Materials Physics and Mechanics. 2023;51(2): 96-111
Research article
http://dx.doi.org/10.18149/MPM.5122023_9

Submitted: August 10, 2022
Approved: September 5, 2022
Accepted: November 15, 2022

Exact elastoplastic analysis of a rotating hollow cylinder made of power-law hardening material

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Abstract. The article is devoted to the elastic-plastic analysis of a rotating hollow cylinder with fixed ends. It is assumed that the strains in a cylinder are infinitesimal and additively decomposed into elastic and plastic components. The elastic component of strain is determined in accordance with Hooke's law. The Tresca's yield condition, the flow rule associated with it and the power law of hardening are adopted in order to calculate plastic strains. The presented analysis covers both loading and unloading stages. Unloading of a cylinder is assumed to be purely elastic. For a number of special cases of the hardening law, an analytical solution of the formulated system of equations is found. Special attention is paid to the calculation of the angular velocity corresponding to the complete transition of a cylinder to the plastic state. Dependencies of the fully-plastic limit angular velocity on the hardening parameters are established.

Keywords: elastoplasticity, stress analysis, Tresca's criterion, power-law hardening, rotating cylinder, exact solution

Acknowledgements. *The authors acknowledge a financial support of this work by the Russian Science Foundation, project No 22-11-00163.*

Citation: Prokudin AN. Exact elastoplastic analysis of a rotating hollow cylinder made of power-law hardening material. *Materials Physics and Mechanics.* 2023;51(2): 96-111. DOI: 10.18149/MPM.5122023_9.

Introduction

A rotating cylinder, along with a rotating disk, is one of the classical problems of solid mechanics. In addition to calculation of stresses and strains, it is of considerable interest to determine the maximum permissible angular velocity. Elastoplastic analysis gives a more accurate estimate of the bearing capacity of a rotating cylinder compared to the theory of elasticity. In this context, a relatively new technology of rotational autofrettage [1,2] also should be mentioned. Accurate modelling of this technology is possible only under the framework of elastoplastic analysis. Next, consider the works which are more relevant to the content of the present article.

The solution of the elastoplastic problem for a linearly hardening rotating hollow cylinder with fixed ends is presented in [3]. A similar problem for an ideal hollow cylinder with free ends was considered by the authors of [4]. In [3,4], the distributions of displacements, stresses, and plastic strains in the cylinder under loading were found. Moreover, elastic limit angular velocity and fully-plastic limit angular velocity was established. In [1,2], rotational autofrettage of a hollow cylinder with fixed ends was modeled considering the Bauschinger effect. The authors obtained the distribution of stresses in the cylinder at the stage of loading, as well as the distribution of residual stresses in the cylinder after its preliminary rotation at a given maximum angular velocity. It was assumed that the cylinder is made of an ideal elastoplastic material [1] or a linearly hardening [2] material. Rotational autofrettage of a hollow cylinder with free ends was studied in [5,6]. In [5], the authors obtained exact solutions for displacements, stresses, and plastic deformations in a rotating hollow cylinder at the stages of loading and unloading. Both isotropic and kinematic (Bauschinger effect) hardening was neglected. It should be noted that in [1–5] the Tresca plasticity condition and the flow law associated with it were adopted. 3D FEM analysis of rotational autofrettage is presented in [6]. The analysis is based on the von Mises' yield condition, the associated flow rule, and the Ramberg-Osgood power-law of hardening. The authors of [6] established a fairly good agreement between the numerical calculations and the analytical solution [5].

A rotating cylinder made of a nonlinear hardening material was studied in [7]. It was established that the parameters of hardening have a significant effect on the distribution of stresses and strains in the cylinder, as well as on the fully-plastic limit angular velocity. In [8], the distribution of residual stresses in a rotating nonlinear hardening cylinder after its preliminary rotation at a given velocity was found. Critical angular velocity is obtained, at which a secondary plastic flow does not occur at the stage of unloading. In [7, 8], the deformation theory of plasticity, the von Mises' yield condition, and the Swift hardening law were used. The solution was carried out using the numerical algorithm based on the shooting method.

The studies [9, 10] are devoted to the elastoplastic analysis of a rotating hollow cylinder in the presence of a stationary temperature gradient between the side surfaces of the cylinder. The problem statement was based on the Tresca condition, the flow law associated with it, and the law of linear hardening. To calculate thermal strains, the heat conduction equation and the Duhamel-Neumann law were adopted. It was assumed that the mechanical and thermophysical parameters of the material do not depend on temperature. It was established [9, 10] that an inhomogeneous temperature field has a significant effect on the evolution of plastic flow in a rotating hollow cylinder.

One way to increase the bearing capacity of a structure and reduce its weight is to use functionally graded materials. Elastoplastic analysis of rotating hollow FGM-cylinders with fixed ends is presented in [11–14]. The results were obtained on the basis of the Tresca condition and the associated flow law, the material of the cylinder was assumed to be perfectly plastic. The power dependence of material properties on the radial coordinate of the cylinder was used. The obtained results [11-14] show that the inhomogeneity of the material has a significant effect on the stress state in the cylinder. Moreover, by a suitable choice of distributions of the mechanical properties of the material, it is possible to significantly increase the maximum angular velocity of the cylinder.

As can be seen from the review, using the Tresca's yield condition, analytical solutions were found for the elastoplastic rotating hollow cylinder in various formulations [1-5,9-14]. In all studies listed above, the material of the cylinder was assumed to be ideal or linearly hardening. As known [15], nonlinear laws more accurately describe the hardening of structural materials however the application of these laws is usually associated with the use of numerical methods [6–8]. This paper is devoted to the elastoplastic analysis of a rotating hollow cylinder based on the Tresca's plasticity condition, the flow law associated with it, and the power law of hardening. For a number of special cases of the hardening law, analytical solutions are obtained.

Problem statement

Consider a hollow cylinder rotating around its own axis. The angular velocity ω of the cylinder slowly increases with time, consequently the angular acceleration can be neglected. It is assumed that the cylinder is in a state of plane strain and holds axial symmetry. Under these kinematic constraints, the only non-zero displacement in the cylinder is the radial displacement u_r, and shear strains and shear stresses are equal to zero. A cylindrical coordinate system (r,θ,z) is used. Following dimensionless variables are introduced

$$\delta=\frac{r_{in}}{r_{out}},\ \beta=\frac{r}{r_{out}},\ \bar{u}=\frac{E}{\sigma_0}\frac{u_r}{r_{out}},\ \bar{\varepsilon}_{ij}=\frac{E}{\sigma_0}\varepsilon_{ij},\ \bar{\varepsilon}_{ij}^e=\frac{E}{\sigma_0}\varepsilon_{ij}^e,\ \bar{\varepsilon}_{ij}^p=\frac{E}{\sigma_0}\varepsilon_{ij}^p,$$
$$\bar{\sigma}_{ij}=\frac{\sigma_{ij}}{\sigma_0},\ \bar{\sigma}_y=\frac{\sigma_y}{\sigma_0},\ H=K\left(\frac{\sigma_0}{E}\right)^n,\ \Omega=\frac{\rho r_{out}^2}{\sigma_0}\omega^2, \tag{1}$$

here r_{in},r_{out} are inner and outer radii of the cylinder, E is Young modulus, σ_0,σ_y are initial and current yield strength, ε_{ij} are total strains, ε_{ij}^e are elastic strains, ε_{ij}^p are plastic strains, σ_{ij} are stresses, K,n are hardening parameters, ρ is density. Next in the article, all formulas are written using dimensionless variables (1), and the overlines are omitted for brevity. For convenience, the loading parameter Ω is called the angular velocity.

