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On the unloading dynamics in an elastic/viscoplastic material predeformed by viscometric twisting

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Abstract. Solving the problem of large elastic-plastic deformation theory, the present paper addresses, involved two stages. We first derive the exact solution to the problem of slow strain growth in a cylindrical layer consisting of an incompressible elastic/viscoplastic material and experiencing viscometric motion subject to no-slip contact between the material and the rigid instrument's walls. Then a striking stick-slip transition at one of the material-instrument interfaces poses the problem of unloading dynamics. Stress jump at the boundary surface causes a shear cylindrical unloading shock wave, which advances into the material and interacts with the elastic-plastic boundary separating viscoplastic flow from reversible deformation region. To solve this dynamic problem the ray method for constructing approximate solutions is adjusted to the case of elastic/viscoplastic material.

Keywords: large strains, elasticity, viscoplasticity, viscometric strains, unloading wave, ray method

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Introduction

The dynamics of solids under impulse or shock loading remains one of the primary concerns of modern mechanics and mathematical physics. Such problems arise when dealing with issues of ensuring the reliability of critical units in mechanisms and machines, upon improving technology for processing materials and manufacturing products, in nonlinear acoustics and acoustic diagnostics, in seismology and seismic exploration. Unsteady impact on the boundaries of deformable bodies causes the propagation of velocity (shock waves) or acceleration (weak waves) discontinuities through the material. They are generally induced by discontinuous initial and boundary conditions; although, a weak wave may terminate into a shock one [1,2] due to the unbounded growth of its strength [3,4]. This phenomenon is essentially nonlinear [5], as also is the dependence of wave velocities on the preliminary strains [6] and shock wave velocities on the amplitude of discontinuities [7-9].

Originally, these and other nonlinear phenomena used to be observed in gas dynamics, which, unlike dynamics of solids, was developed as a nonlinear theory. The situation is more complicated for solids because they exhibit both volumes (as in gas dynamics) and shape-

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changing strains. Volume and shape-changing deformations propagate independently only in the scope of linear elasticity, but they are coupled in the nonlinear case. Sometimes, in order to focus on the aspects of propagating shear strains solely, the material is assumed to be incompressible [10-12]. Herein, we introduce this simplifying constraint too.

When the plastic properties of material should be taken into account along with elastic ones, the problem is getting more challenging. J. Mandel [13] noted that in the context of the Prandtl-Reuss elastoplastic flow model [14] shock waves cannot be uniquely described by a closed system of equations in terms of discontinuities. To avoid this uncertainty, the auxiliary thermodynamic hypothesis on the maximum mechanical energy dissipation on the surfaces across which plastic strains are discontinuous had to be introduced [15,16]. With this hypothesis accepted and small deformations assumed, the first simplest problems of the perfect elastoplasticity [17-20] were solved; algorithms and programs for numerical calculations were developed [21-25]. Among the latter, finite-difference, characteristics, and finite-element calculation methods evolved. But in all cases, the main challenge in calculating the deformation dynamics was associated with discontinuity surfaces traveling through a deformable body, especially the surfaces of velocity discontinuities.

In gas dynamics [3], special shock-fitting algorithms were developed to calculate the location and the amplitude of discontinuities. They typically constitute a two-step scheme of predictor-corrector type relying on the solution of the discontinuity decomposition problem at the intermediate step. In solid dynamics, due to interacting processes of shear and volumetric strain propagation, the problem of the decomposition of a combined discontinuity is hardly ever tractable [17,20]; therefore, various modifications of shock-capturing methods are employed [22]. Nevertheless, it is worth remarking that discontinuities in solids can be fitted by the methods based on integral generalizations of variational inequalities [21] and the methods for embedding near-front asymptotics into finite-difference schemes [26-28]. Asymptotic expansions of solutions behind discontinuity surfaces are constructed either using the perturbation method [29] or the ray method [27]. By so doing, the location of discontinuities and their amplitudes are corrected at each time step of calculations.

The method for constructing approximate solutions of essentially unsteady dynamic problems in the form of near-front expansions was proposed in papers [30] and [31], in which it was termed the ray method. Achenbach and Reddy [30] constructed an approximate solution as a power series with respect to time, while Babicheva et al. [31] did so with respect to the ray coordinate. Obeying compatibility conditions for discontinuities [15,32-34] one can recurrently deduce ordinary differential equations (decay equations) for coefficients of these power series. The recurrence is violated only if the velocity of the discontinuity surface depends on the variable amplitude of discontinuity, i.e., for shock waves in the nonlinear media. Peculiarities of the ray method employment and approximate solutions obtained by means of it were reviewed in [35]. Burenin et. al. [36-38] tailored the method of ray expansions for constructing solutions behind shock wave fronts. In the present paper, we use the ray method in the dynamics of large elastic/viscoplastic deformations [39,40] to construct a ray expansion of a solution behind the front of the unloading shock wave.

Dynamics of elastic/viscoplastic medium unloading was studied in [41-43], where authors obtained the exact solutions of the problems on instantaneous unloading of a material that had undergone large elastic/viscoplastic strains under quasistatic deforming prior to that. The interaction of the unloading shock wave with the moving elastic-plastic boundary was investigated. The success in solving these problems was gained by applying the theory of large elastic-plastic deformations [44,45], in which reversible and irreversible strains were specified by differential equations of their change (transfer). The foundations of the mathematical model for this theory [44,45] were proposed in [46,47]. The content of [44], thus, constitutes analytical and numerical-analytical solutions of boundary-value problems of the large strain theory just mentioned along with a detailed description of the way to obtaining model relations. Herein, we also solve the boundary-value problem in the context of the theory [44], but our concern is constructing an approximate solution behind the front of the unloading shock wave by means of the ray expansions.

We should stress that allowing for the viscous resistance to plastic flow has even facilitated deriving exact solutions [42,43]. Herein, we also take it into account when constructing the ray expansion. Such a simplification is associated with the fact that, due to the viscosity of the material under plastic flow, irreversible deformations are continuous across the surfaces of velocity discontinuities. This leads to the effect that, in contrast to the elastic-plastic medium [16,21], the elastic/viscoplastic one cannot transmit discontinuity surfaces other than those moving with elastic wave velocities [48-50].

Reversible and irreversible deformations are interdependent in the course of deforming. Therefore, the initial stress-strain state cannot be prescribed arbitrarily [42-44] when posing unloading problems. It must be a result of the preliminary loading process. So, any problem of unloading a deformable body that has accumulated large strains must be preceded by solving the problem on the loading of this body providing data on the resulting distribution of reversible and irreversible strains over the body. Such a loading problem can be a quasistatic one, in which inertia forces can be neglected due to the slow process of active deformation. It is precisely such a loading process that we consider here.

Mathematical model of elastic/viscoplastic material

Flow plasticity theory demands decomposing strains in a body into reversible and irreversible parts, which, unlike total strains, are not measurable experimentally. Therefore, introducing them into the theory is hypothetical. However, they are highly important for the theory as they are thermodynamic parameters of state for a deformable body. According to the concept of constructing the thermodynamics of processes, it is reversible and irreversible strains that one should formulate differential equations of their change (transfer) for [44,45,51]. These equations may serve as a definition for the parts of deformation or accompany such parameters of state if the latter is defined out of other considerations. It only matters while doing so that the relations formulated are geometrically consistent.

When deformations are small, the problem of constructing equations of total deformation transfer does not arise because its solution is obvious. In the case of large deformations, [52,53] such a problem is most often reduced to "choosing" the objective derivative of the hypothetically constructed tensor of plastic strains.

For the present study, we adopt the mathematical model proposed in [46,47] and thoroughly discussed in [44,45]. Equations of change in the tensors of reversible $\boldsymbol{e}$ and irreversible $\boldsymbol{p}$ strains can be written in the form [44,45]

$$\frac{D\boldsymbol{e}}{Dt} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon^p} - \frac{1}{2}\left((\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon^p} + \boldsymbol{z}) \cdot \boldsymbol{e} + \boldsymbol{e} \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon^p} - \boldsymbol{z})\right), \quad (1)$$

$$\frac{D\boldsymbol{p}}{Dt} = \boldsymbol{\varepsilon^p} - \boldsymbol{p} \cdot \boldsymbol{\varepsilon^p} - \boldsymbol{\varepsilon}^p \cdot \boldsymbol{p}, \quad (2)$$

where

$$\frac{D\boldsymbol{n}}{Dt} = \frac{d\boldsymbol{n}}{dt} - \boldsymbol{r} \cdot \boldsymbol{n} - \boldsymbol{n} \cdot \boldsymbol{r^T}, \qquad \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla^T\boldsymbol{v}), \qquad \boldsymbol{\omega} = \frac{1}{2}(\nabla\boldsymbol{v} - \nabla^T\boldsymbol{v}),$$

$$\boldsymbol{v} = \frac{d\boldsymbol{u}}{dt} = \frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{v}\nabla\boldsymbol{u}, \qquad \boldsymbol{r} = \boldsymbol{\omega} + \boldsymbol{z},$$

$$\boldsymbol{z} = A^{-1}\{B^2(\boldsymbol{\varepsilon} \cdot \boldsymbol{e} - \boldsymbol{e} \cdot \boldsymbol{\varepsilon}) + B(\boldsymbol{\varepsilon} \cdot \boldsymbol{e}^2 - \boldsymbol{e}^2 \cdot \boldsymbol{\varepsilon}) + \boldsymbol{e} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}^2 - \boldsymbol{e}^2 \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}\},$$

$$A = 8 - 8E_1 - 3E_1^2 - E_2 - \frac{E_1^3}{3} + \frac{E_3}{3}, \quad B = 2 - E_1,$$

$$E_1 = tr(\boldsymbol{e}), \qquad E_2 = \boldsymbol{e} \cdot\cdot\, \boldsymbol{e}, \qquad E_3 = \boldsymbol{e}^2 \cdot\cdot\, \boldsymbol{e}.$$

