

Bearing capacity of spherical thick-walled shell taking into account compressibility and nonlinear plasticity

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Abstract. The stress-strain state of a thick-walled spherical shell is considered under the conditions of compressibility of the material and the nonlinear law of hardening. Using the equations of the relationship between stresses and deformations according to the method of variable elasticity parameters, an integral equation of compatibility of logarithmic deformations is obtained. When performing numerical calculations using the method of simple iterations, the moment of unstable deformation of the spherical shell is determined. The dependences of the relative pressure on the radial displacement of the points of the outer surface of the spherical shell are obtained, taking into account the compressibility of the material and without taking into account the compressibility for an ideal elastic-plastic material and for an elastic-plastic material with nonlinear hardening. According to the results of numerical calculation, failure to account for compressibility introduces a significant error in the calculation of radial displacements of the outer surface. The results of the study will allow us to determine the maximum permissible load of a thick-walled spherical shell corresponding to stable deformation.

Keywords: thick-walled shell, sphere, compressibility of the material, physical nonlinearity, stress-strain state, variable elasticity parameters, shaping

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1. Introduction

Thick-walled axisymmetric shells are currently widely used in various engineering structures. These are high-pressure vessels, ring foundations, pressure pipes, tunnels, and others. Stresses in such structures are distributed unevenly across the thickness, which must be taken into account when calculating strength. The issues of modeling the stress-strain state of shells of various types were considered in [1-7] under conditions of incompressibility of the material, linear hardening, and ideal plasticity.

When conducting a literature review on the subject of the study, one of the little-studied aspects is taking into account the compressibility of the material in the process of elastic-plastic deformation. In addition, it should be noted that the issues of assessing the stress-strain state of shells were considered in the works [8-10]. The analysis of radial and circumferential stresses in a spherical shell was considered in the works [11,12]. The physical nonlinearity of materials manifested in the nonlinear relationship between stresses and deformations under various types of deformation was considered in the works [13-15]. The features of working with the deformation diagram when assessing the stress-strain state were considered in the article [16]. The problem of solving elastic-plastic problems under various hardening laws, and the evaluation of the methods of the solution was considered in the works [17-21]. Aspects of finite element modeling of stamping and shaping problems were investigated in the works [22,23]. The application of the method of variable elasticity parameters in finite element analysis was considered in the work [24]. The issues of modeling and optimization in the field of metallurgy, and mechanical engineering were studied in articles [25-33].

2. Methods

The main equations for calculations beyond the limits of elasticity according to the deformation theory are differential equations of equilibrium, conditions for the compatibility of deformations, the relationship between deformations and stresses, and boundary conditions. When determining the stress-strain state of thick-walled shells, it is also necessary to take into account the physical nonlinearity of the material – the law of hardening during plastic deformation.

The deformation diagram of a material, as a relationship between the intensity of stresses and the intensity of logarithmic deformations, is given by a linear power function [5]:

$$\sigma_i = \begin{cases} 3G e_i & , e_i \leq e_{iT} \\ A e_i^n & , e_i > e_{iT} \end{cases} \quad (1)$$

where $G = E/2(1 + \mu)$ – modulus of elasticity of the second kind, e_i – intensity of logarithmic strains, e_{iT} – intensity of logarithmic strains corresponding to the yield strength, A , n – approximating coefficients of the power function, E – Young's module, μ – Poisson's ratio.

Consider the problem of determining the bearing capacity of a spherical thick-walled shell, taking into account compressibility and nonlinear plasticity. We will solve the problem using the method of variable elasticity parameters. This method is quite universal since it allows solving elastic problems, elastic-plastic, and purely plastic problems in a single formulation.

Consider the stress-strain state of a thick-walled spherical shell under the conditions of compressibility of the material and the nonlinear law of hardening. For a volumetric axisymmetric stress state in a spherical coordinate system ρ, θ, φ at uniform pressure inside the sphere, there are no tangential stresses, derivatives with respect to φ are zero, and the equilibrium conditions have the form:

$$\frac{\partial \sigma_\rho}{\partial \rho} + \frac{1}{\rho} [2\sigma_\rho - (\sigma_\varphi + \sigma_\theta)] = 0;$$

$$\sigma_\varphi = \sigma_\theta$$

or

$$\frac{d\sigma_\rho}{d\rho} = \frac{2(\sigma_\theta - \sigma_\rho)}{\rho}, \quad (2)$$

where σ_ρ , σ_θ , σ_φ – radial, tangential (circumferential) and meridional stresses.