It is assumed that the strains in the cylinder are infinitesimal and additively decomposed into elastic and plastic parts

$$\varepsilon_{rr}=\frac{\partial u}{\partial\beta};\ \varepsilon_{\theta\theta}=\frac{u}{\beta};\ \varepsilon_{zz}=0;$$
$$\varepsilon_{rr}=\varepsilon_{rr}^e+\varepsilon_{rr}^p;\ \varepsilon_{\theta\theta}=\varepsilon_{\theta\theta}^e+\varepsilon_{\theta\theta}^p;\ \varepsilon_{zz}=\varepsilon_{zz}^e+\varepsilon_{zz}^p. \tag{2}$$

The only non-trivial equilibrium equation has form

$$\frac{\partial\sigma_{rr}}{\partial\beta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{\beta}=-\Omega\beta. \tag{3}$$

Elastic strains and stresses are related by Hooke's law

$$\sigma_{rr}=\frac{1}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_{rr}^e+\nu\varepsilon_{\theta\theta}^e+\nu\varepsilon_{zz}^e\right),$$
$$\sigma_{\theta\theta}=\frac{1}{(1+\nu)(1-2\nu)}\left(\nu\varepsilon_{rr}^e+(1-\nu)\varepsilon_{\theta\theta}^e+\nu\varepsilon_{zz}^e\right), \tag{4}$$
$$\sigma_{zz}=\frac{1}{(1+\nu)(1-2\nu)}\left(\nu\varepsilon_{rr}^e+\nu\varepsilon_{\theta\theta}^e+(1-\nu)\varepsilon_{zz}^e\right),$$

where ν is the Poisson ratio.

The Tresca's yield condition reads as follows

$$\sigma_1-\sigma_3=\sigma_y, \tag{5}$$

here σ_{eq} is the equivalent stress, σ_1,σ_3 are the major and minor principal stresses respectively.

Flow rule has form:

$$d\varepsilon_{ij}^p=d\lambda\frac{df}{d\sigma_{ij}}, \tag{6}$$

where $d\varepsilon_{ij}^p$ are the increments of plastic strains, $d\lambda$ is non-negative multiplier, f is plastic function associated with the yield condition (5).

Power law of hardening reads as follows

$$\sigma_y = 1 + H\varepsilon_{eq}^n, \tag{7}$$

here ε_{eq} is the equivalent plastic strain.

The equivalence of the increment of plastic work is

$$\sigma_y \, d\varepsilon_{eq} = \sigma_{rr} \, d\varepsilon_{rr}^p + \sigma_{\theta\theta} \, d\varepsilon_{\theta\theta}^p + \sigma_{zz} \, d\varepsilon_{zz}^p, \tag{8}$$

where $d\varepsilon_{eq}$ is the increment of the equivalent plastic strain.

It is assumed that the side surfaces of the cylinder are traction-free. In this case, the boundary conditions have the form:

$$\sigma_{rr}(\delta) = 0, \sigma_{rr}(1) = 0. \tag{9}$$

The yield condition (5) is satisfied for the first time on the inner surface of the cylinder $\beta = \delta$ at $\Omega = \Omega_p$. With a further increase of the angular velocity, the elastoplastic border $\beta = \beta_{ep}$ moves to the outer surface of the cylinder. At $\Omega = \Omega_{fp}$ the elastic region disappears and the cylinder becomes fully-plastic. The present analysis is restricted to a range $\Omega_p \le \Omega \le \Omega_{fp}$, in which the cylinder consists of the inner plastic region $\left(\beta \in \left[\delta, \beta_{ep}\right]\right)$ and the outer elastic region $\left(\beta \in \left[\beta_{ep}, 1\right]\right)$. Next, the solution in each of the regions is considered.

Elastic region

The distributions of displacements and stresses in the elastic region have the form:

$$\begin{aligned}
u^{el} &= \frac{(1+\nu)d_1}{2\beta} + (1+\nu)(1-2\nu)d_2\beta - \frac{1}{8}\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\Omega\beta^3, \\
\sigma_{rr}^{el} &= -\frac{d_1}{2\beta^2} + d_2 - \frac{(3-2\nu)}{8(1-\nu)}\Omega\beta^2, \\
\sigma_{\theta\theta}^{el} &= \frac{d_1}{2\beta^2} + d_2 - \frac{(1+2\nu)}{8(1-\nu)}\Omega\beta^2, \\
\sigma_{zz}^{el} &= \nu\left(\sigma_{rr}^{el} + \sigma_{\theta\theta}^{el}\right)
\end{aligned} \tag{10}$$

here d_1, d_2 are integration constants.

It is known [1-3] that plastic flow in a rotating hollow cylinder starts on the inner surface where the stress satisfies the inequality $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$. Elastic limit angular velocity is the function of the Poisson ratio ν and the geometric parameter δ [3]:

$$\Omega_p = \frac{4(1-\nu)}{3-2\nu+(1-2\nu)\delta^2}. \tag{11}$$

The integration constants d_1, d_2 may be determined from the system of equations:

$$\begin{cases} \sigma_{\theta\theta}^{el}\left(\beta_{ep}\right) - \sigma_{rr}^{el}\left(\beta_{ep}\right) = 1, \\ \sigma_{rr}^{el}(1) = 0. \end{cases}$$

The meaning of the first equation is that the yield condition (5) should be satisfied on the elastoplastic boundary. The solution of the previous system reads as follows

$$d_1 = \beta_{ep}^2 - \frac{(1-2\nu)}{4(1-\nu)}\Omega\beta_{ep}^4, \quad d_2 = \frac{1}{2}\beta_{ep}^2 - \frac{(1-2\nu)}{8(1-\nu)}\Omega\beta_{ep}^4 + \frac{(3-2\nu)}{8(1-\nu)}\Omega. \tag{12}$$

Plastic region

Assume that the stresses in the plastic region always satisfy the inequality $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$. In this case, Tresca's yield condition (5) has form

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_y. \tag{13}$$

It follows from the yield condition (13) and the flow rule (6) that

$$d\varepsilon_{\theta\theta}^p = -d\varepsilon_{rr}^p, d\varepsilon_{zz}^p = 0.$$

The equivalence of the increment of plastic work (8) yields

$$\sigma_y\, d\varepsilon_{eq} = \sigma_{rr}\, d\varepsilon_{rr}^p + \sigma_{\theta\theta}\, d\varepsilon_{\theta\theta}^p + \sigma_{zz}\, d\varepsilon_{zz}^p = \left(\sigma_{\theta\theta} - \sigma_{rr}\right) d\varepsilon_{\theta\theta}^p.$$

Using the yield condition (13), the previous equation leads to

$$d\varepsilon_{eq} = d\varepsilon_{\theta\theta}^p.$$

Hence, in the case of monotonically increasing load, the plastic strains are expressed as follows

$$\varepsilon_{eq} = \varepsilon_{\theta\theta}^p = -\varepsilon_{rr}^p, \varepsilon_{zz}^p = 0.$$

