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THERMAL STRESSES IN A CIRCULAR PLATE BY A MOVING HEAT SOURCE

K.C. Deshmukh[1*], M.V. Khandait[2], Rajneesh Kumar[3***]**

[1]Department of Mathematics, RTM Nagpur University, Maharashtra, India

[2]Department of Mathematics, Kamala Nehru Mahavidyalaya, Nagpur, Maharashtra, India

[3]Department of Mathematics, Kurukshetra University, Kurukshetra, Harayana, India

*e-mail: kcdeshmukh2000@rediffmail.com

**e-mail: madhavikhandait@rediffmail.com

***e-mail: rajneesh_kuk@rediffmail.com

Abstract. In the present paper, the problem of dynamic thermal stresses in a circular plate by a moving heat source subject to certain boundary conditions is studied. Solutions and numerical results are obtained for the temperature, displacement and stress components. Numerical results are given and illustrated graphically.

1. Introduction

Alzaharnah [1] studied the thermal stresses in thick walled cylinders subjected to a periodic moving heat source. Phythian [2] studied cylindrical heat flow with arbitrary heating rate in which solution of Chen [3] extended to the problem of purely radial heat flow through a hollow cylinder under an arbitrary time dependent heat flux at the outer surface and the zero heat flux at the internal boundary. Kidawa-Kukla [4] has determined temperature distribution in a circular plate heated by a moving heat source. The temperature of plate changes because the heat source moves along circular trajectory on the plate surface. Kulkarni et al. [5] have been succeeded in determining the quasi-static thermal stresses due to internal heat generation in a thin hollow circular disk.

In the present problem, we extend the work of heat conduction studied by Kidawa-Kukla [4] and discuss the thermoelastic behavior of a thin circular plate subjected to the activity of a heat source which changes its place on the plate surface with time. The heat source moves along a circular trajectory round the centre of the plate with constant angular velocity. The analytical solution is obtained by following Ozisik [6] with the help of integral transform and the results are illustrated graphically. The mathematical model is prepared for an aluminum material and illustrated numerically.

2. Problem formulation

Following Kidawa-Kukla [4], consider a circular thin plate of thickness h and radius $r = b$. This plate is subjected to the activity of a moving heat source. The heat source moves on the plate surface along a circular trajectory at radius round centre of the plate with constant angular velocity. The temperature distribution $T(r,\phi,z,t)$ of the plate is described by the differential equation of heat conduction as in Beck et al. [7],

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g(r,\phi,z,t)=\frac{1}{a}\frac{\partial T}{\partial t}, \quad (1)$$

where $T(r,\phi,z,t)$ - temperature, k - thermal conductivity, a - thermal diffusivity, $g(r,\phi,z,t)$ denotes a volumetric energy generation.

In this study, it is assumed that the thermal energy is provided by the moving heat source (which moves along a circular trajectory on the plate surface). The function $g(r,\phi,z,t)$ occurring in equation (1) has the form

$$g(r,\phi,z,t) = \theta\delta(r-r_0)\delta(\phi-\varphi(t))\delta(z-z_0), \tag{2}$$

where θ characterizes the stream of the heat, $\delta(\,)$ is the Dirac delta function, r_0 is the radius of the circular trajectory along which the heat source moves, $\varphi(t)$ is the function describing the movement of the heat source

$$\varphi(t) = \omega t \;, \tag{3}$$

where ω is angular velocity of the moving heat source.

The differential equation (1) is completed by the following initial and boundary conditions:

$$T(r,\phi,z,t){=}0 \;at\; t=0, \tag{4}$$

$$K\frac{\partial T}{\partial r}{=}\alpha_0[T_0-T(r,\phi,z,t)] \;at\, r=b, \tag{5}$$

$$K\frac{\partial T}{\partial z}{=}\alpha_0[T_0-T(r,\phi,z,t)] \;at\; z=h, \tag{6}$$

$$K\frac{\partial T}{\partial z}{=}\alpha_0[T_0-T(r,\phi,z,t)] \;at\; z=0, \tag{7}$$

where α_0, K are the heat transfer coefficients, T_0 is the known temperature of the surrounding medium.