In the transfer equations for reversible (1) and irreversible (2) strains, $\boldsymbol{u}$ and $\boldsymbol{v}$ are the displacement and the velocity vectors respectively. The terms $(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)$ and $\boldsymbol{\varepsilon}^p$ in the right parts of equations (1)-(2) are the sources of strains. The remaining terms in the right parts are the flux terms. They define the interdependence of the thermodynamic parameters $\boldsymbol{e}$ and $\boldsymbol{p}$ in the course of their change. The geometrical consistency of this interdependence requires the definite objective time derivative in (1) and (2). The form of this derivative is indicated for an arbitrary tensor $\boldsymbol{n}$. If the nonlinear part $\boldsymbol{z}$ of the tensor $\boldsymbol{r}$ is null, the objective derivative D/Dt in (1) and (2) becomes the Jaumann derivative [44,45].

The sources in transfer equations (1) and (2) define the growth of reversible $\boldsymbol{e}$ and irreversible $\boldsymbol{p}$ strains and must be related to the reasons for this growth by governing laws. According to (2), there is a possibility for the deformation process with $\boldsymbol{\varepsilon}^p=0$, when $D\boldsymbol{p}/Dt = 0$. In this case, the tensor of irreversible strains is invariable, but its components change as if the body would move without getting deformed. Dependencies (1) and (2) endow the regions of preliminary reversible (elastic) deformation $\boldsymbol{p}=0$ and the unloading regions $\boldsymbol{p}=const\neq 0$ with this property.

Recalling (1) and (2), the Almansi tensor of total strains $\boldsymbol{d}$ is defined as follows [44,45,47]

$$\boldsymbol{d}=\boldsymbol{e}+\boldsymbol{p}-\frac{1}{2}\boldsymbol{e}\cdot\boldsymbol{e}-\boldsymbol{e}\cdot\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{e}+\boldsymbol{e}\cdot\boldsymbol{p}\cdot\boldsymbol{e}. \quad (3)$$

Reasoning from (3), rather it is the tensor $\boldsymbol{s}=\boldsymbol{e}-\boldsymbol{e}\cdot\boldsymbol{e}/2$ that should have been referred to as a tensor of reversible (elastic) strains, in which the tensor $\boldsymbol{e}$ is a main linear part. The tensor $\boldsymbol{e}$ is the one of small elastic strains; it turned out to be convenient for writing transfer equations (1) and (2) by means of it. It also proved to be more useful when writing a counterpart of the Murnaghan formula [44,45,54] in the case of perfect elastoplasticity under study. This formula for the case of an incompressible medium considered below reads [44,45]

$$\boldsymbol{\sigma}=-p_1\boldsymbol{I}-\frac{\partial W}{\partial \boldsymbol{d}}(\boldsymbol{I}-2\boldsymbol{d})\ if\ \boldsymbol{p}=0, \quad (4)$$
$$\boldsymbol{\sigma}=-p_2\boldsymbol{I}-\frac{\partial W}{\partial \boldsymbol{e}}(\boldsymbol{I}-\boldsymbol{e})\ if\ \boldsymbol{p}\neq 0.$$

In (4), $\boldsymbol{\sigma}$ is the Euler-Cauchy stress tensor, p_1 and p_2 are the additional hydrostatic pressures, $\boldsymbol{I}$ is the second-order unit tensor, $W=W(J_1,J_2)$ is the elastic potential (free energy density in the isothermal case). For proceeding calculations, we designate $W=W(J_1,J_2)$ as follows

$$W=-2\mu J_1-\mu J_2+bJ_1^2+(b-\mu)J_1J_2-\chi J_1^3+\cdots, \quad (5)$$
$$J_k=\begin{cases}L_k & if\quad \boldsymbol{p}\equiv 0\\ I_k & if\quad \boldsymbol{p}\neq 0\end{cases},\quad k=1,2,$$
$$L_1=tr(\boldsymbol{d}),\qquad L_2=\boldsymbol{d}\cdot\cdot\,\boldsymbol{d},\qquad I_1=tr(\boldsymbol{e})-\frac{1}{2}\boldsymbol{e}\cdot\cdot\,\boldsymbol{e},\quad I_2=\boldsymbol{e}\cdot\cdot\,\boldsymbol{e}-\boldsymbol{e}^2\cdot\cdot\,\boldsymbol{e}+\frac{1}{4}\boldsymbol{e}^2\cdot\cdot\,\boldsymbol{e}^2.$$

In (5), μ is the shear modulus; b and χ are the higher-order elastic moduli. The invariants I_1 and I_2 of the reversible strain tensor are specified in such a way as to ensure the limiting transition from the second formula in (4) to the first one as the plastic strains tend to zero.

The relation (2) allows deforming without the growth of irreversible strains, i.e. elastic deforming. We remark that this assumption idealizes the deformable material. This is achievable only in the scope of perfect elastoplasticity. The real body possesses viscosity, so upon imposing stress it acquires unrecoverable strains due to the creeping onset. However, differential equation (2) does not contradict this either. Ignoring this phenomenon, we further assume the deformable material to be perfectly elastic-plastic [14]. In such a material, irreversible (plastic) strains begin going up when the stresses reach the loading surface $f(\boldsymbol{\sigma},\boldsymbol{\varepsilon}^p,k)=0$. As a loading surface, we take the unified condition of maximum tangential stresses [14]

$$f(\sigma_i,\varepsilon_k^p,k)=max|\sigma_i-\sigma_j|-2k-2\eta max|\varepsilon_k^p|. \quad (6)$$

In (6), σ_i and ε_k^p are the principal values of the stress and the strain rate tensors respectively, k is the yield stress, and η is the coefficient of viscous resistance to plastic flow. Imposing the condition of the von Mises maximum principle [14], we have the associated law of plastic flow

$$\boldsymbol{\varepsilon^p} = \lambda \frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \lambda > 0. \tag{7}$$

In order to get the closed system of equations both in the region of elastic deformation and in the region of plastic flow it suffices to complete the preceding relations with the equation of motion or the balance equation

$$\nabla \cdot \boldsymbol{\sigma} = \rho \frac{d\boldsymbol{v}}{dt}, \tag{8}$$

$$\nabla \cdot \boldsymbol{\sigma} = 0\,. \tag{9}$$

Whenever forces of inertia are feasible to be ignored in (8) so as to have (9), a quasistatic approximation is allowed in solving the problem.

Quasistatic deformation

Let the material, the deformation properties of which obey the relations stated in Section 2, fills the layer between two coaxial rigid surfaces. The equations of these surfaces in the cylindrical coordinate system (r, φ, z) are the following: $r = r_0$, $r = R (r_0 < R)$. Deforming is caused by revolving the inner cylinder $r \leq r_0$ about its axis $r = 0$ while the surface $r = R$ is motionless. The material is considered to be initially unstrained. Assume that the no-slip conditions are satisfied at the boundary surfaces $r = r_0$ and $r = R$ until the shear stress exceeds some given threshold value $|\sigma_{r\varphi}| \leq \sigma_0$

$$\boldsymbol{u}(R, t) = \boldsymbol{v}(R, t) = 0, \; \sigma_{r\varphi}(r_0, t) = -\alpha t. \tag{10}$$

The latter condition specifies the load on the deformable material; σ_0 and α are the prescribed constants of the problem. We put $\sigma_0 > k$. Due to the imposed condition of incompressibility, the paths of the material's particles are concentric circles, and the only nonzero component of the velocity vector is v_φ. According to (1)-(3), the kinematics of the medium in this case is determined by the relations

$$u_r = r(1 - cos\psi), \quad u_\varphi = rsin\psi, \tag{11}$$

$$d_{rr} = -\frac{f^2}{2}, \quad d_{r\varphi} = \frac{f}{2}, \qquad f = r\frac{\partial\psi}{\partial r}, \qquad v_\varphi = r\omega = r\frac{\partial\psi}{\partial t},$$

$$\varepsilon_{r\varphi} = \frac{r}{2}\frac{\partial^2\psi}{\partial r\partial t}, \qquad \omega_{r\varphi} = -\frac{\partial\psi}{\partial t} - \frac{r}{2}\frac{\partial^2\psi}{\partial r\partial t}, \qquad r_{r\varphi} = -\frac{\partial\psi}{\partial t} + \frac{2\varepsilon_{r\varphi}(1 - e_{\varphi\varphi})}{e_{rr} + e_{\varphi\varphi} - 2},$$

where $\psi = \psi(r, t)$ is the central angle of twisting of medium particles, and $\omega = v_\varphi / r$ is the angular velocity.