It follows from the kinematic relations (2) and the previous expressions that

$$\varepsilon_{rr}^e = \varepsilon_{rr} + \varepsilon_{eq}, \ \ \varepsilon_{\theta\theta}^e = \varepsilon_{\theta\theta} - \varepsilon_{eq}, \ \ \varepsilon_{zz}^e = 0.$$

Thus, the Hooke's law (4) can be transformed into the form:

$$\sigma_{rr} = \frac{1}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta} + (1-2\nu)\varepsilon_{eq}\right),$$

$$\sigma_{\theta\theta} = \frac{1}{(1+\nu)(1-2\nu)}\left(\nu\varepsilon_{rr} + (1-\nu)\varepsilon_{\theta\theta} - (1-2\nu)\varepsilon_{eq}\right),$$

$$\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right).$$

Using the previous expressions, the equilibrium equation (3) takes the form:

$$\frac{\partial^2 u}{\partial\beta^2} + \frac{1}{\beta}\frac{\partial u}{\partial\beta} - \frac{u}{\beta^2} = -\frac{(1+\nu)(1-2\nu)}{(1-\nu)}\Omega\beta - \frac{(1-2\nu)}{(1-\nu)}\left(2\frac{\varepsilon_{eq}}{\beta} + \frac{d\varepsilon_{eq}}{d\beta}\right).$$

From the solution of the previous equation, one may find:

$$u^{pl} = \frac{(1+\nu)c_1}{2\beta} + (1+\nu)(1-2\nu)c_3\beta - \frac{(1+\nu)(1-2\nu)}{8(1-\nu)}\Omega\beta^3 - \frac{(1-2\nu)}{(1-\nu)}\beta\int_\delta^\beta \frac{\varepsilon_{eq}}{\beta}d\beta,$$

$$\sigma_{rr}^{pl} = -\frac{c_1}{2\beta^2} + c_2 - \frac{(3-2\nu)}{8(1-\nu)}\Omega\beta^2 - \frac{1}{1-\nu^2}\int_\delta^\beta \frac{\varepsilon_{eq}}{\beta}d\beta, \tag{14}$$

$$\sigma_{\theta\theta}^{pl} = \frac{c_1}{2\beta^2} + c_2 - \frac{(1+2\nu)}{8(1-\nu)}\Omega\beta^2 - \frac{1}{1-\nu^2}\left(\int_\delta^\beta \frac{\varepsilon_{eq}}{\beta}d\beta - \varepsilon_{eq}\right),$$

here c_1, c_2 are integration constants.

The integration constants c_1 and c_2 may be determined from the system of equations:

$$\begin{cases} \sigma_{rr}^{pl}(\delta) = 0, \\ \varepsilon_{eq}(\beta_{ep}) = 0. \end{cases}$$

The solution of the previous system is as follows

$$c_1 = \beta_{ep}^2 - \frac{(1-2\nu)}{4(1-\nu)}\Omega\beta_{ep}^4, \quad c_2 = \frac{\beta_{ep}^2}{2\delta^2} - \frac{(1-2\nu)}{8(1-\nu)}\Omega\frac{\beta_{ep}^4}{\delta^2} + \frac{(3-2\nu)}{8(1-\nu)}\Omega\delta^2. \tag{15}$$

The coordinate of the elastoplastic boundary is determined from the condition: $u^{el}\left(\beta_{ep}\right)=u^{pl}\left(\beta_{ep}\right)$.

Using (10), (12), (14) and (15), the previous condition may be written in the form:

$$\left(1-2\nu\right)\Omega\beta_{ep}^{4}-4\left(1-\nu\right)\beta_{ep}^{2}+\frac{8\delta^{2}}{\left(1-\delta^{2}\right)\left(1+\nu\right)}\int_{\delta}^{\beta_{ep}}\frac{\varepsilon_{eq}}{\beta}d\beta+\left(3-2\nu\right)\Omega\delta^{2}=0. \tag{16}$$

The equation (16) is nonlinear for the border radius β_{ep} and can be solved numerically by Newton's method for assigned values of the mechanical parameters and the angular velocity $\left(\Omega>\Omega_{p}\right)$.

With the complete transition of the cylinder to the plastic state, the condition $\beta_{ep}=1$ is satisfied. Consequently, the equation (16) transforms into

$$\left(\left(1-2\nu\right)+\left(3-2\nu\right)\delta^{2}\right)\Omega_{fp}+\frac{8\delta^{2}}{\left(1-\delta^{2}\right)\left(1+\nu\right)}\int_{\delta}^{1}\frac{\varepsilon_{eq}}{\beta}d\beta-4\left(1-\nu\right)=0. \tag{17}$$

The equation (17) has an analytical solution for the linear-hardening material $\left(n=1\right)$. If $n\neq 1$, then the equation (17) is nonlinear for Ω_{fp} and can be solved by Newton's method.

Proposition. The obtained elastoplastic solution (10), (12), (14), (15) and (16) is continuous across the border $\beta=\beta_{ep}$ between elastic and plastic regions.

Proof. In the present article the following continuity conditions at $\beta=\beta_{ep}$ are used

$$\begin{cases} u \ =0, \\ \varepsilon_{eq}=0, \\ \sigma_{\theta\theta}^{el}-\sigma_{rr}^{el}=1, \end{cases}$$

where $[\![\,.\,]\!]$ means jump of a variable, i.e., the difference between function values on the elastoplastic border. Next, we prove that the previous system of conditions ensures the continuity of all functions.

Obviously, the condition $u \ =0$ implies $\varepsilon_{\theta\theta} \ =0$. From the condition $\varepsilon_{eq}=0$ it follows that $[\![\varepsilon_{rr}^{pl}]\!]=[\![\varepsilon_{\theta\theta}^{pl}]\!]=[\![\varepsilon_{zz}^{pl}]\!]=0$. As a result, it can be concluded that $[\![\varepsilon_{\theta\theta}^{el}]\!]=0$. One can see that $\sigma_{\theta\theta}^{pl}-\sigma_{rr}^{pl}=\sigma_{\theta\theta}^{el}-\sigma_{rr}^{el}$, which in turn means that $\sigma_{rr} \ = \ \sigma_{\theta\theta}$. Combining the previous equation with the equations $[\![\varepsilon_{\theta\theta}^{el}]\!]=0, [\![\varepsilon_{zz}^{el}]\!]=0$ and using the inverse Hooke's law

$$\varepsilon_{rr}^{el}=\sigma_{rr}-\nu\sigma_{\theta\theta}-\nu\sigma_{zz},$$
$$\varepsilon_{\theta\theta}^{el}=\sigma_{\theta\theta}-\nu\sigma_{rr}-\nu\sigma_{zz},$$
$$\varepsilon_{zz}^{el}=\sigma_{zz}-\nu\sigma_{rr}-\nu\sigma_{\theta\theta},$$

the following system of equations can be obtained

$$\begin{cases} \sigma_{rr} \ - \ \sigma_{\theta\theta} \ =0, \\ -\nu \ \sigma_{rr} \ + \ \sigma_{\theta\theta} \ -\nu \ \sigma_{zz} \ =0, \\ -\nu \ \sigma_{rr} \ -\nu \ \sigma_{\theta\theta} \ + \ \sigma_{zz} \ =0. \end{cases}$$

The solution of the previous system is $\sigma_{rr} \ = \ \sigma_{\theta\theta} \ = \ \sigma_{zz} \ =0$ because its determinant equals $\left(1+\nu\right)\left(1-2\nu\right)>0$. The continuity of the elastic strain ε_{rr}^{el} follows from the continuity of stresses and the inverse Hooke's law. Finally, $[\![\varepsilon_{rr}^{el}]\!]=0$ and $[\![\varepsilon_{rr}^{pl}]\!]=0$ imply that $\varepsilon_{rr}=0$. Proof is complete.