Integrating equation (2), we obtain the integral equation of equilibrium in the form:

$$\sigma_\rho = \int_R^\rho \frac{2(\sigma_\theta - \sigma_\rho)}{\rho} d\rho + \sigma_{\rho R}, \quad (3)$$

where R – radius of the outer surface of the sphere, $\sigma_{\rho R}$ – radial stress on the outer surface of the sphere.

The conditions for the compatibility of logarithmic strains in the case of central symmetry are represented as [6]:

$$\frac{de_\theta}{d\rho} = \frac{1 - \exp(e_\theta - e_\rho)}{\rho}; \quad (4)$$

$$e_\varphi = e_\theta.$$

The equations of the relationship between stresses and deformations, in accordance with the method of variable elasticity parameters, have the form:

$$\left. \begin{aligned} e_\rho &= \frac{1}{E^*} [\sigma_\rho - 2\mu^* \sigma_\theta]; \\ e_\theta &= \frac{1}{E^*} [\sigma_\theta - \mu^* (\sigma_\theta + \sigma_\rho)]. \end{aligned} \right\}, \quad (5)$$

where E^* and μ^* – variable elasticity parameters.

In the equation of compatibility of logarithmic deformations (4), we will replace:

$$\frac{de_\theta}{d\rho} = \frac{1}{\exp(e_\theta)} \frac{d(\exp(e_\theta))}{d\rho},$$

we write this equation in the form:

$$\frac{d(\exp(e_\theta))}{d\rho} = \frac{1}{\rho} \exp(e_\theta) - \frac{1}{\rho} \exp(2e_\theta - e_\rho). \quad (6)$$

Considering equation (6) as a nonlinear differential equation of the first degree of the form and solving it by the Bernoulli method, we obtain a general solution:

$$\exp(e_\theta) = -\rho \int \frac{1}{\rho^2} \exp(2e_\theta - e_\rho) d\rho + C\rho.$$

Given the boundary conditions: $\rho = R$, $e_\theta = e_{\theta R} = \ln(R/R_0)$, then:

$$e_\theta = \ln \left(-\rho \int_R^\rho \frac{1}{\rho^2} \exp(2e_\theta - e_\rho) d\rho + \rho/R_0 \right).$$

Using the equations of the relationship between stresses and deformations (5), we obtain an integral equation of the compatibility of logarithmic strains in stresses:

$$\sigma_\theta = \frac{\mu^*}{(1-\mu^*)} \sigma_\rho + \frac{E^*}{(1-\mu^*)} \ln \left(-\rho \int_R^\rho \frac{1}{\rho^2} \exp \left(\frac{2\sigma_\theta}{E^*} - \frac{(1+2\mu^*)}{E^*} \sigma_\rho \right) d\rho + \rho/R_0 \right). \quad (7)$$

The solution for determining the stress-strain state of a spherical shell, in accordance with the method of variable elasticity parameters, is carried out by the method of successive approximations according to a recurrent scheme using equations (3) and (7) for given boundary conditions, that is, with a known position of the outer edge of the sphere R and radial stress on the outer edge $\sigma_{\rho R} = 0$:

$$\sigma_\theta^{(k+1)} = \frac{\mu^{*(k)}}{(1-\mu^{*(k)})} \sigma_\rho^{(k)} +$$

$$+ \frac{E^{*(k)}}{(1-\mu^{*(k)})} \ln \left(-\rho \int_R^\rho \frac{1}{\rho^2} \exp \left(\frac{2\sigma_\theta^{(k)}}{E^{*(k)}} - \frac{(1+2\mu^{*(k)})}{E^{*(k)}} \sigma_\rho^{(k)} \right) d\rho + \rho/R_0 \right);$$

$$\sigma_\rho^{(k+1)} = \int_R^\rho \frac{2(\sigma_\theta^{(k+1)} - \sigma_\rho^{(k)})}{\rho} d\rho,$$

where the values with the index (k) и $(k+1)$ denote, respectively, their values in the k -th and $(k+1)$ -th approximations. Numerical integration is carried out from R to $r^{(j)}$, where $r^{(j)}$ – the inner radius of the pipe in the process of deformation. In the zero approximation, for $j = 0$, we assume that $r^{(j)} = r_0$, where r_0 is the inner radius of the pipe in the initial state without load