Taking into account that for such problems diagonal components of the strain tensors are small quantities of a higher order of smallness compared to off-diagonal components [44], hereinafter we restrict ourselves to terms of the first order with respect to diagonal components and of the second order with respect to off-diagonal components. Such a restriction, though not mandatory, allows us to obtain analytical relations for the displacements. From (4) and (5) we derive stress components

$$\sigma_{rr} = -p - 2\mu + 2(b + \mu)e_{rr} + 2be_{\varphi\varphi} + \mu e_{r\varphi}^2, \tag{12}$$

$$\sigma_{\varphi\varphi} = -p - 2\mu + 2(b + \mu)e_{\varphi\varphi} + 2be_{rr} + \mu e_{r\varphi}^2,$$

$$\sigma_{zz} = -p - 2\mu + 2b(e_{rr} + e_{\varphi\varphi}) - 2\mu e_{r\varphi}^2, \quad \sigma_{r\varphi} = 2\mu e_{r\varphi}, \qquad \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{\sigma_{r\varphi}} = \frac{e_{rr} - e_{\varphi\varphi}}{e_{r\varphi}}.$$

At the beginning of the loading process (from the instant $t = 0$), the material deforms reversibly ($p_{ij} = 0$). According to relations (12) and (3), the stress components, in this case, can be written as follows

$$\sigma_{rr} = \sigma_{zz} = -(p + 2\mu) - \frac{(b+\mu)}{2} f^2 = -s(r,t), \tag{13}$$
$$\sigma_{\varphi\varphi} = -s(r,t) + \mu f^2, \quad \sigma_{r\varphi} = \mu f.$$

Substituting (13) into (9), we get two balance equations

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \quad \frac{\partial \sigma_{r\varphi}}{\partial r} + 2\frac{\sigma_{r\varphi}}{r} = 0. \tag{14}$$

Then, integrating equations (14) subject to the boundary conditions (10), we obtain a solution for the case when material acquires the elastic deformation

$$\psi = \frac{c(t)}{2\mu}\left(\frac{1}{R^2} - \frac{1}{r^2}\right), \ \omega = \frac{\dot{c}(t)}{2\mu}\left(\frac{1}{R^2} - \frac{1}{r^2}\right), \ \dot{c}(t) = \frac{dc(t)}{dt}, \tag{15}$$
$$\sigma_{r\varphi} = \frac{c(t)}{r^2}, \quad \sigma_{rr} = \sigma_{zz} = \frac{c^2(t)}{4\mu}\left(\frac{1}{r_0^4} - \frac{1}{r^4}\right) + \sigma_{rr}^0,$$
$$\sigma_{\varphi\varphi} = \frac{c^2(t)}{4\mu}\left(\frac{1}{r_0^4} + \frac{3}{r^4}\right) + \sigma_{rr}^0, \qquad c(t) = -\alpha t r_0^2,$$
$$e_{r\phi} = d_{r\phi} = \frac{c(t)}{2\mu r^2}, \quad e_{rr} = -\frac{3}{2} e_{r\phi}^2, \quad e_{\phi\phi} = \frac{e_{r\phi}^2}{2},$$

where $\sigma_{rr}^0 = \sigma_{rr}(r_0, t)$ is a prescribed value of the stress component σ_{rr} at the surface $r = r_0$.

Solution (15) is valid within the time interval from $t = 0$ to $t = t_0$ until the plasticity condition (6) is fulfilled at the rigid wall $r = r_0$, which, under the restrictions applied, takes the form $\sigma_{r\varphi}(r_0, t_0) = -k$. Substituting the value $\sigma_{r\varphi}(r_0, t_0)$ into the last equation and using (15), we find the moment of the plastic flow onset $t_0 = k/\alpha$.

Thenceforth, the region of viscoplastic flow $V^{(P)}: r_0 \le r \le m(t)$ develops from the boundary $r = r_0$. The surface $r = m(t)$ is an elastic-plastic boundary. In the region $V^{(E)}: m(t) \le r \le R$ reversible deforming continues.

By integrating the balance equations (14) in the region of recoverable deforming and using condition (10), we find

$$\sigma_{r\varphi}^{(E)} = \frac{c(t)}{r^2}, \ \psi^{(E)} = \frac{c(t)}{2\mu}\left(\frac{1}{R^2} - \frac{1}{r^2}\right), \ \omega^{(E)} = \frac{\dot{c}(t)}{2\mu}\left(\frac{1}{R^2} - \frac{1}{r^2}\right). \tag{16}$$

Hereinafter, the superscript "E" is related to the region $V^{(E)}$ while the superscript "P" denotes the region $V^{(P)}$.

In the case under study, the yielding criterion (6) reads

$$f\left(\sigma_{r\varphi}, \varepsilon_{r\varphi}^p\right) = \sigma_{r\varphi}^2 - \left(k - \eta\left|\varepsilon_{r\varphi}^p\right|\right)^2 = 0. \tag{17}$$

In virtue of the associated flow rule (7), the condition (17) leads to

$$\sigma_{r\varphi} = -k + \eta \varepsilon_{r\varphi}^p, \quad \lambda = \frac{\varepsilon_{r\varphi}^p}{\eta \varepsilon_{r\varphi}^p - k}. \tag{18}$$

By integrating the balance equations (14) in conjunction with the continuity conditions for the stress components at the elastic-plastic boundary $r = m(t)$, we obtain the components $\sigma_{r\varphi}$, $e_{r\varphi}$ in the plastic flow region

$$\sigma_{r\varphi}^{(P)} = \frac{c(t)}{r^2}, \ e_{r\varphi} = \frac{c(t)}{2\mu r^2}. \tag{19}$$

Comparing (18) and (19), we calculate the plastic strain rate

$$\varepsilon_{r\varphi}^p = \frac{1}{\eta}\left(k + \frac{c(t)}{r^2}\right). \tag{20}$$

The unknown function of integration $c(t)$ has to be determined from the condition of zero plastic strain rate at the elastic-plastic boundary $r = m(t)$

$$c(t) = -k m(t)^2. \tag{21}$$

On the other hand, following the last boundary condition in (10)

$$c(t) = -\alpha t r_0^2, \tag{22}$$

that allows us to get the expression for the surface $m(t)$

$$m(t) = r_0\sqrt{\alpha t/k}. \tag{23}$$

According to the relations (1)-(3), the yielding criterion (17), and the associated flow rule (18), in the region of plastic flow the kinematics of the medium are determined by the relations

$$\varepsilon_{r\varphi} = \varepsilon^e_{r\varphi} + \varepsilon^p_{r\varphi} = \frac{\partial e_{r\varphi}}{\partial t} + \frac{\partial p_{r\varphi}}{\partial t},\ \varepsilon^p_{rr} = \frac{\partial p_{rr}}{\partial t} + 2p_{r\varphi}\left(r_{\varphi r} + \varepsilon^p_{r\varphi}\right) - 2p_{r\varphi}\frac{\partial \psi}{\partial t}, \quad (24)$$

$$\varepsilon^p_{\varphi\varphi} = \frac{\partial p_{\varphi\varphi}}{\partial t} + 2p_{r\varphi}\left(r_{r\varphi} + \varepsilon^p_{r\varphi}\right) + 2p_{r\varphi}\frac{\partial \psi}{\partial t}, \qquad \varepsilon^p_{rr} = -\varepsilon^p_{\varphi\varphi} = \frac{\varepsilon^p_{r\varphi}\left(e_{rr} - e_{\varphi\varphi}\right)}{2e_{r\varphi}}.$$

Components of the velocity and the strain rate tensor are determined by (11).