Thermal stresses. Following Kulkarni et al. [5], we assume that for small thickness h the plate is in a plane state of stress. In fact "the smaller the thickness of the plate compared to its diameter, the nearer to a plane state of stress is the actual state". Then the displacement equations of thermoelasticity have the form

$$U_{i,kk}+\left(\frac{1+v}{1-v}\right)e,_i = 2\left(\frac{1+v}{1-v}\right)\cdot a_t\cdot T,_i\,;\; e=U_{k,k};\quad k,i=1,2\,,$$

where U_i- displacement component, $e-$ dilatation, $T-$ temperature, and v and a_t are respectively, the Poisson's ratio and the linear coefficient of thermal expansion of the plate material.

Introducing $U_i=\psi_{,i}$, $i=1,2,$ we have $\nabla_1{}^2\psi=(1+v)a_tT$,

where $\nabla_1{}^2 = \dfrac{\partial^2}{\partial r_1{}^2}+\dfrac{\partial^2}{\partial r_2{}^2}$,

$$\sigma_{ij}=2\mu(\psi_{,ij}-\delta_{ij}\psi_{,kk}),\quad i,j,k=1,2,$$

where μ is the Lamé constant and δ_{ij} is the Kronecker symbol.

In the axially-symmetric case $\psi=\psi(r,z,t)$, $T=T(r,z,t)$.

The differential equation governing the displacement potential function $\psi(r,z,t)$ is given as

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}=(1+\nu)\cdot a_t T \tag{8}$$

with $$\psi=\frac{\partial\psi}{\partial r}=0 \ at \ r=b, \text{ for all time } t, \tag{9}$$

where ν is the Poisson's ratio and a_t is the coefficient of linear thermal expansion.
The stresses σ_{rr} and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr}=\frac{-2\mu}{r}\frac{\partial\psi}{\partial r}, \tag{10}$$

$$\sigma_{\theta\theta}=-2\mu\frac{\partial^2\psi}{\partial r^2}. \tag{11}$$

The surface of the circular plate at $r=b$ is assumed to be traction free. The boundary conditions can be taken as

$$\sigma_{rr}=0 \quad \text{at } r=b \tag{12}$$

Also, in the plane state of stress within the circular plate

$$\sigma_{rz}=\sigma_{zz}=\sigma_{\theta z}=0. \tag{13}$$

Initially,

$$T=\psi=\sigma_{rr}=\sigma_{\theta\theta}=0 \ at \ t=0 \tag{14}$$

The equations (1) to (14) constitute the mathematical formulation of the thermoelastic problem.

3. Analysis

Transient heat conduction problem. To obtain the expression for the temperature function $T(r,\phi,z,t)$, we develop the Fourier transforms in first step and the Hankle transform in second step defined in [5] respectively as,

First step:

$$\bar{T}(r,\phi,\eta_p,t)=\int_{z'=0}^{h}K(\eta_p,z')T(r,\phi,z',t)dz',\ T(r,\phi,z,t)=\sum_{p=1}^{\infty}K(\eta_p,z)\bar{T}(r,\phi,\eta_p,t),$$

where kernel of the transform is

$$K(\eta_p,z)=\frac{\sqrt{2}\left(\eta_p\cos\eta_p z+\frac{\alpha_0}{k}\sin\eta_p z\right)}{\left[\left(\eta_p^2+\left(\frac{\alpha_0}{k}\right)^2\right)\left(h+\frac{\alpha_0}{k}\left\{\frac{1}{\eta_p^2-\left(\frac{\alpha_0}{k}\right)^2}\right\}\right)+\frac{\alpha_0}{k}\right]^{1/2}}$$

where η_p's are roots of transcendental equation

$$\tan \eta_p h = \frac{2\left(\frac{\alpha_0}{k}\right)\eta_p}{\eta_p^2 - \left(\frac{\alpha_0}{k}\right)^2};$$

$$\overline{\overline{T}}(r,\nu',\eta_p,t) = \int_{\phi'=0}^{2\pi} \cos\nu'(\phi-\phi')\overline{T}(r,\phi',\eta_p,t)d\phi',\ \overline{T}(r,\phi,\eta_p,t) = \frac{1}{\pi}\sum_{\nu'=0}^{\infty}\overline{\overline{T}}(r,\nu',\eta_p,t).$$