As calculations have shown, the results of calculations do not depend on the choice of the values of the initial approximation for σ_ρ , σ_θ , и σ_φ , therefore, in the initial approximation we take:

$$\sigma_\rho^{(0)} = 0; \sigma_\theta^{(0)} = 0; \sigma_\varphi^{(0)} = 0; E^{*(0)} = E; \mu^{*(0)} = \mu,$$

where E – modulus of elasticity of the material from which the sphere is made, μ – Poisson's ratio.

The deformed state in the $(k+1)$ -th approximation is determined by:

$$e_\rho^{(k+1)} = \frac{1}{E^{*(k)}} [\sigma_\rho^{(k+1)} - 2\mu^{*(k)}\sigma_\theta^{(k+1)}];$$

$$e_\theta^{(k+1)} = \frac{1}{E^{*(k)}} [\sigma_\theta^{(k+1)} - \mu^{*(k)}(\sigma_\theta^{(k+1)} + \sigma_\rho^{(k+1)})].$$

In the case of the transition of the material to the plasticity zone, the secant modulus in $(k+1)$ -th approximations is determined by:

$$E_{sec}^{(k+1)} = \frac{A (e_i^{(k+1)})^n}{e_i^{(k+1)}},$$

where the strain intensity is determined in the $(k+1)$ -th approximation:

$$e_i^{(k+1)} = \frac{2}{3} (e_\rho^{(k+1)} - e_\theta^{(k+1)}).$$

Then the value of the variable elasticity parameters is specified:

$$E^{*(k+1)} = \frac{E_{sec}^{(k+1)}}{1 + \frac{1-2\mu}{3E} E_{sec}^{(k+1)}}; \mu^{*(k+1)} = \frac{\frac{1}{2} - \frac{1-2\mu}{3E} E_{sec}^{(k+1)}}{1 + \frac{1-2\mu}{3E} E_{sec}^{(k+1)}}.$$

To control the convergence of the process, the values of stress intensities are compared:

$$\frac{\sigma_i^{(k+1)} - \sigma_i^{(k)}}{\sigma_i^{(k+1)}} 100\% \leq \Delta\sigma_i\%.$$

The calculation is continued until the specified percentage accuracy is reached. After achieving the specified accuracy of the calculation of the stress-strain state, the position of the inner surface of the spherical shell is clarified:

$$r^{(j+1)} = r_0 \exp(e_{\theta r}^{(j)}),$$

where $e_{\theta r}^{(j)}$ – the value of the tangential logarithmic deformation at the upper bound of the numerical integration, that is, when $\rho = r^{(j)}$.

After clarifying the inner radius during deformation and changing the upper limit of numerical integration, the calculation of the stress-strain state is repeated until the specified accuracy of determining the inner radius is reached:

$$\frac{r^{(j+1)} - r^{(j)}}{r^{(j+1)}} 100\% \leq \Delta r\%.$$

After the final determination of the stress-strain state, it is possible to determine the internal pressure in the sphere at which a given increase in the outer radius occurred:

$$p = -\sigma_{\rho r},$$

where $\sigma_{\rho r}$ – the value of the radial stress on the inner surface of the pipe in the conclusion of all calculations.

To determine the bearing capacity of a thick-walled shell, it is necessary to determine when the pressure reaches the point of the extremum. To do this, the condition must be met:

$p = P_{max}$ by $dp = 0$.

Thus, the value of P_{max} determines the bearing capacity of the sphere. By changing the outer radius R with a certain step and determining the pressure p at each step, it is possible to plot the dependence of p on R . Analyzing the resulting graph, it is possible to determine the moment when, with an increase in the outer radius, the deformation pressure begins to fall, that is, the pressure reaches a maximum and the pressure increment with an increase in the radius becomes less than zero:

$dp < 0$.