It should be noted that the obtained solution (14), as well as equations (16) and (17), are valid for any law of isotropic hardening. If the function $\varepsilon_{eq}(\beta)$ is known, then distributions of stresses and plastic strains can be easily derived for any value of the angular velocity Ω in the range $\Omega_p<\Omega\le\Omega_{fp}$. The last step of the solution process is to determine the plastic strain ε_{eq}. It follows from (5), (7) and (15) that

$$\begin{gathered}\varepsilon_{eq}+H_0\,\varepsilon_{eq}^n+\nu_0\Psi=0,\\ \nu_0=1-\nu^2,\, H_0=\nu_0 H,\, \Psi=1-\frac{c_1}{\beta^2}-\frac{(1-2\nu)}{4(1-\nu)}\Omega\beta^2.\end{gathered}\tag{18}$$

In general case, the equation (18) is transcendental and has no solution for arbitrary $n\in\mathbb{R}$. However, if $n\in\mathbb{Q}$, then the equation (18) is reduced to algebraic one. According to the Abel–Ruffini theorem, there is no solution in radicals to algebraic equations of degree five or higher. Hence the equation (18) has solution only in the following cases $n=1/4,1/3,1/2,2/3,3/4,1,4/3,3/2,2,3,4$.

The results of experiments [16] show that hardening parameter usually is less than one for metals $n<1$. Consequently, the further analysis is limited to the following values $n=1/4,1/3,1/2,2/3,3/4,1$.

The trivial case $n=1$ corresponds to the linear-hardening material. The solution of (18) reads as

$$\varepsilon_{eq}=-\frac{\nu_0\Psi}{1+\nu_0 H}.\tag{19}$$

Using (19), the integral in the equation (17) may be evaluated in elementary functions, which in turn leads to dependence for plastic limit angular velocity

$$\Omega_{fp}=\frac{4\left(1-\delta^2\right)\left(1-\nu^2\right)H-8\delta^2\log\delta}{\left(1-\delta^2\right)\left(4\delta^2+(1+\nu)\left(1-2\nu+\delta^2(3-2\nu)\right)H\right)}.\tag{20}$$

The found solution (19), (20) certainly coincides with the previously known solution [3]. If $H_0=0$, then the solution (19) reduces to $\varepsilon_{eq}=-\nu_0\Psi$ and describes perfectly plastic material.

Next, special cases of nonlinear hardening are considered. The equation of the form $x+ax^{p/q}+b=0\;(p,q\in\mathbb{N})$ is reduced to the equation $y^q+a\,y^p+b=0$ by using the apparent substitution $y=x^{1/q}$.

In the case $n=1/2$, the equation (18) has two roots but only one of them satisfies the condition $\varepsilon_{eq}(\beta_{ep})=0$. This root read as follows

$$\varepsilon_{eq}=\frac{1}{4}\left(\sqrt{H_0^2-4\nu_0\Psi}-H_0\right)^2.$$

It should be noted that using previous expression, the indefinite integral $\int\varepsilon_{eq}/\beta\,\mathrm{d}\beta$ may be evaluated through elliptical integrals which are supported by most computer algebra systems.

If $n=1/3$, then the equation (18) has only one real root

$$\varepsilon_{eq}=\frac{1}{18}\left(\gamma-3^{1/3}2^{2/3}H_0\,\gamma^{-1}\right)^3,\quad \gamma=\left(\sqrt{3}\sqrt{27v_0^2\Psi^2+4H_0^3}-9v_0\Psi\right)^{1/3}.$$

In the case $n=2/3$, the single real root of the equation (18) has the form

$$\varepsilon_{eq}=\frac{1}{27}\left(2^{-1/3}\,\kappa+2^{1/3}H_0^2\,\kappa^{-1}-H_0\right)^3,\quad \kappa=\left(3\sqrt{3}\sqrt{27v_0^2\Psi^2+4v_0\Psi H_0^3}-27v_0\Psi-2H_0^3\right)^{1/3}.$$

If $n=1/4$, then only one root of the equation (18) satisfies the condition $\varepsilon_{eq}\left(\beta_{ep}\right)=0$. This root read has form

$$\varepsilon_{eq}=\left(-\frac{1}{2}\lambda+\frac{1}{2}\sqrt{2H_0\lambda^{-1}-\lambda^2}\right)^4,$$

$$\lambda=\sqrt{4\left(\frac{2}{3}\right)^{1/3}v_0\Psi\,t^{-1}+\frac{t}{2^{1/3}3^{2/3}}},\quad t=\left(9H_0^2+\sqrt{3}\sqrt{27H_0^4-256v_0^3\Psi^3}\right)^{1/3}.$$

In the case $n=3/4$, the equation (18) has two real roots but only one of them satisfies the condition $\varepsilon_{eq}\left(\beta_{ep}\right)=0$. This root may be written as follows

$$\varepsilon_{eq}=\left(\frac{1}{2}\sqrt{\varphi+\frac{H_0^3}{4\chi}}-\frac{1}{2}\chi-\frac{H_0}{4}\right)^4,\quad \chi=\sqrt{\frac{H_0^2}{4}-4\frac{2^{1/3}\left(-v_0\Psi\right)^{2/3}}{3^{1/3}d}+\frac{\left(-v_0\Psi\right)^{1/3}}{2^{1/3}3^{2/3}}d},$$

$$\varphi=\frac{H_0^2}{2}+4\frac{2^{1/3}\left(-v_0\Psi\right)^{2/3}}{3^{1/3}d}-\frac{\left(-v_0\Psi\right)^{1/3}}{2^{1/3}3^{2/3}}d,\quad d=\left(\sqrt{81H_0^4-768v_0\Psi}-9H_0^2\right)^{1/3}.$$

For the cases $n=1/4,1/3,2/3,3/4$, the integrals in (15)–(17) are evaluated numerically.

Validity of solution

The obtained elastoplastic solution is valid only if the following inequality is satisfied in the plastic region during loading process

$$\sigma_{\theta\theta}>\sigma_{zz}>\sigma_{rr} \tag{21}$$

Validity of this inequality strongly depends on the values of parameters δ,ν,H,n. It is easy to verify that for particular values of parameters the inequality (21) is violated. For example, if $\nu\le 0$ then at the vicinity of the inner boundary of a cylinder the elastic stress state satisfies the inequality $\sigma_{\theta\theta}>\sigma_{rr}\ge\sigma_{zz}$. It follows from the boundary condition $\sigma_{rr}\left(\delta\right)=0$ and the relation $\sigma_{zz}=\nu\left(\sigma_{rr}+\sigma_{\theta\theta}\right)$. As a result, the inequality (21) will be violated in the plastic region. On the other hand, in the very thick-walled cylinders with relatively low but positive Poisson ratio, the radial stress may exceed axial one. This issue is illustrated in Fig. 1 where the distributions of stresses in a cylinder are presented for $\delta=0.2,\nu=0.25,H=0,\Omega=3.$