Second step:

$$\overline{\overline{\overline{T}}}(\beta_m,\nu',\eta_p,t) = \int_{r'=0}^{b} K_0(\beta_m,r')\overline{\overline{T}}(r',\nu',\eta_p,t)dr',\ \overline{\overline{T}}(r,\nu',\eta_p,t) = \sum_{m=1}^{\infty} K_0(\beta_m,r)\overline{\overline{\overline{T}}}(\beta_m,\nu',\eta_p,t),$$

where the kernel of the transform is

$$K_0(\beta_m,r) = \frac{\sqrt{2}}{b}\frac{1}{\left[1+\left(\frac{\alpha_0}{k\beta_m}\right)^2\right]^{1/2}}\frac{J_0(\beta_m r)}{J_0(\beta_m b)}$$

and β_1, β_2, β_3 ... ... are the roots of the transcendental equation

$$J_0'(\beta_m b) = \frac{\alpha_0}{k\beta_m}J_0(\beta_m b) \text{ or } -k\beta_m J_0'(\beta_m b) + \alpha_0 J_0(\beta_m b) = 0.$$

On applying the above transforms and their inverses to equations (1) to (7) the temperature distribution function is obtained as

$$T(r,\phi,z,t) = \frac{a}{k\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} K_0(\beta_m,r)\, K(\eta_p,z)\int_{t=0}^{t} e^{-a(\beta_m^2+\eta_p^2)(t-\tau)}$$

$$\Big\{[\theta r_0 K_0(\beta_m,r_0)\cos\nu'(\varphi-\phi(\tau))K(\eta_p,z_0)]$$

$$+\alpha_0 T_0\Bigg[-\frac{bk\, dK_0(\beta_m,b)}{dr}\int_{z'=0}^{h}\int_{\varphi'=0}^{2\pi} K(\eta_p,z')\cos\nu'(\varphi-\varphi')dz'd\varphi'$$

$$(\{K(\eta_p,0)+K(\eta_p,h)\}\int_{r'=0}^{b}\int_{\varphi'=0}^{2\pi} r'K_0(\beta_m,r'))\cos\nu'(\varphi-\varphi')dr'd\varphi')\Bigg]d\tau\Big\}, \quad (15)$$

Using equation (15) in (8), one obtains the displacement function as

$$\psi(r,\ \phi,z,\ t) = -\frac{\sqrt{2}a(1+\nu)a_t}{k\pi\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{\left[\frac{1}{\beta_m}J_0(\beta_m r) - \frac{1}{\beta_m}J_0(\beta_m b) - (r-b)\frac{\alpha_0}{k\beta_m}J_0(\beta_m b)\right]}{\beta_m^2\left[1+\left(\frac{\alpha_0}{k\beta_m}\right)^2\right]^{1/2} J_0(\beta_m b)}$$

$$K(\eta_p,z)\int_{\tau=0}^{t} e^{-a(\beta_m^2+\eta_p^2)(t-\tau)} \Big\{[\theta\, r_0 K_0(\beta_m,r_0)\cos v'(\phi-\varphi(\tau))\, K(\eta_p,z_0)]$$

$$+\alpha_0 T_0\Big\lfloor -\frac{bk\, dK_0(\beta_m,b)}{dr}\int_{z'=0}^{h}\int_{\phi'=0}^{2\pi} K(\eta_p,z')\cos v'(\phi-\phi')dz'\,d\phi'$$

$$(\{K(\eta_p,0)+K(\eta_p,h)\}\int_{r'=0}^{b}\int_{\varphi'=0}^{2\pi} r'K_0(\beta_m,r'))\cos v'(\varphi-\varphi')dr'd\varphi')\Big\rfloor d\tau\Big\}, \quad (16)$$