Utilizing kinematic relations (24) and formula (20), we obtain the component of plastic strains $p_{r\varphi}$

$$p_{r\varphi} = \frac{1}{\eta}\left(kt - \frac{\alpha r_0^2 t^2}{2r^2}\right) + g(r). \quad (25)$$

Recalling the condition of zero plastic strain rate at the elastic-plastic boundary $r = m(t)$ and relation (23), we derive the unknown function $g(r)$

$$g(r) = -\frac{k^2 r^2}{2\alpha\eta r_0^2}. \quad (26)$$

In accordance with relations (3), we have $d_{r\varphi} = e_{r\varphi} + p_{r\varphi}$. Then, combining (11) and (23)-(26), we deduce the differential equation governing the twist angle $\psi(r,t)$ in the region of plastic flow

$$\frac{r}{2}\frac{\partial \psi}{\partial r} = \frac{1}{\eta}\left(kt - \frac{\alpha t^2 r_0^2}{2r^2}\right) - \frac{k^2 r^2}{2\alpha\eta r_0^2} - \frac{\alpha t r_0^2}{2\mu r^2}.$$

After integration in conjunction with the condition of continuity of function $\psi(r,t)$ across the elastic-plastic boundary $r = m(t)$, we find $\psi(r,t)$ in the yielding region

$$\psi^{(P)} = \frac{2kt}{\eta} ln\frac{r}{m(t)} + \frac{\alpha r_0^2 t}{2\mu}\left(\frac{1}{r^2} - \frac{1}{R^2}\right) + \frac{\alpha r_0^2 t^2}{2\eta}\left(\frac{1}{r^2} - \frac{1}{m(t)^2}\right) + \frac{k^2}{2\alpha\eta r_0^2}\left(m(t)^2 - r^2\right). \quad (27)$$

Referring to (25) and (26), we have

$$p_{r\varphi} = \frac{1}{\eta}\left(kt - \frac{\alpha r_0^2 t^2}{2r^2} - \frac{k^2 r^2}{2\alpha r_0^2}\right). \quad (28)$$

Using equations of variation of strain components (1)-(2) and conditions (17)-(18), we obtain a system of equations for diagonal components of elastic e_{rr}, $e_{\varphi\varphi}$ and plastic p_{rr}, $p_{\varphi\varphi}$ strain tensors in the yielding region. These components are small quantities of a higher order of smallness compared to off-diagonal ones

$$\frac{\partial p_{r\phi}}{\partial t} = \varepsilon^p_{r\varphi},\ \ \frac{\partial p_{\varphi\varphi}}{\partial t} = -\varepsilon^p_{r\varphi}\frac{p_{\varphi\varphi} - e^2_{r\varphi}}{e_{r\varphi}} + \frac{4\varepsilon_{r\varphi}p_{r\varphi}}{2 + e^2_{r\varphi}}\left(1 + p_{\varphi\varphi} - \frac{1}{2}e^2_{r\varphi} - 2e_{r\varphi}p_{r\varphi}\right), \quad (29)$$

$$e_{rr} = p_{\varphi\varphi} - \frac{3e^2_{r\varphi}}{2} - 2e_{r\varphi}p_{r\varphi}, \qquad p_{rr} + p_{\varphi\varphi} = -2p^2_{r\varphi}, \qquad e_{rr} + e_{\varphi\varphi} = -e^2_{r\varphi}.$$

System (29) can be integrated numerically. The stress component σ_{rr} is found from the first balance equation (14) with the account of its value at the surface $r = r_0$. Then, from the relations (12), the hydrostatic pressure p and the stress components $\sigma_{\varphi\varphi}$ and σ_{zz} are determined.

Unloading wave. Ray method

The moment the stress $\left|\sigma_{r\varphi}\right|$ at the surface $r = r_0$ touches the critical value σ_0, the stick boundary condition shifts abruptly to the condition of friction between the material and the rigid cylindrical surface. Various conditions of dry or viscous friction can be taken as a law of friction. Here, we adopt the friction condition according to Prandtl law: $\left|\sigma_{r\phi}(r_0,t)\right| = \sigma_s, \sigma_s = const, \quad \sigma_s < k$.

According to (19), a change in the boundary condition occurs at the instant of time $t = t_s = \sigma_0/\alpha$. The jump in the boundary condition $\left[\sigma_{r\phi}(r_0,t_s)\right] = \sigma_s - \sigma_0 \neq 0$ causes the cylindrical shock wave Σ_1, which location in space is determined by the equation $r = r_1(t) = r_0 + \int_{t_s}^{t} G(\xi)d\xi$.

A shock wave is thought to be a surface of strong discontinuity, i.e. such a surface that displacements are continuous over it, while velocities and stresses suffer a finite discontinuity. The surface of strong discontinuity [48] could be interpreted as a limiting layer of thickness ΔS ($\Delta S \to 0$); within this layer velocities and stresses change from the values v_i^+, σ_{ij}^+ to the values v_i^-, σ_{ij}^-, remaining monotonic and continuous inside the layer. Across the surfaces of weak discontinuity that also emerge in what follows, stresses and velocities remain continuous, but some of their partial derivatives undergo a discontinuity. As is customary, the surface of weak discontinuity, across which a discontinuity in the first-order derivative of the velocity occurs, is called a weak wave of the first order or an acceleration wave.

Thus, since the emergence of shock wave, the material of the deformable layer is divided into three parts. It keeps deforming reversibly in the region $V^{(E)}: m(t) \le r \le R$, while viscoplastic flow proceeds in the region $V^{(P)}: r_1(t) \le r \le m(t)$. In the unloading domain $V^{(1)}: r_0 \le r \le r_1(t)$ material deforms reversibly anew, but with previously accumulated unrecoverable strains. According to (28) and (29), the component of irreversible strains $p_{r\varphi}(r)$ alone is invariable upon unloading process

$$p_{r\varphi}(r) = \frac{1}{\eta}\left(k\tau_1(r) - \frac{\alpha r_0^2 \tau_1^2(r)}{2r^2} - \frac{k^2 r^2}{2\alpha r_0^2}\right), \tag{30}$$

where $\tau_1(r) = t_s + \int_{r_0}^{r} d\xi / G(\xi)$ is the time the wave front Σ_1 reaches a position r. Other components of plastic strain tensor $p_{rr}(r,t)$, $p_{\varphi\varphi}(r,t)$ keep changing with time due to changing elastic strains $e_{r\phi}(r,t)$. Hence, there is an internal redistribution between elastic and plastic strains; components $p_{rr}(r,t)$, $p_{\varphi\varphi}(r,t)$ can both increase and decrease in magnitude. Behind the unloading shock wave equations of motion may be written as

$$\frac{\partial \sigma_{r\varphi}}{\partial r} + 2\frac{\sigma_{r\varphi}}{r} = \rho r \ddot{\psi}, \quad \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = -\rho r \dot{\psi}^2. \tag{31}$$

The former equation in (31) is meant for determining the kinematics of unloading deforming, while the latter can be used for calculating the additional hydrostatic pressure $p(r,t)$ based on the solution obtained for the function $\psi(r,t)$. Therefore, the boundary-value problem for the former equation in (31) should be solved first. From (11), (12), and (30), we get the following expression for the stress component $\sigma_{r\varphi}$

$$\sigma_{r\varphi} = \mu r \psi_{,r} - \frac{2\mu}{\eta}\left(k\tau_1(r) - \frac{\alpha r_0^2}{2r^2}\tau_1^2(r) - \frac{k^2 r^2}{2\alpha r_0^2}\right). \tag{32}$$

Substituting (32) into the former equation in (31), we obtain a second-order partial differential equation of a hyperbolic type. The boundary conditions for such an equation will be the condition of displacement continuity across the unloading wave $r = r_1(t)$ and the friction condition at the boundary surface $r = r_0$, respectively

$$[\psi]|_{\Sigma_1} = (\psi^+ - \psi^-)|_{r=r_1(t)} = 0, \tag{33}$$

$$\sigma_{r\varphi}(r_0, t) = -\sigma_s. \tag{34}$$

Hereinafter, squire brackets denote the jump across the discontinuity surface for any function enclosed in them; $\psi^+ = \psi^+(r_1(t), t)$ and $\psi^- = \psi^-(r_1(t), t)$ are the values of function $\psi(r,t)$ just ahead of the discontinuity surface and immediately behind it, respectively.

In an elastic/viscoplastic medium, irreversible strains are continuous across the discontinuity surface $[p_{r\varphi}] = 0$ [48-50]. Therefore, the velocity $G(t)$ of the unloading shock wave can depend only on preliminary strains and jumps of reversible strains [42,50]. In all the previous relations, terms of third and higher powers of $e_{r\varphi}$ were omitted. If we restrict ourselves only to the preceding relations, we derive that the unloading wave speed is constant $G = \sqrt{\mu/\rho}$ and $\tau_1(r) = t_s + (r - r_0)/G$. The equation of motion in this case takes the simplest form

$$\psi_{,rr} + \frac{3}{r}\psi_{,r} - \frac{\ddot{\psi}}{G^2} = \frac{2}{\eta}\left(\frac{k}{Gr} - \frac{\alpha r_0^2 \tau_1(r)}{Gr^3} + \frac{2k\tau_1(r)}{Gr^2} - \frac{k^2 r^2}{2\alpha r_0^2}\right). \tag{35}$$

Even such a simplified equation (35) is not exactly solvable. Therefore, we solve it approximately by means of the ray method subject to the boundary conditions (33) and (34). It is worth remarking that the ray method allows for solving the problem even without simplifying assumptions made. They are not required for the method. If one includes in (12) third and higher order powers of $e_{r\varphi}$, the velocity of the unloading shock wave G will not be constant. Such a case for a nonlinear elastic medium was considered in [26-28,36-38]. A review of publications devoted to the features of applying the ray method to constructing approximate solutions of solid dynamics problems one can find in [35]. In the case being considered, to construct an approximate solution for equation (35) with boundary conditions (33) and (34), it is convenient to apply the ray method in its classical form, as it was proposed more than half a century ago [30]. In addition, since the ray is a curve invariably orthogonal to the discontinuity surface, for the problem in question it coincides with the coordinate axis Or.

