wall at the point $s = 0.2$: a – up to $\tau = 3$, b – up to $\tau = 26$

Thus, the obtained results are consistent with the known data from the literature. We also note that the constructed mathematical model can be numerically investigated in standard mathematical packages and does not require the development of special solution methods.

7. Conclusions

Problems solved:

1. A mathematical model of the dynamics of a curved pipeline as a membrane shell filled with an ideal compressible fluid is constructed.
2. The form (4), (8) of the approximate solution of the model equations is proposed, in which the dimension of the problem is reduced by one (the angular coordinate is excluded).
3. The radial coordinate is eliminated from the model equations, by consecutive changes of variables and transformations. A reduced mathematical model is constructed (5), (11), (16), the desired functions of which depend only on one spatial coordinate and time.
4. For two relations between the parameters of the mathematical model, an asymptotic analysis of the equations of the reduced model is performed, which led to two different systems of equations (20), (24) with different characteristics.
5. The resulting systems of model equations are transformed into systems of first-order equations. This result was obtained by replacing unknown functions and isolating Riemann invariants. The characteristics of the obtained systems of equations are found and the main types of waves propagating in a curved tube are selected.
6. Numerical experiments have been performed. The results obtained correspond to the data from the literature sources.

Thus, within the framework of the assumptions made, a simpler mathematical model is constructed than that used in [1,2]. It is shown that it has sufficient generality to cover a wide class of pipe vibration phenomena described in the literature on the basis of the rod theory.

In applications, the proposed mathematical model can be applied to calculate transient and oscillatory modes in curved pipelines, mentioned, for example, in [9,20,37]. In addition, this model can be used to create data arrays for training neural networks of pipeline monitoring systems [4].

References

- [1] Rukavishnikov VA, Tkachenko OP. Numerical and asymptotic solution of the equations of propagation of hydroelastic vibrations in a curved pipe. *Journal of Applied Mechanics and Technical Physics*. 2000;41(6): 1102-1110.
- [2] Tkachenko OP, Ryabokon' AS. Numerical estimates of the adequacy of the mathematical model of hydroelastic oscillations in curved pipeline. *Computational Continuum Mechanics*. 2017;10(1): 90-102. (In Russian)
- [3] Liu R, Xiong H, Wu X, Yan Sh. Numerical studies on global buckling of subsea pipelines. *Ocean Engineering*. 2014;78: 62-72.
- [4] Ho M, El-Borgi S, Patil D, Song G. Inspection and monitoring systems subsea pipelines: A review paper. *Structural Health Monitoring*. 2020;19(2): 606-645.
- [5] Kwon HJ. Computer simulations of transient flow in a real city water distribution system. *KSCE Journal of Civil Engineering*. 2007;11(1): 43-49.
- [6] Zhukowski NE. *About water hammer in water pipes*. Moscow: State Publisher of Technical-Theoretical Literature; 1949. (In Russian)
- [7] Aldoshin GT. On the history of hydroelasticity from Euler to the present day. *Mechanics of Solid*. 2007;27: 184-191. (In Russian)
- [8] Skalak R. An extension of the theory of water hammer. *Transaction of the ASME*. 1956;78(1): 105-116.
- [9] Wiggert DC, Otwell RS, Hatfield FJ. The effect of elbow restraint of pressure transients. *Journal of Fluids Engineering*. 1985;107(3): 402-406.
- [10] Aldoshin GT. Water hammer in a deformed pipeline. *Vestnik of Leningrad University. Series I. Mathematics. Mechanics. Astronomy*. 1961;V.Ch: 93-102. (In Russian)
- [11] Li S, Karney BW, Liu G. FSI research in pipeline systems – A review of the literature. *Journal of Fluids and Structures*. 2015;57: 277-297.
- [12] Tijsseling AS. Water hammer with fluid-structure interaction in thick-walled pipes. *Computers and Structures*. 2007;85(11-14): 844-851.
- [13] Daude F, Tijsseling AS, Galon P. Numerical investigations of water-hammer with column-separation induced by vaporous cavitation using a one-dimensional Finite-Volume approach. *Journal of Fluids and Structures*. 2018;83: 91-118.
- [14] Srinil N. Analisis and prediction of vortex-induced vibrations of variable-tension vertical risers in linearly sheared currents. *Applied Ocean Research*. 2011;33(1): 41-53.
- [15] Chen W, Ji C, Xu D, Srinil N. Wake patterns of freely vibrating side-by-side circular cylinders in laminar flows. *Journal of Fluids and Structures*. 2019;89: 82-95.
- [16] Ma B, Srinil N. Two-dimensional vortex-induced vibration suppression through the cylinder transverse linear/nonlinear velocity feedback. *Acta Mechanica*. 2017;228(12): 4369-4389.
- [17] Svetlitsky VA. *Dynamics of Rods*. Berlin: Springer; 2005.
- [18] Thorley ARD. Pressure transients in hydraulic pipelines. *Journal of Basic Engineering*. 1969;91(3): 453-460.
- [19] Walker JS, Phillips JW. Pulse propagation in fluid-filled tubes. *Journal of Applied Mechanics*. 1977;44(1): 31-35.