This indicates that the moment of unstable deformation of the shell with subsequent destruction has come. The value of the maximum pressure in this case determines the bearing capacity of the spherical shell.

3. Results and Discussion

Let us compare the results of calculations obtained on the example of a thick-walled spherical shell, performed taking into account the compressibility of the material and in the case of an incompressible material, for an ideal elastic-plastic material and for an elastic-plastic material with nonlinear hardening. The inner radius of the shell $r_0 = 50$ mm, outer radius $R_0 = 100$ mm.

In the case of an ideal elastic-plastic material, the deformation diagram of the material is given by the equation:

$$\sigma_i = \begin{cases} 3G & \text{by } e_i \leq e_{iT} \\ \sigma_T & \text{by } e_i > e_{iT} \end{cases}$$

Figures 1 and 2 show the results of changes in relative internal pressure ($\bar{p} = p/\sigma_T$ or $\bar{p} = p/\sigma_{0.2}$) from the radial displacement of the points of the outer surface of the spherical shell ($U_R = R - R_0$).

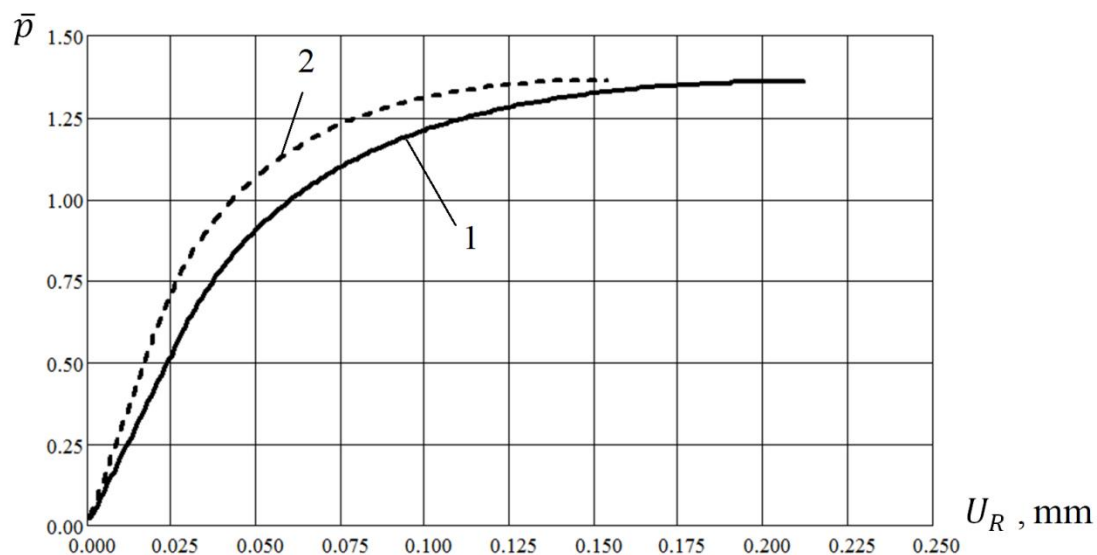


Fig. 1. Graph of the relative pressure dependence on the radial displacement of the points of the outer surface of the spherical shell.

- 1 – taking into account the compressibility of the material ($\mu = 0.3$);
- 2 – without taking into account the compressibility of the material ($\mu = 0.5$)