The function $\psi(r,t)$ is continuous across the discontinuity surface $[\psi]=0$, but its derivatives and, hence, velocity, strains, and stresses undergo ordinary discontinuity. Let us introduce a ray series for the function $\omega(r,t)=\dot{\psi}(r,t)$ behind the unloading shock wave as follows

$$\omega(r,t)=\omega^-|_{t=\tau_1}+\dot{\omega}^-|_{t=\tau_1}(t-\tau_1)+\frac{1}{2}\ddot{\omega}^-|_{t=\tau_1}(t-\tau_1)^2+\cdots=$$
$$=(\omega^+-[\omega])|_{t=\tau_1}+(\dot{\omega}^+-[\dot{\omega}])|_{t=\tau_1}(t-\tau_1)+\frac{1}{2}(\ddot{\omega}^+-[\ddot{\omega}])|_{t=\tau_1}(t-\tau_1)^2+\cdots. \quad (36)$$

Expansion (36) is Taylor's series in time t about the instant of arrival of the wave front at a given point of space. The ray series for the stresses and the angle of twisting are written similarly to (36), and all the quantities desired are expressed in terms of jumps in the angular velocity and its derivatives $[\partial^{n-1}\omega/\partial t^{n-1}]$ $(\omega=\dot{\psi},\ n=1,2,\dots)$. All the quantities ahead of the discontinuity surface (marked by index "+") are considered to be known. They are provided by the solution of the quasistatic problem on viscoplastic flow in the region $r_1(t)\le r\le m(t)$. In order to calculate the jump of $\omega(r,t)$ and discontinuities of its derivatives of order n over the shock wave, one should differentiate the former equation in (31) $n-1$ times with respect to time, then write the result on other sides of the discontinuity surface and take a difference of the expressions obtained employing geometrical and kinematical conditions of compatibility [14,32-34]. Thus, we obtain a system of first-order inhomogeneous linear differential equations

$$\frac{\delta\xi_n}{\delta t}+\frac{3G}{2r_1}\xi_n=\Phi_n(t,r_1,\xi_1,\xi_2,\dots,\xi_{n-1}),\quad \xi_n=\left[\frac{\partial^n\psi}{\partial t^n}\right]\Big|_{\Sigma_1}. \quad (37)$$

In (37), $\delta/\delta t$ is the δ-derivative with respect to time (Thomas derivative [32]); Φ_n is the recurrently calculated function that depends on discontinuities of lower-order derivatives, parameters of motion, and strain state ahead of the discontinuity surface

$$\Phi_1=\frac{G}{\eta r_1}\left(k-\frac{\alpha r_0^2 t}{r_1^2}\right),\ \ \Phi_2=\frac{\alpha r_0^2 G}{\eta r_1^3}-\frac{3G^2A^{(1)}}{8r_1^{7/2}}-\frac{2kG^2}{\eta r_1^2}, \quad (38)$$
$$\Phi_3=-\frac{3B^{(1)}G^2}{8r_1^{7/2}}+\frac{15A^{(1)}G^3}{64r_1^{9/2}}+\frac{2kG^3}{\eta r_1^3}.$$

Ordinary differential equations (37) are referred to as decay equations. The constants of integration $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ arising in turn at each step are determined by virtue of boundary condition (34). Condition (33) is automatically satisfied due to the structure of the ray series. Finally, we obtain the following relations behind the unloading shock wave

$$\xi_1=\frac{A^{(1)}}{r_1^{3/2}}+\frac{2k}{3\eta}-\frac{2\alpha r_0^2}{\eta}\left(\frac{2}{Gr_1}-\frac{t}{r_1^2}\right),\ \ \xi_2=\frac{B^{(1)}}{r_1^{3/2}}+\frac{3A^{(1)}G}{8r_1^{5/2}}-\frac{4kG}{\eta r_1}-\frac{2\alpha r_0^2}{\eta r_1^2}, \quad (39)$$
$$\xi_3=\frac{C^{(1)}}{r_1^{3/2}}+\frac{3B^{(1)}G}{8r_1^{5/2}}-\frac{15A^{(1)}G^2}{128r_1^{7/2}}-\frac{4kG^2}{\eta r_1^2},$$

$$\psi^{(1)}(r,t) = \psi^{(P)}\big|_{t=\tau_1} + \left(\dot{\psi}^{(P)} - \xi_1\right)\big|_{t=\tau_1}(t-\tau_1) + \frac{1}{2}\left(\ddot{\psi}^{(P)} - \xi_2\right)\big|_{t=\tau_1}(t-\tau_1)^2 +$$
$$+\frac{1}{6}\left(\dddot{\psi}^{(P)} - \xi_3\right)\big|_{t=\tau_1}(t-\tau_1)^3 + \cdots,$$
$$\sigma_{r\varphi}^{(1)}(r,t) = -\frac{\alpha r_0^2\tau_1}{r^2} + \frac{\mu r\xi_1(\tau_1)}{G} +$$
$$+\left(-\frac{\alpha r_0^2}{r^2} - \left[\dot{\sigma}_{r\varphi}^{(1)}\right]\Big|_{t=\tau_1}\right)(t-\tau_1) - \frac{1}{2}\left[\ddot{\sigma}_{r\varphi}^{(1)}\right]\Big|_{t=\tau_1}(t-\tau_1)^2 + \cdots,$$
$$\left[\dot{\sigma}_{r\varphi}^{(1)}\right] = -\frac{3\mu}{2}\xi_1 - \frac{\mu \mathrm{r}_1}{G}\xi_2 - \frac{\mu}{\eta}\left(k - \frac{\alpha r_0^2 t}{r_1^2}\right),$$
$$\left[\ddot{\sigma}_{r\varphi}^{(1)}\right] = -\frac{\mu \mathrm{r}_1}{G}\xi_3 - \frac{3\mu}{2}\xi_2 - \frac{3\mu G}{8\mathrm{r}_1}\xi_1 + \frac{3\alpha\mu r_0^2 Gt}{4\eta r_1^3} - \frac{7k\mu G}{4\eta \mathrm{r}_1} + \frac{3\alpha\mu r_0^2}{2\eta r_1^2},$$
$$A^{(1)} = \frac{\sqrt{r_0}G(\sigma_0 - \sigma_s)}{\mu} + \frac{2r_0^{3/2}}{\eta}\left(\frac{2\alpha r_0}{G} - \alpha t_s - \frac{k}{3}\right),$$
$$B^{(1)} = \frac{15G^2(\sigma_s - \sigma_0)}{8\mu\sqrt{r_0}} + \frac{\sqrt{r_0}G(13k + 7\sigma_0)}{4\eta} + \frac{\alpha\sqrt{r_0}G}{\mu} + \frac{\alpha r_0^{3/2}}{2\eta},$$
$$C^{(1)} = -\frac{15GB^{(1)}}{8r_0} - \frac{105G^2A^{(1)}}{128r_0^2} + \frac{8kG^2}{\eta\sqrt{r_0}} + \frac{6\alpha G\sqrt{r_0}}{\eta}.$$

The superscript "(1)" indicates that the value refers to the domain $V^{(1)}: r_0 \le r \le r_1(t)$. Calculations according to the above recurrent scheme can be continued if necessary. For instance, by calculating $\Phi_4(t, r_1)$ and writing down the decay equation for $\xi_4(t)$, we can solve the initial-boundary value problem for this ordinary differential equation and so on. Here we restrict ourselves to three steps of the method.

Solution (39) is valid up to the instant $t = t_m$, at which the unloading wave encounters elastic-plastic boundary $r = m(t)$

$$t_m = t_s + \frac{1}{2G^2}\left(\frac{\alpha r_0^2}{k} - 2r_0G + \sqrt{\frac{4\alpha r_0^2 G}{k}(Gt_s - r_0) + \frac{\alpha^2 r_0^4}{k^2}}\right). \tag{40}$$

Figures 1-3 show the distribution of the shear stress $\sigma_{r\varphi}(r,t)$, angle of twisting $\psi(r,t)$, and angular velocity $\omega^{(1)}(r,t)$ (2a) at the instant $t = t_m$ the wave Σ_1 encounters elastic-plastic boundary. Figure 2(b) depicts the variation of unloading wave intensity with time. Here, we consider the case when the region with irreversible deformations is much narrower than the elastic one and makes up approximately 1/5 of the layer thickness. The calculation was carried out for the following values of the constants: $\rho_0 = 8.96 \times 10^3$ kg×m^3, $\mu = 41.5$ GPa, $k = 68.5$ MPa, $\eta = 4.83$ GPa×s, $\sigma_0 = 71.5$ MPa, $\sigma_s = 58.2$ MPa, $R/r_0 = 1.1$, $\alpha = 6343$ Pa/s.