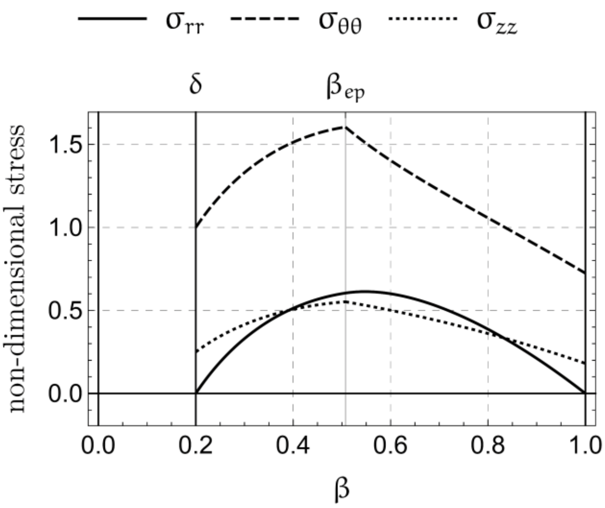


Fig. 1. Violation of the inequality (21) in the plastic region $(\delta = 0.2, \nu = 0.25, H = 0, \Omega = 3)$

It is interesting to note that the inequality (21) in the plastic region was assumed by other authors who studied rotating hollow cylinders with fixed ends [1–3] but the range of validity of this assumption has not been examined in detail before. The current section is aimed to fill this gap.

The stress state in the plastic region is

$$\begin{aligned} &\sigma_{\theta\theta} - \sigma_{rr} = 1 + \Delta, \\ &\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right), \end{aligned} \tag{22}$$

where $\Delta = H\varepsilon_{eq}^{n}, \Delta \geq 0$.

In the following, it is assumed that Poisson ratio is positive $\nu > 0$.

Stress state (22) should satisfy the inequality (21). Consider first part of the inequality (21). The previous equations can be combined as follows

$$\sigma_{zz} = 2\nu\sigma_{\theta\theta} - \nu(1 + \Delta).$$

Then the inequality $\sigma_{\theta\theta} > \sigma_{zz}$ can be transformed into the form

$$\sigma_{\theta\theta} > -\frac{\nu}{1 - 2\nu}(1 + \Delta).$$

Since $\sigma_{\theta\theta} > 0, \nu > 0, \nu < 1/2, \Delta > 0$ then the previous inequality is valid.

Next, consider the second part of the inequality (21). It follows from (22) that

$$\sigma_{zz} = 2\nu\sigma_{rr} + \nu(1 + \Delta).$$

The inequality $\sigma_{zz} > \sigma_{rr}$ implies that

$$\sigma_{rr} < \frac{\nu}{1 - 2\nu} + \frac{\nu}{1 - 2\nu}\Delta. \tag{23}$$

Consequently, the elastoplastic solution is valid if the radial stress satisfies the inequality (23). The careful estimation of the radial stress and the equivalent plastic strain is very complicated due to the nonlinearity of the solution. Consider the simpler inequality

$$\max \sigma_{rr} < \frac{\nu}{1 - 2\nu}. \tag{24}$$

It is obvious that the previous inequality implies the inequality (23). Next, the maximum possible value of the radial stress should be estimated. It follows from (15) that the stresses increase with decreasing equivalent plastic strain and reach a maximum when $\varepsilon_{eq} = 0$. This corresponds to the limiting case $H \to \infty$ when the plastic solution degenerates into the elastic one

$$\hat{\sigma}_{rr}^{pl} = \lim_{H\to\infty} \sigma_{rr}^{pl} = -\frac{c_1}{2\beta^2} + c_2 - \frac{(3-2\nu)}{8(1-\nu)}\Omega\beta^2. \tag{25}$$

The dependence of the angular velocity on the border radius follows from (16)

$$\Omega(\beta_{ep}) = \frac{4(1-\nu)\beta_{ep}^2}{(1-2\nu)\beta_{ep}^4 + (3-2\nu)\delta^2}.$$

By using previous expression and (15), the limiting distribution (25) of the radial stress may be transformed into the form

$$\hat{\sigma}_{rr}^{pl} = \frac{(3-2\nu)\beta_{ep}^2}{2\left((3-2\nu)\delta^2 + (1-2\nu)\beta_{ep}^4\right)} \frac{(1-\beta^2)(\beta^2-\delta^2)}{\beta^2}. \tag{26}$$

Next, one has to evaluate the maximum value of the function (26). It follows from the boundary conditions (9) and Rolle's theorem that for any fixed border radius β_{ep} the radial stress has a point where the first derivative is zero. It is easy to verify that this point is $\beta^* = \sqrt{\delta} \in [\delta, 1]$ which is a maximum point.

Consider the derivative of (26) with respect to β_{ep}.

$$\frac{\partial \hat{\sigma}_{rr}^{pl}}{\partial \beta_{ep}} = (3-2\nu)\frac{\left((3-2\nu)\delta^2\beta_{ep} - (1-2\nu)\beta_{ep}^5\right)}{\left((3-2\nu)\delta^2 + (1-2\nu)\beta_{ep}^4\right)^2}\frac{(1-\beta^2)(\beta^2-\delta^2)}{\beta^2}.$$

The equation $\frac{\partial \hat{\sigma}_{rr}^{pl}}{\partial \beta_{ep}} = 0$ has an only positive root $\beta_{ep}^* = \left(\frac{3-2\nu}{1-2\nu}\right)^{1/4}\sqrt{\delta}$. The second derivative analysis shows that β_{ep}^* is a maximum point. As a result, the radial stress has a maximum at point $(\beta = \beta^*, \beta_{ep} = \beta_{ep}^*)$ and the maximum value is

$$\max \sigma_{rr}^{pl} = \max \hat{\sigma}_{rr}^{pl} = \hat{\sigma}_{rr}^{pl}(\beta^*, \beta_{ep}^*) = \frac{(1-\delta)^2}{4\delta}\sqrt{\frac{3-2\nu}{1-2\nu}}.$$

Combining the previous expression with the inequality (24) one may conclude that the inequality (21) is valid for any power-hardening material with $H > 0, n > 0, \nu > \nu_t(\delta)$ where

$$\nu_t(\delta) = \frac{3(1-\delta)^2}{2\left(2 - 4\delta + 2\delta^2 + \sqrt{1 - 4\delta + 18\delta^2 - 4\delta^3 + \delta^4}\right)}. \tag{27}$$

One may notice that $\nu_t(0) = \frac{1}{2}, \nu_t(1) = 0.$ It should be remarked that under certain conditions the point β_{ep}^* may be out of range $[\delta, 1]$, namely $\beta_{ep}^* > 1$. As a result, the actual maximum value of the radial stress may be less than predicted above. However, the analysis showed that this issue does not significantly affect the function $\nu_t(\delta)$.

Next, consider the range of validity of the presented solution in the special case of ideal elastoplastic material. According to the (15), (18) and (19), if $H = 0$ then the radial stress in the plastic region takes the form

$$\sigma_{rr}^{pl} = \frac{1}{2}(\delta^2 - \beta^2)\Omega + \log(\beta/\delta). \tag{28}$$

It should be noted that

$$\frac{\partial \sigma_{rr}^{pl}}{\partial \Omega} = \frac{1}{2}\left(\delta^2 - \beta^2\right) \le 0.$$

The last inequality implies that the radial stress decreases with increase in Ω. Consequently, for any given point $\beta \in (\delta, 1]$ in the cylinder the radial stress reaches a maximum at the moment when elastoplastic border reaches this point.