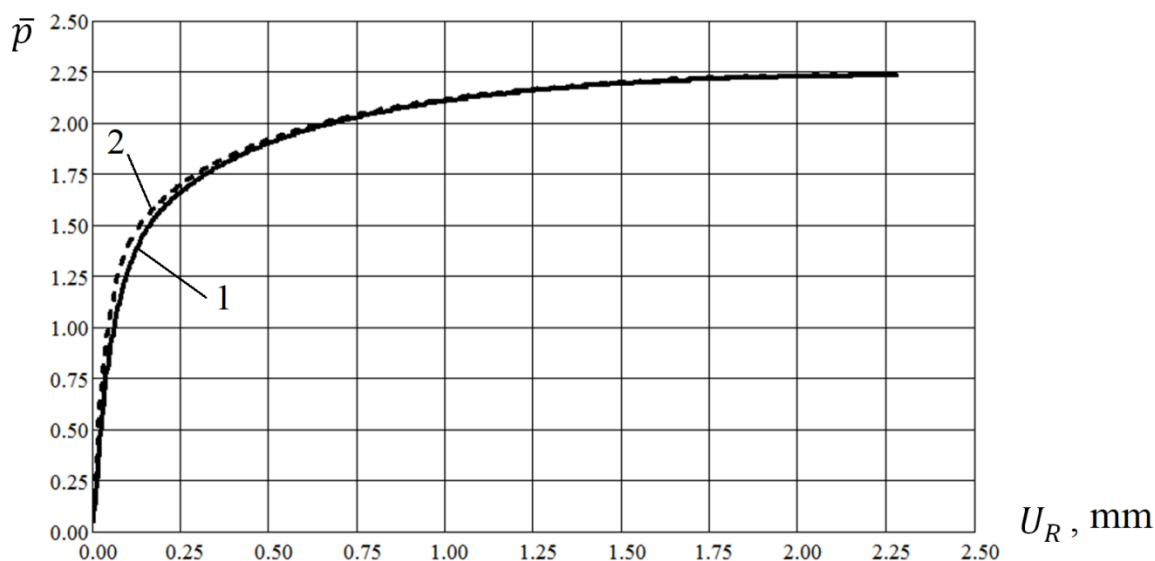


Fig. 2. Graph of the relative pressure dependence on the radial displacement of the points of the outer surface of the spherical shell

(elastic-plastic material with nonlinear hardening $E=71000$ MPa, $\sigma_{0,2}=224$ MPa, $\sigma_B=400$ MPa, $\delta\% = 15$)

1 – taking into account the compressibility of the material ($\mu = 0.3$);

2 – without taking into account the compressibility of the material ($\mu = 0.5$)

As follows from the comparison of curves 2 with curves 1 (see Figs. 1 and 2), the assumption of incompressibility of the material gives a very small error in determining the bearing capacity of a thick-walled spherical shell. Thus, calculations show that for an ideal elastic-plastic material, this error is 0.35%, and for an elastic-plastic material with nonlinear hardening – 0.013%.

A significant error is obtained when calculating radial displacements. Thus, for an ideal elastic-plastic material, this error for radial displacements of the outer surface of the sphere during destruction is 40%, and for an elastic-plastic material with nonlinear hardening – 7%.

A deeper analysis of the results obtained for an ideal elastic-plastic material showed that they agree very well with the analytical solution for a constrained thick-walled pipe presented in books [2,7]. As with the analytical solution, it can be shown that curve 1 in Fig. 1 can be obtained from curve 2 by multiplying the radial displacement value by $2(1 - \mu)$.

This analysis proves the adequacy of the proposed mathematical model and the reliability of the results obtained in this article.

Figures 3 and 4 show the results of changing the effect of the compressibility of the material on the calculated value of internal pressure depending on the radial displacement of the points of the outer surface of the spherical shell.

The effect of compressibility was estimated as a percentage of the design pressure in the case of compressible material according to the formula

$$\Delta p\% = \frac{(p^* - p_*)}{p_*} 100\%,$$

where p_* – pressure under the condition of compressibility of the material, p^* – pressure under the condition of incompressibility of the material