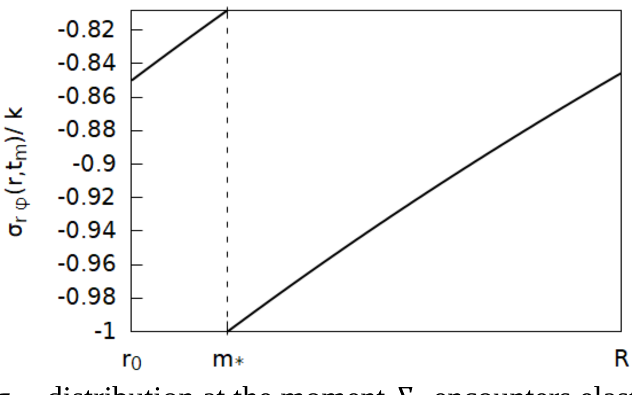

Fig. 1. Shear stress $\sigma_{r\varphi}$ distribution at the moment Σ_1 encounters elastic-plastic boundary $r = m_*$

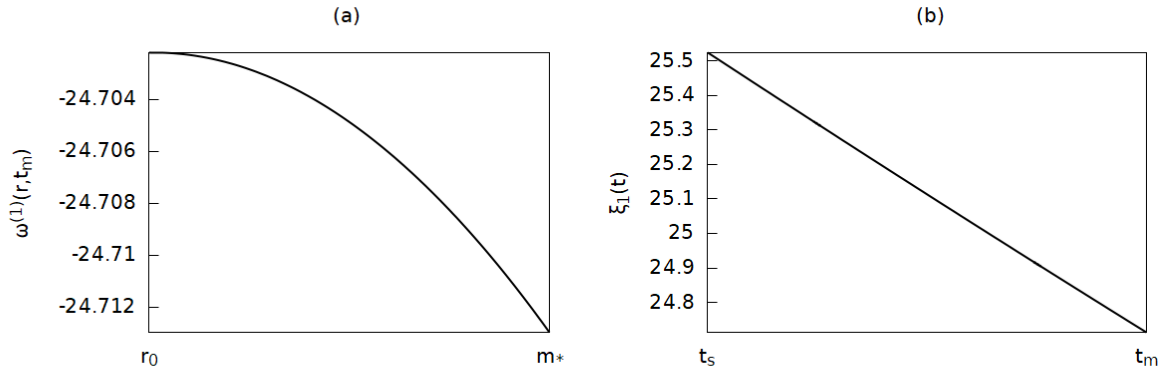

Fig. 2. Angular velocity $\omega^{(1)}$ distribution at the moment Σ_1 encounters elastic-plastic boundary $r = m_*$ (a); variation of unloading wave intensity with time (b)

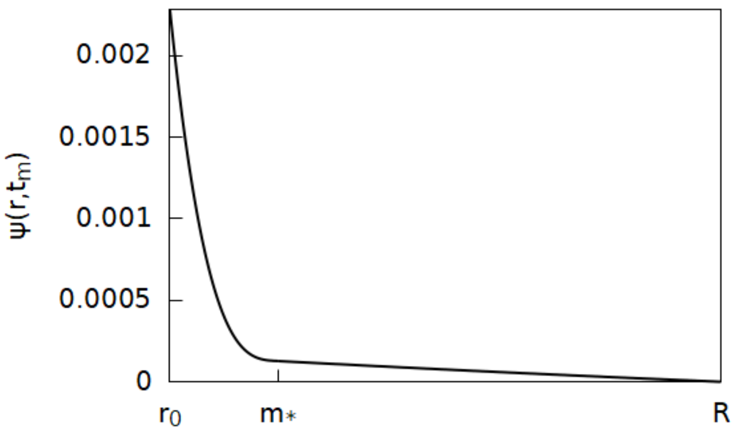

Fig. 3. Angle of twisting ψ distribution at the moment Σ_1 encounters elastic-plastic boundary $r = m_*$

Reflection of the unloading wave from the elastic-plastic boundary

At the point of time $t = t_m$, plastic flow ceases, and the region with accumulated irreversible strains is bounded by the surfaces $r = r_0$ and $r = m_*$, where $m_* = m(t_m) = r_0\sqrt{\alpha t_m / k}$. At exactly the same time from the surface $r = m_*$ two discontinuity surfaces, the surface $\Sigma_2 : r_2 = m_* - G(t - t_m)$, and $\Sigma_3 : r_3 = m_* + G(t - t_m)$, start propagating in opposite directions to the inner and to the outer cylinder respectively (Fig. 4).

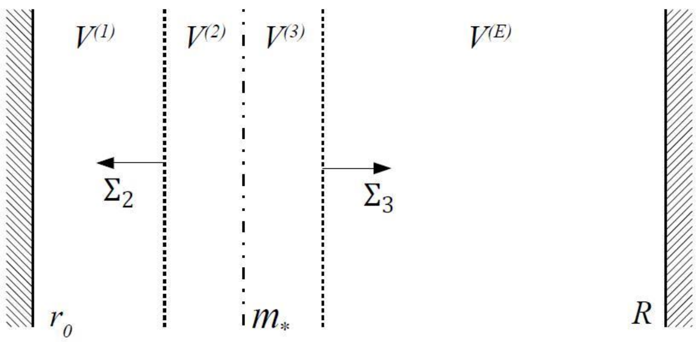

Fig. 4. Wave pattern after reflection of unloading wave from elastic-plastic boundary m_*

In the region with accumulated irreversible strains motion of the medium is governed by equation (35), while in the region of reversible deformation the equation of motion reads

$$\psi_{,rr}+\tfrac{3}{r}\psi_{,r}-\tfrac{\ddot{\psi}}{G^2}=0. \tag{41}$$

Boundary conditions for equations (35) and (41) are the condition of continuity of displacements across $\Sigma_2\colon r=r_2(t)$ and $\Sigma_3\colon r=r_3(t)$ as well as matching displacements and their derivatives at the elastic-plastic boundary $r=m_*$. The latter ensures the smoothness of a solution in the domain between Σ_2 and Σ_3 at each instant of time beyond t_m

$$[\psi]|_{\Sigma_2}=[\psi]|_{\Sigma_3}=0,\quad \psi^{(2)}\Big|_{m_*}=\psi^{(3)}\Big|_{m_*},\quad \psi^{(2)}_{,r}\Big|_{m_*}=\psi^{(3)}_{,r}\Big|_{m_*}. \tag{42}$$

As above, a superscript in parentheses indicates the value is calculated in the region behind the wave with a corresponding number. The solution for the desired function $\omega(r,t)$ behind the waves Σ_2 and Σ_3 is represented by the ray series similar to (36)

$$\omega^{(2)}=\left(\omega^{(1)}-\eta_1\right)\Big|_{\tau_2}+\left(\dot{\omega}^{(1)}-\eta_2\right)\Big|_{\tau_2}(t-\tau_2)+\tfrac{1}{2}\left(\ddot{\omega}^{(1)}-\eta_3\right)\Big|_{\tau_2}(t-\tau_2)^2+\cdots, \tag{43}$$

$$\omega^{(3)}=\left(\omega^{(E)}-\zeta_1\right)\Big|_{\tau_3}-\zeta_2|_{\tau_3}(t-\tau_3)-\tfrac{1}{2}\zeta_3|_{\tau_3}(t-\tau_3)^2+\cdots, \tag{44}$$

$$\eta_i=\left[\frac{\partial^i\psi}{\partial t^i}\right]\Bigg|_{\Sigma_2},\quad \tau_2=t_m+(m_*-r)/G,\qquad \zeta_i=\left[\frac{\partial^i\psi}{\partial t^i}\right]\Bigg|_{\Sigma_3},\quad \tau_3=t_m+(r-m_*)/G.$$

Differential equations for the ray series coefficients can be obtained in accordance with the procedure described in Section 4. After integration, we substitute the result into (43), (44), compare it with the boundary conditions (42), and obtain the desired solution

$$\psi^{(2)}(r,t)=\psi^{(1)}\Big|_{\tau_2}+\omega^{(1)}\Big|_{\tau_2}(t-\tau_2)+\tfrac{1}{2}\dot{\omega}^{(1)}\Big|_{\tau_2}(t-\tau_2)^2+ \tag{45}$$
$$+\frac{1}{6}\left(\ddot{\omega}^{(1)}\Big|_{\tau_2}-\eta_3(\tau_2)\right)(t-\tau_2)^3+\cdots,$$

$$\sigma^{(2)}_{r\varphi}(r,t)=\sigma^{(1)}_{r\varphi}\Big|_{\tau_2}+\dot{\sigma}^{(1)}_{r\varphi}\Big|_{\tau_2}(t-\tau_2)+\frac{1}{2}\left(\ddot{\sigma}^{(1)}_{r\varphi}\Big|_{\tau_2}-\frac{\mu r}{G}\eta_3(\tau_2)\right)(t-\tau_2)^2+\cdots,$$

$$\psi^{(3)}(r,t)=\psi^{(E)}\Big|_{\tau_3}+\left(\omega^{(E)}\Big|_{\tau_3}-\zeta_1(\tau_3)\right)(t-\tau_3)-\tfrac{1}{2}\zeta_2(\tau_3)(t-\tau_3)^2- \tag{46}$$
$$-\frac{1}{6}\zeta_3(\tau_3)(t-\tau_3)^3+\cdots,$$