Using (16) in equations (10) and (11), one obtains the thermal stresses respectively as

$$\sigma_{rr}=\frac{2\sqrt{2}\mu a(1+\nu)\,a_t}{rk\pi b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{K(\eta_p,z)\Big\lfloor J_1(\beta_m r)+\dfrac{\alpha_0}{k\beta_m}J_0(\beta_m b)\Big\rfloor}{\beta_m^2\left[1+\left(\dfrac{\alpha_0}{k\beta_m}\right)^2\right]^{1/2} J_0(\beta_m b)}$$

$$\int_{\tau=0}^{t} e^{-a(\beta_m^2+\eta_p^2)(t-\tau)} \Big\{[\theta\, r_0 K_0(\beta_m,r_0)\cos v'(\phi-\varphi(\tau))\, K(\eta_p,z_0)]$$

$$+\alpha_0 T_0\Big\lfloor -\frac{bk\, dK_0(\beta_m,b)}{dr}\int_{z'=0}^{h}\int_{\phi'=0}^{2\pi} K(\eta_p,z')\cos v'(\phi-\phi')dz'\,d\phi'$$

$$(\{K(\eta_p,0)+K(\eta_p,h)\}\int_{r'=0}^{b}\int_{\phi'=0}^{2\pi} r'\,K_0(\beta_m,r'))\cos v'(\phi-\phi')\,dr'd\phi')\Big\rfloor d\tau\Big\}, \quad (17)$$

$$\sigma_{\theta\theta}=\frac{2\sqrt{2}\mu a(1+\nu)\,a_t}{k\pi b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\frac{K(\eta_p,z)\,[J_1'(\beta_m r)]}{\beta_m\left[1+\left(\dfrac{\alpha_0}{k\beta_m}\right)^2\right]^{1/2} J_0(\beta_m b)}$$

$$\int_{\tau=0}^{t} e^{-a(\beta_m^2+\eta_p^2)(t-\tau)} \Big\{[\theta\, r_0 K_0(\beta_m,r_0)\cos v'(\phi-\varphi(\tau))\, K(\eta_p,z_0)]$$

$$+\alpha_0 T_0\Big\lfloor -\frac{bk\, dK_0(\beta_m,b)}{dr}\int_{z'=0}^{h}\int_{\phi'=0}^{2\pi} K(\eta_p,z')\cos v'(\phi-\phi')dz'\,d\phi'$$

$$(\{K(\eta_p,0)+K(\eta_p,h)\}\int_{r'=0}^{b}\int_{\phi'=0}^{2\pi} r'\,K_0(\beta_m,r'))\cos v'(\phi-\phi')\,dr'd\phi')\Big\rfloor d\tau\Big\}, \quad (18)$$

Numerical results and discussions. Calculation results for the response of temperature, displacement and stresses are carried out along the radial and axial direction for the fix time and the constant surrounding temperature. The aluminum material is chosen for numerical calculations. In the calculation process, the material constants necessary to be known are:

Radius of the circular plate $b=1\ m$;

Thickness of a circular plate $h=0.2\ m$;

Angular velocity $\omega = 2\pi, \ t \to \tau = 4$ sec;
Heat transfer coefficient α_0 = 10;
Temperature of surrounding medium T_0 = 20 ^{0}C;
r_0 = 0.5; z_0 = 0.1;
Thermal conductivity $k = 0.86\ (m^2 s^{-1})$;
Thermal diffusivity $a = 84.18 \times 10^{-6} (m^2 s^{-1})$;
Density $\rho = 2707\ \ kg/m^3$;
Specific heat $c_p = 896\ \ J/kgK$;
Poisson ratio $\nu = 0.35$;
Coefficient of linear thermal expansion $a_t = 22.2 \times 10^{-6}\ \frac{1}{K}$;
Lamé constant $\mu = 26.67$.

Roots of the transcendental equation. The $\beta_1 = 3.005$, $\beta_2 = 6.772$, $\beta_3 = 10.06$, $\beta_4 = 13.258$, $\beta_5 = 16.428$ are positive roots of transcendental equation

$$-k\beta_m J_0'(\beta_m b) + \alpha_0 J_0(\beta_m b) = 0;$$

$\eta_1 = 20.805$, $\eta_2 = 34.654$, $\eta_3 = 49.435$, $\eta_4 = 64.613$, $\eta_5 = 79.985$ are the positive roots of the transcendental equation

$$\tan \eta_p h = \frac{2\left(\frac{\alpha_0}{k}\right)\eta_p}{\eta_p^2 - \left(\frac{\alpha_0}{k}\right)^2}.$$

Set for convenience: $X = \frac{2a}{k\pi}$, $Y = \frac{\sqrt{2}a(1+\upsilon)a_t}{k\pi b}$ and $Z = \frac{2\sqrt{2}a\mu(1+\upsilon)a_t}{k\pi b}$.