The dependence of Ω according to border radius β_{ep} follows from (16) and takes the form

$$\Omega\left(\beta_{ep}\right) = \frac{4(1-\nu)\left(1 + 2\log\left(\beta_{ep}\,\delta^{-1}\right) - \beta_{ep}^2\right)}{3 - 2\nu - 4(1-\nu)\delta^2 + (1-2\nu)\left(2 - \beta_{ep}^2\right)\beta_{ep}^2}. \tag{29}$$

The dependence of the radial stress at the elastoplastic border according to the border coordinate β_{ep} may be obtained by combining (28) and (29) and reads as follows

$$\hat{\sigma}_{rr}^{pl}\left(\beta_{ep}\right) = \log\left(\beta_{ep}\,\delta^{-1}\right) + \frac{1}{2}\left(\delta^2 - \beta_{ep}^2\right)\frac{4(1-\nu)\left(1 + 2\log\left(\beta_{ep}\,\delta^{-1}\right) - \beta_{ep}^2\right)}{3 - 2\nu - 4(1-\nu)\delta^2 + (1-2\nu)\left(2 - \beta_{ep}^2\right)\beta_{ep}^2}.$$

The function $\hat{\sigma}_{rr}^{pl}\left(\beta_{ep}\right)$ reaches a maximum at some point $\beta_{ep}^* \in (\delta, 1)$. As a result, the inequality (21) is valid for $\nu > \nu_t(\delta)$, where the point β_{ep}^* and the value of ν_t may be evaluated numerically from the system of nonlinear equations

$$\begin{cases} \dfrac{\partial \hat{\sigma}_{rr}^{pl}}{\partial \beta_{ep}}\left(\beta_{ep}^*\right) = 0, \\ \hat{\sigma}_{rr}^{pl}\left(\beta_{ep}^*\right) = \dfrac{\nu}{1-2\nu}. \end{cases}$$

for any value of $\delta \in (0, 1)$.

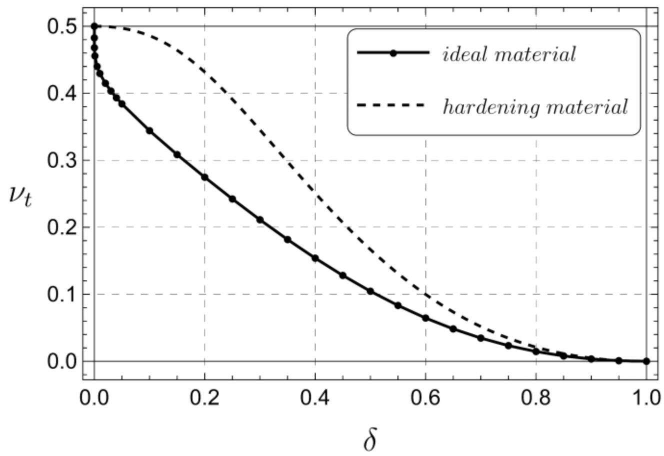


Fig. 2. The dependence $\nu_t(\delta)$ for ideal and hardening elastoplastic material

The plots of the function $\nu_t(\delta)$ for hardening and ideal elastoplastic material are presented in Fig. 2. The plot for ideal material is constructed by numerical calculations. From a technical point of view, relatively thin-walled cylinders are more commonly used in practice. For $\delta \ge 0.5$ the presented elastoplastic solution for a nonlinear hardening material is correct if $\nu > 1/6$ which is valid for the vast majority of structural materials. The classical solution [3] for an ideal elastoplastic material has a much broader range of validity in comparison to the presented solution. This is especially noticed for very thick-walled cylinders. On the other hand, if $\delta \geq 0.9$ then both solutions have very broad $(\nu > 0.0034)$ and almost the same ranges of applicability.

Unloading

The maximum angular velocity and final border radius are denoted as $\Omega_{\max}$ and $\hat{\beta}_{ep}$. It is assumed that unloading of a cylinder is purely elastic and the elastic properties E, ν of a material are the same as under loading. In this case the distributions of residual stresses at stand-still can be evaluated as the difference between the stresses at $\Omega = \Omega_{\max}$ and the ones occurring in a fictitious elastic material with unlimited elastic behavior at the same angular velocity. Using the purely elastic solution (10) and the boundary conditions (9) together with the solutions (10),(12) and (14),(15) in elastic and plastic region respectively, the distributions of residual stresses can be obtained in the following form

$$\sigma_{rr}^{res} = \begin{cases} \dfrac{b}{\beta^2} - \dfrac{b}{\delta^2} - \dfrac{1}{1-\nu^2} \displaystyle\int_{\delta}^{\beta} \dfrac{\hat{\varepsilon}_{eq}}{\beta} d\beta, & \left(\delta \leq \beta \leq \hat{\beta}_{ep}\right); \\ \dfrac{b}{\beta^2} - b, & \left(\hat{\beta}_{ep} \leq \beta \leq 1\right); \end{cases}$$

$$\sigma_{\theta\theta}^{res} = \begin{cases} -\dfrac{b}{\beta^2} - \dfrac{b}{\delta^2} - \dfrac{1}{1-\nu^2} \left(\displaystyle\int_{\delta}^{\beta} \dfrac{\hat{\varepsilon}_{eq}}{\beta} d\beta - \hat{\varepsilon}_{eq} \right), & \left(\delta \leq \beta \leq \hat{\beta}_{ep}\right); \\ -\dfrac{b}{\beta^2} - b, & \left(\hat{\beta}_{ep} \leq \beta \leq 1\right). \end{cases}$$

$$b = -\frac{\hat{\beta}_{ep}^2}{2} + \frac{(3-2\nu)\delta^2 + (1-2\nu)\hat{\beta}_{ep}^4}{8(1-\nu)} \Omega_{\max}, \quad \hat{\varepsilon}_{eq} = \varepsilon_{eq}\left(\Omega_{\max}\right).$$

Results

Consider a rotating hollow cylinder with the following parameter values $\delta = 0.5, \nu = 0.3$. It should be noted that these parameters satisfy the condition of validity for all values of hardening parameters (Section 5 and Fig. 2). Firstly, the obtained results are illustrated by the dependency of plastic limit angular velocity Ω_{fp} on hardening parameters n and H. This relation is plotted in Fig. 3. It can be seen that Ω_{fp} increases with increase in H. This dependence is the more pronounced, the lower the value of the power parameter n. For example, if $n=1$, then the increase of Ω_{fp} for $H \in [0,1]$ is ≈ 29 %. On the other hand, if $n = 1/4$ the increase is ≈ 42 %.

Series of Figs. 4-9 show the distribution of the circumferential stress and the equivalent plastic strain a cylinder for several values of H and n. It is assumed that the cylinder is fully-plastic. The plots are limited to the circumferential stress because the it is mostly affected by the parameters. It is seen that with increase in H the stress increases and the plastic strain decreases. In same time, the increase of power parameter n has inverse effect. It is interesting to note that for small values of n and large values of H the sufficiently large region in a vicinity of outer surface of the cylinder almost has no plastic strains (Figs. 4, 5) even in the fully-plastic stress state.