$$\sigma^{(3)}_{r\varphi}(r,t)=-\frac{\alpha\tau_3 r_0^2}{r^2}+\frac{\mu r}{G}\zeta_1(\tau_3)+\left(-\frac{\alpha r_0^2}{r^2}+\frac{3\mu}{2}\zeta_1(\tau_3)+\frac{\mu r}{G}\zeta_2(\tau_3)\right)(t-\tau_3)+$$
$$+\frac{\mu r}{2G}\left(\zeta_3(\tau_3)+\frac{3G}{2r}\zeta_2(\tau_3)+\frac{3G^2}{8r^2}\zeta_1(\tau_3)\right)(t-\tau_3)^2+\cdots,$$

$$\eta_1=\eta_2=0,\eta_3=\frac{C^{(2)}}{r_2^{3/2}},\zeta_1=\frac{A^{(3)}}{r_3^{3/2}},\zeta_2=\frac{B^{(3)}}{r_3^{3/2}}+\frac{3GA^{(3)}}{8r_3^{5/2}},$$

$$\zeta_3=\frac{C^{(3)}}{r_3^{3/2}}+\frac{3GB^{(3)}}{8r_3^{5/2}}-\frac{15A^{(3)}G^2}{128r_3^{7/2}},$$

$$A^{(2)}=B^{(2)}=0,\qquad C^{(2)}=\frac{\alpha^2 r_0^4}{2\eta k m_*^{5/2}}+\frac{kG^2}{2\eta\sqrt{m_*}}-\frac{3G\alpha r_0^2}{4\eta m_*^{3/2}},$$

$$A^{(3)}=A^{(1)}+\frac{4\sqrt{m_*}}{\eta}\left(\frac{2}{3}km_*-\frac{\alpha r_0^2}{G}\right),\qquad B^{(3)}=B^{(1)}-\frac{5kG\sqrt{m_*}}{\eta}-\frac{\alpha r_0^2}{2\eta\sqrt{m_*}},$$

$$C^{(3)}=C^{(1)}-\frac{\alpha^2 r_0^4}{2\eta k m_*^{5/2}}-\frac{21kG^2}{16\eta\sqrt{m_*}}-\frac{33\alpha r_0^2 G}{32\eta\sqrt{m_*}}.$$

As it follows from (45)-(46) Σ_3 is a shock wave, while Σ_2 is a weak wave of second order across which stresses and velocities remain continuous, but an acceleration derivative jump occurs. In view of the fact the region with irreversible strains occupies a small part of the layer, Σ_2 will reflect from the cylinder wall $r = r_0$ before Σ_3 reaches $r = R$. The reflection causes propagation of the weak wave of second order $\Sigma_4 : r_4 = r_0 + G(t - t_r),\ \ t_r = t_m + (m_* - r_0)/G$. If we restrict ourselves by cubic terms in the ray solution no new discontinuities occur when Σ_4 travels through the elastic-plastic boundary $r = m_*$, but the calculation of preliminary deformations is implemented differently on the left/right of the elastic-plastic boundary. Preliminary deformations are given by the solution behind Σ_2 in the region $r_0 \le r \le m_*$ and by the solution behind Σ_3 in the region $m_* \le r \le r_3(t)$. Let us write down the solution behind the wave Σ_4

$$\psi^{(4)}(r,t) = \psi^{(i)}\big|_{\tau_4} + \omega^{(i)}\big|_{\tau_4}(t - \tau_4) + \tfrac{1}{2}\dot{\omega}^{(i)}\big|_{\tau_4}(t - \tau_4)^2 + \tag{47}$$

$$+\frac{1}{6}\left(\ddot{\omega}^{(i)}\big|_{\tau_4} - \theta_3(\tau_4)\right)(t - \tau_4)^3 + \cdots,$$

$$\sigma_{r\varphi}^{(4)}(r,t) = \sigma_{r\varphi}^{(i)}\Big|_{\tau_4} + \dot{\sigma}_{r\varphi}^{(i)}\Big|_{\tau_4}(t - \tau_4) + \frac{1}{2}\left(\ddot{\sigma}_{r\varphi}^{(i)}\Big|_{\tau_4} + \frac{\mu r}{G}\theta_3(\tau_4)\right)(t - \tau_4)^2 + \cdots,$$

$$\theta_3 = [\ddot{\omega}]|_{\Sigma_4} = \frac{C^{(2)}}{r_4^{3/2}},\ \ \tau_4 = t_r + (r - r_0)/G,$$

in which $i = 2$ in the region $r_0 \le r \le m_*$ and $i = 3$ in the region $m_* \le r \le r_3(t)$. The next change in the wave pattern occurs when the wave Σ_3 reflects from the outer boundary $r = R$ in the form of shock wave $\Sigma_5 : r_5 = R - G(t - t_R),\ t_R = t_m + (R - m_*)/G$. There are no difficulties to calculate the stress-strain pattern behind Σ_5 by means of the algorithm described and boundary condition (1) at the surface $r = R$, preliminary deformations ahead Σ_5 are prescribed by (46). Further, upon weak wave Σ_4 and shock wave Σ_5 interaction at the instant $\tilde{t} = t_m + (R - r_0)/G$ wave pattern remains unchanged, but from this moment one has to take into account that stress-strain state ahead Σ_4 is determined by the solution behind Σ_5 while preliminary deformations for the wave Σ_5 are determined according to (47).

At this point the analytical study is considered complete, and the calculation of the further deforming, if necessary, it is advisable to carry out numerically, using the analytical expressions obtained to approximate the solution at the nodes of the near-front domain.

Concluding remarks

The research conducted broadens our insight into the processes occurring in the material under complex deforming within a wide strain rate range. The problem considered features a two-stage deforming mechanism. In the first stage which involved slow loading of a cylindrical elastic/viscoplastic layer the exact solution of the corresponding boundary-value problem in the scope of large deformation theory has been obtained by applying the quasistatic approach. In the second stage calculation of the dynamic unloading triggered by instantaneous stress drop below the yield limit at the material and the rigid wall interface has been implemented. This event induced the cylindrical unloading shock wave, across which the plastic strain rate vanishes instantly and the viscoplastic flow ceases. Upon its reflection from the elastic-plastic boundary, a weak wave, propagating over the region with accumulated time invariable irreversible deformations, and a shock wave, moving over the elastic region, have been formed. Further unsteadiness in the layer is associated with the reflection of these waves from the instrument walls and the elastic-plastic boundary.

The solution to the dynamic problem has been constructed by means of truncated ray expansions about the time of the disturbance arrival at a given point of a domain. This method has allowed us to calculate the stress-strain state behind the discontinuity surfaces arising in the least cumbersome manner avoiding a need to solve the partial differential equation (which cannot be integrated exactly in the case of axial symmetry), but reducing the problem of integrating a system of ordinary differential equations of the first order that determine the change of velocity jump and the jumps of its time-derivatives up to the second order over the wave fronts. Higher-order jumps can be determined according to the same algorithm with no major difficulties except for increasing computational load.

However, some limitations are worth mentioning. First of all, truncated ray series demonstrate a close approximation to the solution only in the vicinity of the wave front, which is well applicable for narrow layers, as is one considered afore, and for short time range elapsed after the moment of wave motion commence. To construct a uniformly valid solution for a domain of a more considerable extent, methods of regularization of ray expansions can be applied, both analytical [35] and numerically analytical [26-28]. In the latter case, the analytical solution is incorporated into a numerical finite-difference scheme.

Second of all, the unloading wave velocity turned out to be constant owing to allowing the reversible strains to be small. In the case of finite strains situation is more complicated; the shock wave velocity and in turn the location of the wave front will be affected by strains ahead of the wave and discontinuities amplitudes, and the evolution behavior of the discontinuity of each order is coupled with that of higher order. In addition, the wave pattern becomes more intricate because two shear shock waves will appear in a prestrained medium: plane and circularly polarized ones.

Nevertheless, despite the noted limitations, the results of the present work lay the groundwork for studying more complex unsteady boundary-value problems involving large deformations.