The numerical calculation has been carried out with the help of computational mathematical software Mathcad-2000 and graphs are plotted with the help of Excel (MS Office-2007).

Figure 1: Due to moving heat source along a circular trajectory round center of the plate with constant angular velocity ω, heat absorbs towards the center and fluctuations take place between centre to outer circular edge of the plate.

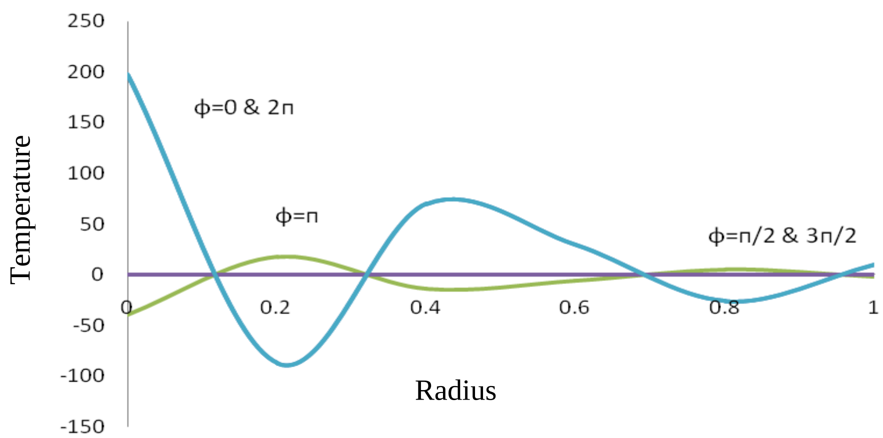

Fig. 1. Temperature function.

Figure 2: Since the circular plate is clamped, displacement at outer circular edge $r = 1$ is zero. The maximum displacement takes place at center $r = 0$

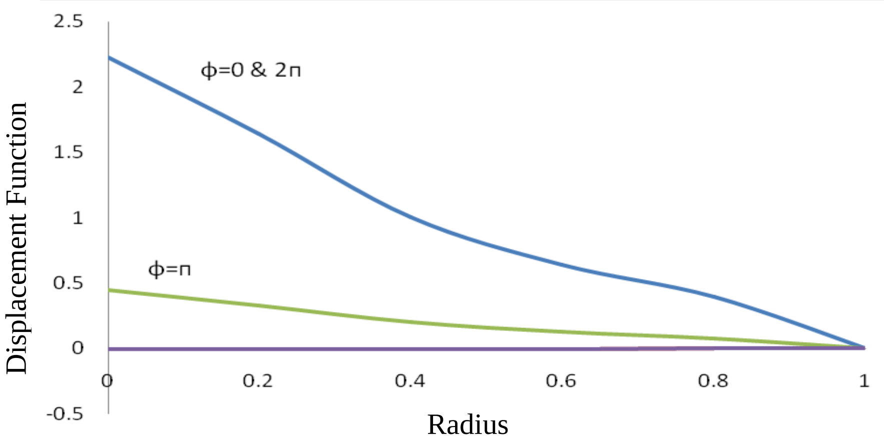

Fig. 2. Displacement function.

Figure 3: The radial stress function is zero at outer circular edge $r = 1$. It develops compressive stresses at $\phi = 0$, $\phi = 2\pi$ where as tensile stresses appear at $\phi = \pi$.