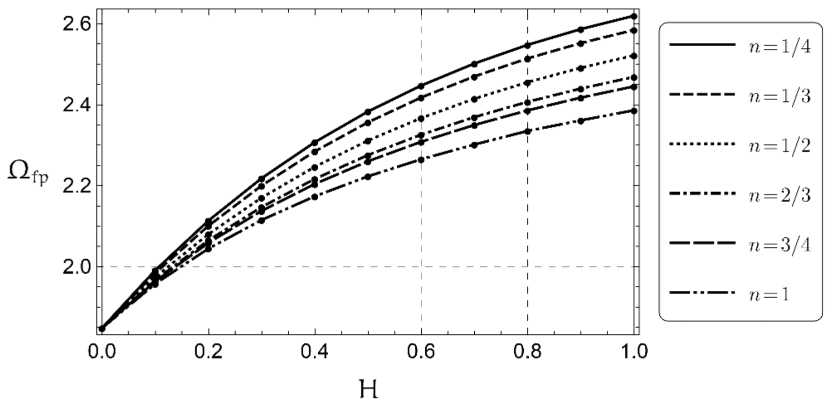
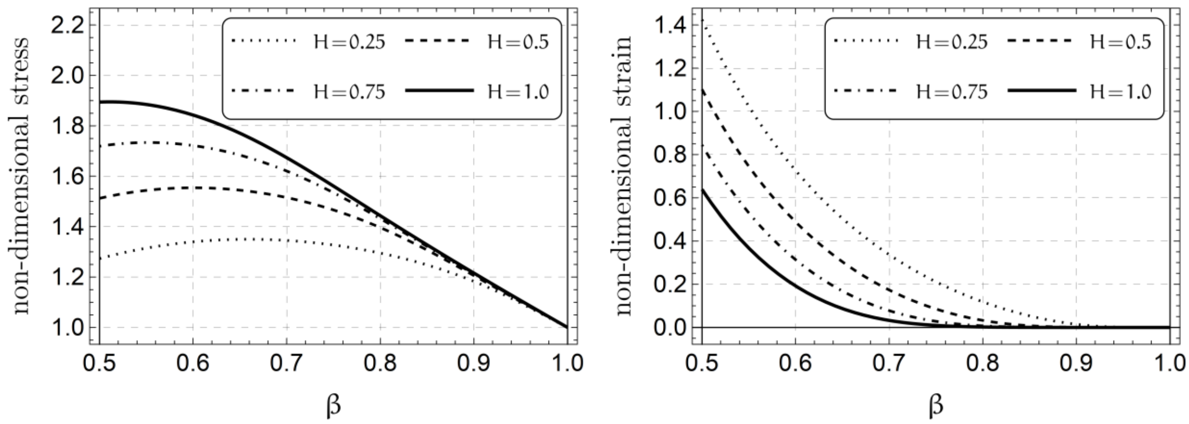
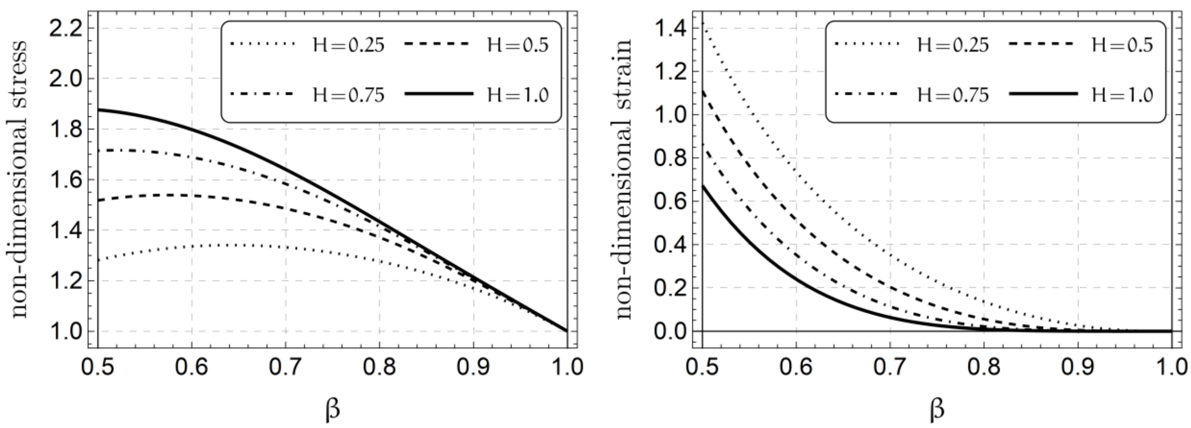


Fig. 3. The dependency of $\Omega_{fp}(H)$ for several values of n



(a) (b)

Fig. 4. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=1/4$



(a) (b)