- [20] Otwell RS. The effect of elbow translations on pressure transient analysis of piping systems. *ASME PVP, Fluid Transients and Fluid-Structure Interaction*. 1982;64: 127-136.
- [21] Hatfield FJ, Wiggert DC, Otwell RS. Fluid structure interaction in piping by component synthesis. *Journal of Fluids Engineering*. 1982;104(3): 318-325.
- [22] Williams DJ. Water hammer in non-rigid pipes: Precursor waves and mechanical damping. *Journal of Mechanical Engineering Science*. 1977;19(6): 237-242.
- [23] Klar A, Marshall AM. Shell versus beam representation of pipes in the evaluation of tunneling effects on pipelines. *Tunnelling and Underground Space Technology*. 2008;23(4): 431-437.
- [24] Volmir AS. *Shells in the flow of liquid and gas: problems of hydroelasticity*. Moscow: Science; 1979. (In Russian)
- [25] Novozhilov VV, Radok JRM. *Thin Shell Theory*. Netherlands: Springer; 2014.
- [26] Liu X, Zhang H, Li M, Xia M, Zheng W, Wu K, et al. Effects of steel properties on the local buckling response of high strength pipelines subjected to reverse faulting. *Journal of Natural Gas Science and Engineering*. 2016;33: 378-387.
- [27] Yan Y, Shao B, Wang J, Yan X. A study on stress of buried oil and gas pipeline crossing a fault based on thin shell FEM model. *Tunnelling and Underground Space Technology*. 2018;81: 472-479.
- [28] Tkachenko OP. The mathematical model of propagation of the pressure wave in the fluid stream within the curved underground pipeline. *Computational Technologies*. 1996;1(3): 78-86. (In Russian)
- [29] Vlasov VZ. The principles of constructing a general technical theory of shells. In: *Selected Works*. Moscow: Publishing house of the Academy of Sciences of the USSR; 1963. p. 467-503. (In Russian)
- [30] Kartvelishvili NA. *Dynamics of pressure pipelines*. Moscow: Energy; 1979. (In Russian)
- [31] Rukavishnikov VA, Tkachenko OP. Dynamics of a fluid-filled curvilinear pipeline. *Applied Mathematics and Mechanics*. 2018;39(6): 905-922.
- [32] Samul VI. *Fundamentals of the theory of elasticity and plasticity*. Moscow: Science; 1982. (In Russian)
- [33] Sedov LI. *Mechanics of Continuous Media*. Singapore: World Scientific Publishing Company; 1997.
- [34] Tkachenko OP. The resolving equations of the mathematical model of curved pipeline. *Bulletin of the Yakovlev Chuvash State Pedagogical University. Series: Mechanics of Limit State*. 2017;4: 114-124. (In Russian)
- [35] Nayfeh AH. *Perturbation Methods*. New York: Wiley; 1973.
- [36] Godunov SK. *Equations of mathematical physics*. Moscow: Science; 1979. (In Russian)
- [37] Mironova TB, Prokofiev AB, Shachmatov EV. Development of the finite element model of vibroacoustical characteristics of pipe system. *Vestnik of Samara University. Aerospace and Mechanical Engineering*. 2008;7(3): 157-162. (In Russian)

THE AUTHORS

Tkachenko O.P.

e-mail: olegt1964@gmail.com

ORCID: 0000-0003-1806-0274

Ryabokon A.S.

e-mail: anyuta.riabokon@yandex.ru

ORCID: 0000-0003-2713-4917