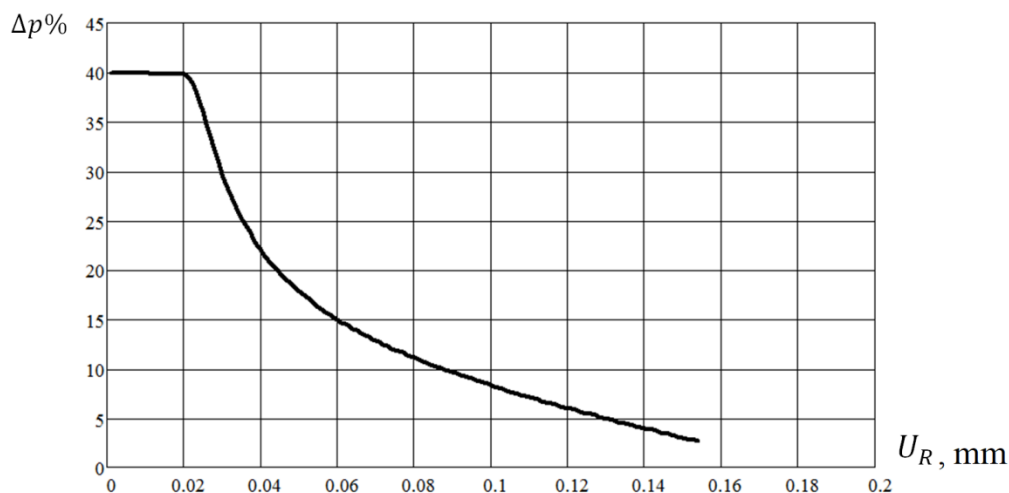


Fig. 3. Graph of the change in the effect of compressibility of the material on the calculated value of internal pressure depending on the radial displacement of the points of the outer surface of the spherical shell (ideal elastic-plastic material)

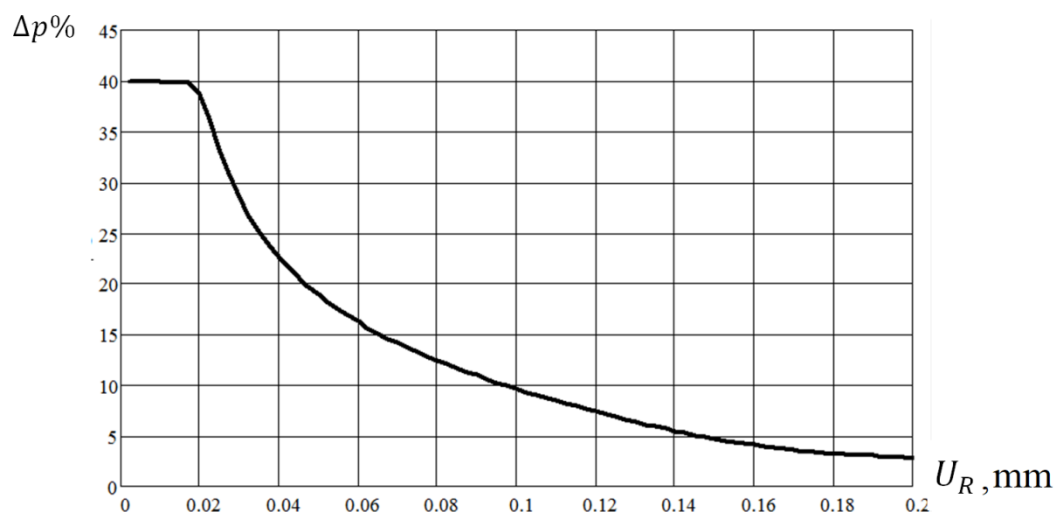


Fig. 4. Graph of changes in the effect of compressibility of the material on the calculated value of internal pressure depending on the radial displacement of the points of the outer surface of the spherical shell (elastic-plastic material with nonlinear hardening)

The analysis of the graphs presented in Figs. 3 and 4 show that the compressibility of the material has the maximum effect on the calculated value of the internal pressure during elastic deformation. As soon as the radial displacements exceed a certain value at which plastic deformations occur in the walls of the spherical shell, the compressibility effect decreases sharply, asymptotically approaching zero. Moreover, the nature of the change in the influence of compressibility practically does not depend on hardening, but depends only on the intensity of deformations and repeats the nature of the change in the variable elasticity parameter μ^* .

It can be shown that the maximum value of $\Delta p\%_{max}$ explicitly depends on the Poisson's ratio and is defined by the expression $\Delta p\%_{max} = (1 - 2\mu)100\%$.

4. Conclusion

Thus, these studies have shown the effectiveness and universality of the application of the method of variable elasticity parameters for solving complex elastic-plastic problems, taking

into account the compressibility and nonlinear plasticity of the material. According to the presented ratios, the bearing capacity of the sphere is determined by the maximum load that the sphere can withstand without destruction. Since the load and the radial stress at the inner boundary of the sphere are equal in modulus, the maximum pressure is determined primarily by the radial stress, which practically does not depend on the coefficient μ . As a result, the compressibility of the material affects the amount of deformation of the neck formation but does not affect the bearing capacity.

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