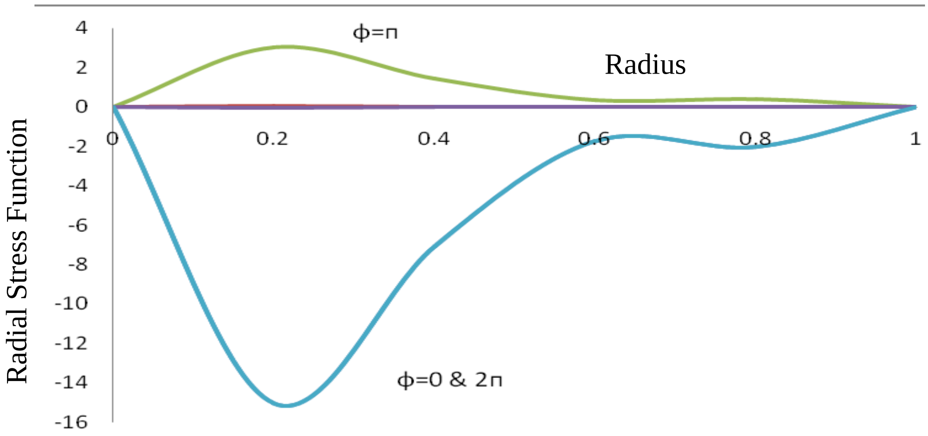

Fig. 3. Radial stress function.

Figure 4: The angular stress function shows variation between $0 \le r \le 1$. It develops compressive stresses at $\phi = \pi$ where as tensile stresses at $\phi = 0$, $\phi = 2\pi$.

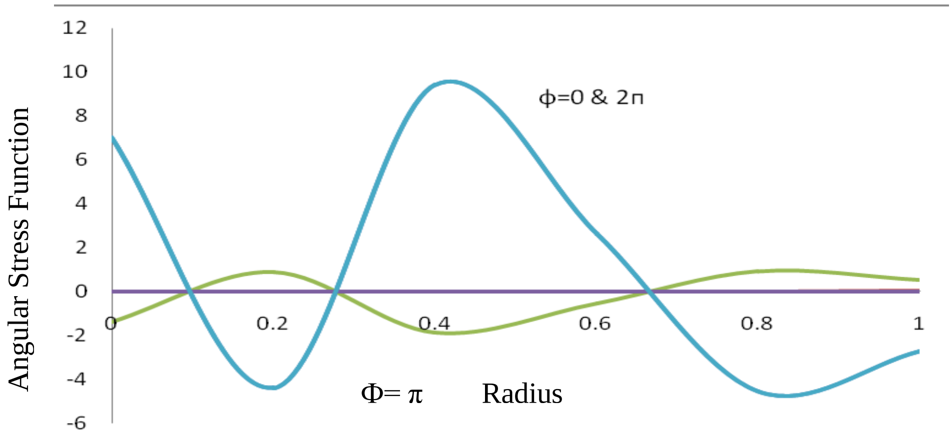

Fig. 4. Angular stress function.

4. Conclusion

In this paper, we consider the heat conduction problem studied by Kidawa-Kukla [3]. Here, we modified the solution of the problem studied by Kidawa-Kukla [3] in an analytical form with the help of integral transform technique and discussed the thermal stresses.

In this paper, an analytical model to describe the three dimensional temperature field for a circular plate with a heat source which moves over its surface is established with the help of integral transform technique. The thermal stresses $\sigma_{rr}, \sigma_{\theta\theta}$ have been studied due to the heat source $g(r,\phi,z,t)$ which is an instantaneous point heat source of the stream of the heat of strength θ which is situated at the point (r_0, $\varphi(t)$, z_0) inside the plate, releases its heat spontaneously at time $t = \tau$ whereas $\varphi(t)$, is the function describing the movement of the heat source $\varphi(t) = \omega t$ which causes cycling heating of various plate areas. We discussed the radial stress function σ_{rr} and angular stress function $\sigma_{\theta\theta}$ at the outer boundary of the plate which built in-edge.

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References

[1] I. Alzaharnah // *AIP Conference Proceedings* **1315** (2011) 739.
[2] J.E. Phythian // *AIAA Journal* **1(4)** (1963) 925.
[3] S-Y. Chen // *Journal of the Aerospace Sciences* **28(4)** (1961) 336.
[4] J. Kidawa-Kukla // *Scientific Research of the Institute of Mathematics and Computer Science* **1(7)** (2008) 71.
[5] V.S. Kulkarni, K.C. Deshmukh, S.D. Warbhe // *Journal of Thermal Stresses* **31(8)** (2008) 698.
[6] M.N. Özisik, *Boundary Value Problems of Heat Conduction* (International Textbook Company, Scranton, Pennsylvania, 1968).
[7] J.V. Beck, K.D. Cole, A. Haji-Sheikh, B. Litkouhi, *Heat Conduction using Green's Functions (*Hemisphere Publishing Corporation, Philadelphia, 1992).