Fig. 5. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=1/3$

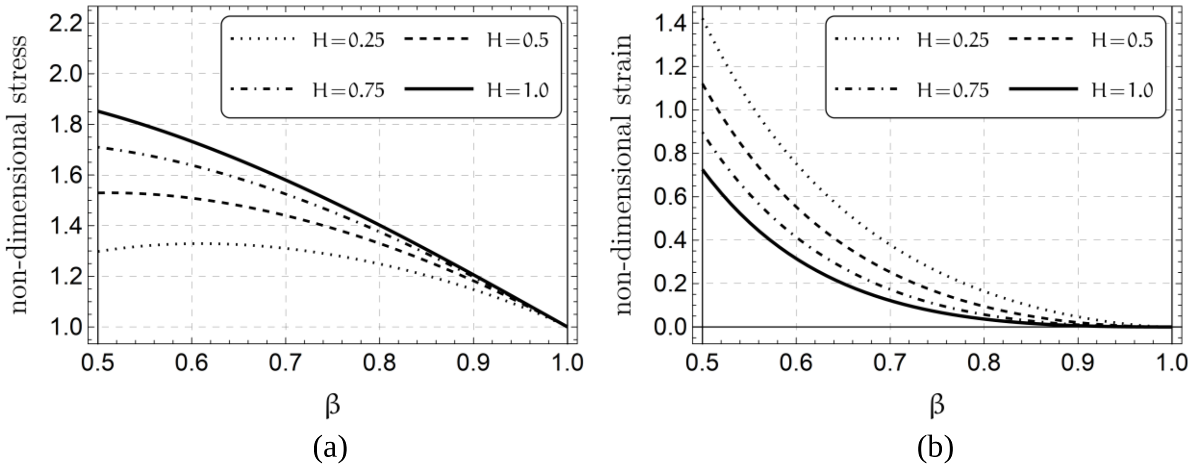


Fig. 6. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=1/2$

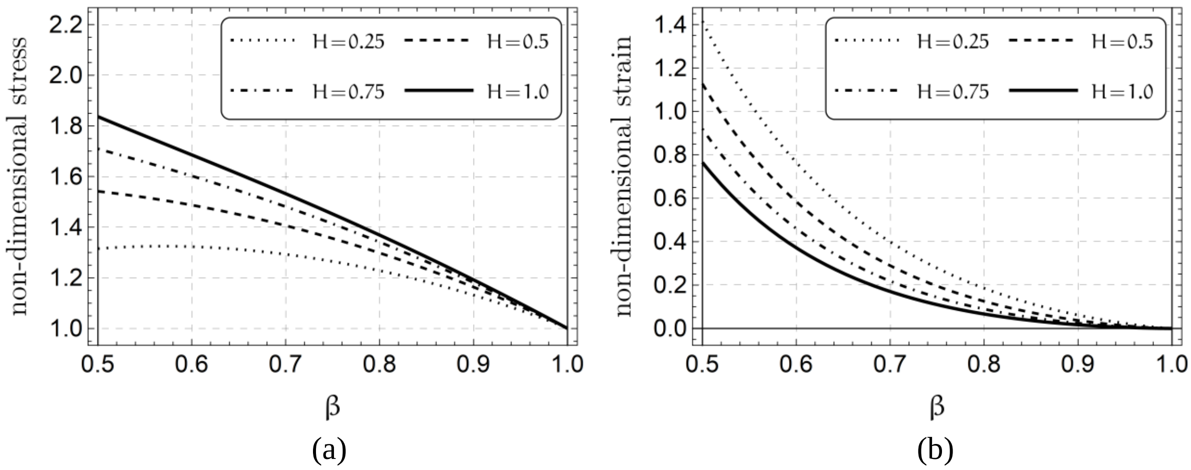


Fig. 7. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=2/3$

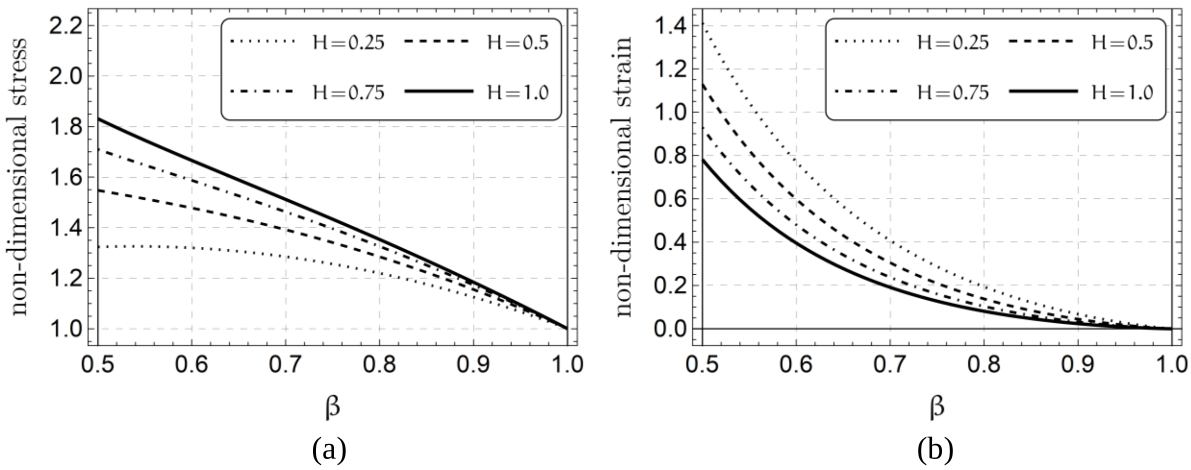
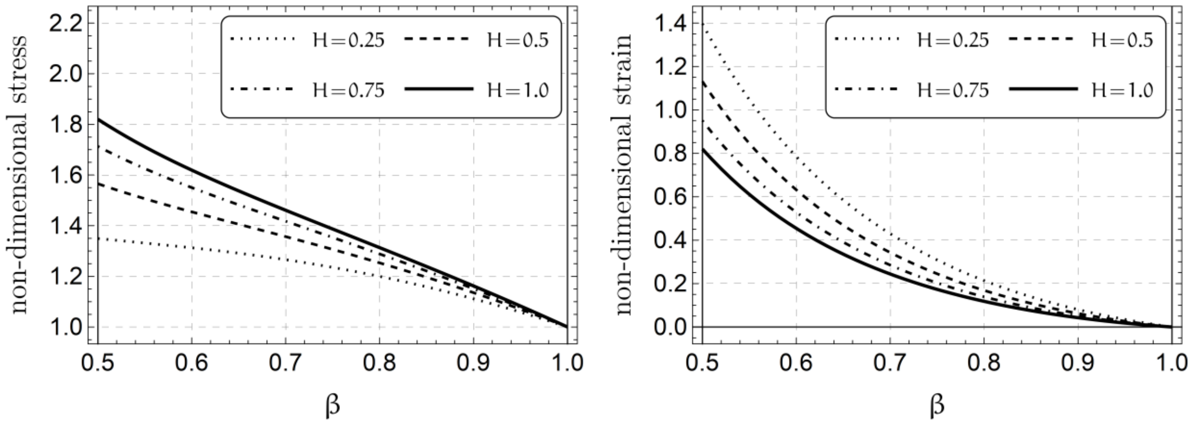


Fig. 8. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=3/4$



(a) (b)

Fig. 9. The distribution of (a) the circumferential stress and (b) the equivalent plastic strain for the several values of H and $n=1$

Finally, the rotational autofrettage of a hollow cylinder with fixed ends is investigated. A cylinder with $\delta=0.5$ made of 2024-T3 Aluminum [16] is considered. Mechanical parameters have following values $\nu=0.3, \sigma_0=200\,MPa, K/\sigma_0=2.8, E/\sigma_0=360, n=0.25$. Hence, non-dimensionless hardening parameter $H=0.6428$. Plastic flow starts at $\Omega_p=1.12$ and the cylinder becomes fully-plastic at $\Omega_{fp}=2.472$. Fig. 10 shows the distribution of residual circumferential stress after its previous rotation for several values of maximum angular velocity $\Omega_{\max}$, namely $\Omega_{\max}=1.5, 1.8, 2.1, \Omega_{fp}$. One can see that the rotational autofrettage process forms the compressive residual stresses in a vicinity of inner boundary of a cylinder. The magnitude of residual stresses increases with increase in maximum angular velocity. In the same time, if $\Omega_{\max}$ is close to plastic limit angular velocity Ω_{fp} then the tensile residual stresses form in an outer boundary of a cylinder.

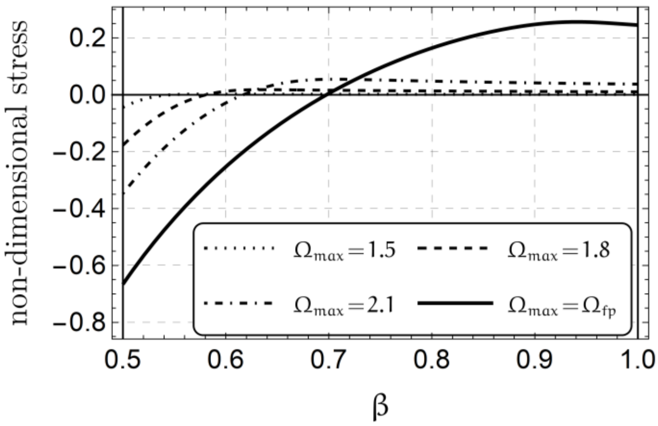


Fig. 10. The distribution of the residual circumferential stress for the several values of $\Omega_{\max}$

Conclusions

The paper considered an elastoplastic problem in a rotating hollow cylinder with fixed ends under the Tresca yield condition and the power law of hardening. For a number of special cases of the hardening law, analytical solutions are obtained. The dependence of the plastic limit angular velocity of the cylinder on the parameters of hardening of is established. Despite the fact that for many materials the hardening parameter n has values close to those considered in the present article, the obtained results are not universal and cannot describe an arbitrary material. For further research, it is of interest to obtain an approximate analytical solution for an arbitrary value of n.

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