References

1. Rozdestvenskii BL, Janenko NN. *Systems of quasilinear equations and their applications to gas dynamics*. Providence, Rhode Island: American Mathematical Society; 1983.
2. Gel'fand IM. Some problems of the theory of quasilinear equations. *Amer. Math. Soc. Transl. Ser*. 1963;29(2): 295-381.
3. Godunov SK, Zabrodin AV, Ivanov MY, Kraiko AN, Prokopov GP. *Numerical solution of multidimensional problems of gas dynamics*. Moscow: Nauka; 1976. (In Russian)
4. Hopf E. On the right weak solution of the Cauchy problem for a quasilinear equation of first order. *J. Math. Mech*. 1969;19(6): 483-487.
5. Engelbrecht J. *Nonlinear wave processes of deformation in solids*. London: Pitman; 1983.
6. Truesdell C. General and exact theory of waves in finite elastic strain. *Arch. Rat. Mech. Anal*. 1961;8(4): 263-296.
7. Bland DR. *Nonlinear dynamic elasticity*. Waltham, Massachusetts: Blaisdell; 1969.
8. Burenin AA, Chernyshov AD. Shock waves in an isotropic elastic space. *J. Appl. Math. Mech*. 1978;42(4): 711-717.
9. Kulikovskii A, Sveshnikova E. *Nonlinear waves in elastic media*. Boca Raton, Florida: CRC Press; 1995.
10. Chy B-T. Transverse shock waves in incompressible elastic solids. *J. Mech. Phys. Solids*. 1967;15(1): 1-14.
11. Wesolowski Z, Bürger W. Shock waves in incompressible elastic solids. *Rheol. Acta*. 1977;16(2): 155-160.
12. Fu YB, Scott NH. The evolutionary behavior of plane transverse weak nonlinear shock waves in unstrained incompressible isotropic elastic non-conductors. *Wave Motion*. 1989;11(4): 351-365.
13. Mandel J. Ondes plastique dans un milieu indéfini à trios dimensions. *J. Mec*. 1962;1(1): 3-30. (In French)
14. Bykovtsev GI, Ivlev DD. *Theory of Plasticity*. Vladivostok: Dal'nauka; 1998. (In Russian)

15. Bykovtsev GI, Kretova LD. Shock wave propagation in elastic-plastic media. *J. Appl. Math. Mech*. 1972;36(1): 94-103.
16. Burenin AA, Bykovtsev GI, Rychkov VA. Velocity discontinuity surfaces in the dynamics of irreversibly compressible media. In: *Problems of Continuum Mechanics (to the 60th anniversary of academician V.P. Myasnikov)*. Vladivostok: Inst. of Automation and Control Processes FEB RAS; 1996. p.116-127. (In Russian)
17. Kameniarzh IA. Simple waves and collapse of a discontinuity in an elastic-plastic medium with mises condition. *J. Appl. Math. Mech*. 1972; 36(2): 296-305.
18. Clifton RJ. Dynamic plasticity. *Trans ASME: J. Appl. Mech*. 1983;50(4b): 941-952.
19. Druyanov BA, Svyatova EA. The problem of the structure of a discontinuity in a strain-hardening plastic medium. *J. Appl. Math. Mech.* 1986;51(6): 808-810.
20. Balashov DB. Decay of discontinuity in linearly strain-hardening elastic-plastic medium. *Mech. Solids*. 1993; 28(2): 113-120.
21. Sadovskii VM. *Discontinuous solutions in the dynamics of elastic-plastic media*. Moscow: Fizmatlit; 1997. (In Russian)
22. Ivanov GV, Volchkov YM, Bogul'skii IO, Anisimov SA, Kurguzov VD. *Numerical solution of dynamic problems of elastic-plastic deformation of solid bodies.* Novosibirsk: Sib. Univ.; 2002. (In Russian)
23. Wilkins ML. Calculations of elastic-plastic flow. In: Adler B, Fernbach S, Rotenberg M. (eds.). *Methods of Computational Physics, Vol. 3: Fundamental Methods in Hydrodynamics*. New York: Academic Press; 1964; p.211-264.
24. Kukudzhanov VN. *Numerical continuum mechanics*. Berlin: Walter de Gruyter; 2013.
25. Bathe K-J, Shyder MD, Cimento AP, Rolph WD. 1980, On some current procedures and difficulties in finite element analysis of elasto-plastic response. *Computers and Structures.* 1980;12: 607-624.
26. Burenin AA, Zinov'ev PV. On the problems of identifying discontinuity surfaces in numerical methods of dynamics of deformable media. In: *Problems of Mechanics (dedicated to the 90th anniversary of A.Yu. Ishlinskii).* Moscow: Fizmatlit;2003. p.146-155. (In Russian)
27. Gerasimenko EA, Zavertan AV. Ray expansions of solutions around fronts as a shock-fitting tool for shock loading simulation. *Comput. Math. and Math. Phys.* 2009;49(4): 698-709.
28. Burenin AA, Sevastyanov GM, Shtuka VI. On the localization of discontinuities in calculations of incompressible elastic media dynamics. *Computational Continuum Mechanics.* 2016; 9(4): 400-411. (In Russian)
29. Whizam GB. *Linear and nonlinear waves*. New York: John Wiley & Sons; 1974.
30. Achenbach ID, Reddy DP. Note of wave propagation in linearly viscoelastic media. Z. anqew. *Maht. und. Phus. (ZAMP).* 1967;18(1): 141-144.
31. Babicheva LA, Bykovtsev GI, Verveiko ND. Ray method of solving dynamic problems in elastic-viscoplastic media. *J. Appl. Math. Mech*. 1973;37(1): 132-141.
32. Thomas T. *Plastic flow and fracture in solids*. New York: Academic Press; 1961.
33. Grinfel'd MA. *Methods of continuum mechanics in the theory of phase transitions.* Moscow: Nauka; 1990. (In Russian)
34. Grinfel'd MA. Ray method of calculating the wave front intensity in nonlinearly elastic material. *J. Appl. Math. Mech*. 1978;42(5): 958-977.
35. Rossikhin YA, Shitikova MV. Ray method for solving dynamic problems connected with propagation of wave surfaces of strong and weak discontinuities. *Appl. Mech. Rev*. 1995;48(1): 1-39.
36. Burenin AA. Possibility of constructing approximate solutions to nonstationary problems in the dynamics of elastic media under shock impacts. *Dal'nevostochn. Mat. Sb*. 1999;8: 49-72. (In Russian)

37. Burenin AA, Rossikhin YA, Shitikova MV. A ray method for solving boundary value problems connected with the propagation of finite amplitude shock waves. In: *Proceedings of the Int. Symp. on Nonlinear Theory and its Applications*. 1993. p.1085-1088.
38. Rossikhin YA, Burenin AA, Potianikhin DA. Shock Waves Via Ray Expansions. In: Altenbach H, Öchsner A. (eds.) *Encyclopedia of Continuum Mechanics*. Berlin: Springer; 2019.
39. Kovtanyuk LV. On the forcing of an elastoviscoplastic material through an inflexible circular cylindrical die. *Doklady Physics*. 2005;50: 112-114.
40. Kovtanyuk LK, Shitikikov AV. Theory of large elastoplastic deformations of materials taking account of temperature and reological effects. *Vestnik DVO Ross. Akad. Nauk*. 2006;4: 87-93. (In Russian)
41. Burenin AA, Kovtanyuk LV, Lushpei AV. The transient retardation of rectilinear viscoplastic flow when the loading stresses are abruptly removed. *J. Appl. Math. Mech*. 2009;73(4): 478-482.
42. Burenin AA, Kovtanyuk LV, Kulaeva DV. Interaction of a one-dimensional unloading wave with an eastoplastic boundary in an elastoviscoplastic medium. *J. Appl. Mech. Tech. Phys.* 2012;53(1): 90-97.
43. Kovtanyuk LV, Rusanov MM. On collision of an unloading wave with advancing elastoplastic boundary in a flat heavy layer. *J. Appl. Ind. Math*. 2015;9: 519-526.
44. Burenin AA, Kovtanyuk LV. *Large irreversible strains and elastic aftereffect.* Vladivostok: Dal'nauka; 2013. (In Russian)
45. Begun AS, Burenin AA, Kovtanyuk LV, Lemza AO. On the mechanisms of production of large irreversible strains in materials with elastic, viscous and plastic properties. *Arch. Appl. Mech*. 2020;90: 829-845.
46. Shitikov AV, Bykovtsev GI. Finite deformations in an elastoplastic media. *Sov. Phys. Dokl*. 1990;35(3): 297-299.
47. Burenin AA, Bykovtsev GI, Kovtanyuk LV. A simple model of finite strain in an elastoplastic medium. *Doklady Physics*. 1996;41(3): 127-129.
48. Bezglasnyi PA, Verveiko ND. On the propagation of shock waves in an elastoviscoplastic medium. *Izv. Akad. Nauk SSSR, Mech. Tverd. Tela*. 1971;5: 71-76.
49. Rossikhin YA, Shitikova MV. Discontinuity Surfaces in Elasto-Visco-Plastic Media. In: Altenbach H, Öchsner A. (eds.) Encyclopedia of Continuum Mechanics. Berlin: Springer; 2019.
50. Verveiko ND, Shapiro AI, Krupenko SE. *The origin and motion of the tips of cracks behind the fronts of elastoviscoplastic waves.* Voronezh: Quarta Publishers; 2017. (In Russian)
51. Haase R. *Thermodynamics of irreversible processes*. Reading, Massachusetts: Addison-Welsey; 1968.
52. Lee EH. Elastic-plastic deformations at finite strains. *J. Appl. Mech*. 1969;36(1): 1-6.
53. Levitas VI. *Large deformation of materials with complex rheological properties at normal and high pressure*. New York: Nova Science Publishers; 1996.
54. Lurie AI. *Nonlinear theory of elasticity*. Amsterdam, North Holland: Elsevier; 1990.

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