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INSTABILITY OF SOLUTION OF THE DYNAMIC SLIDING FRICTIONAL CONTACT PROBLEM OF COUPLED THERMOELASTICITY

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Abstract. We consider one-dimensional thermoelastic contact problem on vertical indentation of a rigid thermally insulated half-plane moving horizontally with constant speed over an elastic coating (strip), while bottom side of the latter is bonded to a rigid foundation. Thermal flux generated by friction is directed to the strip. Temperature, displacement and stress distributions along the depth of the coating are derived in the form of infinite series over eigenfunctions. It is shown that the thermoelastodynamic instability of the obtained solutions is present in all time range and at any velocities of the half-plane sliding over the surface of the coating.

1. Problem statement

For studying dynamics of sliding frictional thermoelastic contact we consider following plane strain problem. A rigid half-plane $B\ (h \le x < \infty)$ is sliding with constant velocity V over the surface $(x = h)$ of an elastic coating represented by infinite strip of width $h\ (0 \le x \le h)$ with its lower boundary $(x = 0)$ perfectly bonded to a rigid foundation represented by a half-plane $A\ (-\infty < x \le 0)$ [1]. During sliding, Coulomb friction takes place, but wear of the coating is neglected. The half-plane B deforms elastic coating, moving along the x axis according to law $u(h,t) = -\Delta(t)$, $t > 0$, where $u(h,t)$ is vertical displacement of the surface of the coating at $x = h$. At initial time moment, temperature of the coating is zero: $T(x,0) = T_0 = 0$ $(0 \le x \le h)$, where $T(x,t)$ is temperature distribution by width of the coating. The half-plane B is thermally insulated, and the thermal flux $K\dfrac{\partial T(h,t)}{\partial x} = -fV\sigma(h,t)$ generated by friction flows into the elastic coating. As the lower boundary of the coating lies on the undeformable foundation $(x = 0)$ represented by the half-plane A, the elastic displacements on that boundary are zero: $u(0,t) = 0$. Also, the lower boundary is maintained at temperature $T(0,t) = 0$. Before initial time moment, coating was resting, so initial conditions for u and $\dfrac{\partial u}{\partial t}$ are zero: $u(x,0) = \dfrac{\partial u(x,0)}{\partial t} = 0$. Here $\sigma(x,t)$ are compression stresses in the coating, K is thermal conductivity. $\Delta(t)$ is taken in form

$$\Delta(t)=\Delta_0\begin{cases}0, & -\infty<t<0\\ -1+e^{\varepsilon t}, & 0\le t\le t_\varepsilon\\ 1, & t_\varepsilon\le t<\infty\end{cases},\qquad \dot{\Delta}(t)=\Delta_0\begin{cases}0, & -\infty<t<0\\ \varepsilon e^{\varepsilon t}, & 0\le t\le t_\varepsilon\\ 0, & t_\varepsilon<t<\infty\end{cases}\qquad \varepsilon>0\,, \tag{1}$$

where $\varepsilon>0$, $t_\varepsilon=\varepsilon^{-1}\ln 2$, Δ_0 is depth of the half-plane *B* indentation into the elastic coating ($0<\Delta_0<h$).

Variation of $u(x,t)$ and $\sigma(x,t)$ in the coating is governed by the equation of theory of elasticity [2]

$$\frac{\partial\sigma}{\partial x}-\rho\frac{\partial^2 u}{\partial t^2}=0\,,\ 0<x<h\,,\ t>0\,, \tag{2}$$

and temperature $T(x,t)$ is governed by heat conduction equation [3]

$$\frac{\partial^2 T}{\partial x^2}-\frac{1}{\kappa}\frac{\partial T}{\partial t}=0\,,\ 0<x<h\,,\ t>0\,, \tag{3}$$

where ρ, κ are density and thermal diffusivity of the material of the coating, respectively. Connection between $\sigma(x,t)\,,u(x,t)$ and $T(x,t)$ is established by Hooke's law, taken in the following form [4]:

$$\sigma(x,t)=\frac{2\mu(1-\nu)}{1-2\nu}\frac{\partial u}{\partial x}-\frac{2\mu(1+\nu)}{1-2\nu}\alpha T(x,t)\,, \tag{4}$$

where μ, ν, α are shear modulus, Poisson's ratio and coefficient of expansion for the material of the coating, respectively.

To construct solution of the problem, we substitute expression (4) into the differential equation (2), for convenience. Thus, we obtain differential equation of the movement of the elastic medium with respect to $u(x,t)$ and $T(x,t)$

$$\frac{\partial^2 u}{\partial x^2}-\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}=\frac{1+\nu}{1-\nu}\alpha\frac{\partial T}{\partial x}\,,\ 0\le x\le h\,,\ t>0\,,\ a=\sqrt{\frac{2\mu(1-\nu)}{\rho(1-2\nu)}}\,, \tag{5}$$

where a is the elastic wave propagation speed in the coating.

2. Solution of the problem

The transient initial boundary contact problem stated here can be considered as coupled thermoelasticity problem [5], as $\sigma(x,t)$ and $T(x,t)$ are connected not only through (4) and (5), but also in the boundary condition for the upper boundary of the coating..

Using the Laplace integral transform, the solution for this problem $T(x,t)$, $u(x,t)$, $\sigma(x,t)$ can be expressed with contour integrals:

$$T(x,t)=\frac{1-\nu}{1+\nu}\frac{\gamma\hat{V}}{\alpha h}\frac{1}{2\pi i}\int_\Gamma D(z)N_T(x,z)R^{-1}(z)\exp z\tilde{t}dz\,,\ \tilde{t}=\frac{t}{t_\kappa}\,,\ t_\kappa=\frac{h^2}{\kappa}\,, \tag{6}$$

$$u(x,t)=-\frac{1}{2\pi i}\int_\Gamma D(z)N_u(x,z)R^{-1}(z)\exp\left(z\tilde{t}\right)dz\,, \tag{7}$$

$$\sigma(x,t)=-\frac{2\mu(1-\nu)}{(1-2\nu)h}\left[\sum_{n=1}^{\infty}t_a\left(\dot{\Delta}\left(s_n^-(x,t)\right)+\dot{\Delta}\left(s_n^+(x,t)\right)\right)+\frac{1}{2\pi i}\int_\Gamma D(z)N_\sigma(x,z)R_\sigma^{-1}(z)\exp\left(z\tilde{t}\right)dz\right], \tag{8}$$

$$\Delta_0^{-1}D(z)=z^{-1}\left(2\exp\left(-zt_{\varepsilon\kappa}\right)-1\right)+\left(z-\varepsilon t_{\kappa}\right)^{-1}\left(1-\exp\left(-\left(z-\varepsilon t_{\kappa}\right)t_{\varepsilon\kappa}\right)\right),$$

$$N_T(x,z)=\sqrt{z}\left(1-\gamma^2 z\right)\operatorname{ch}\gamma z\operatorname{sh}\left(\sqrt{z}xh^{-1}\right),$$

$$R(z)=\left(1-\gamma^2 z\right)\operatorname{ch}\sqrt{z}\operatorname{sh}\gamma z-\gamma\hat{V}\left(\operatorname{ch}\sqrt{z}\operatorname{ch}\gamma z-\gamma\sqrt{z}\operatorname{sh}\sqrt{z}\operatorname{sh}\gamma z-1\right),$$

$$N_u(x,z)=\left(1-\gamma^2 z\right)\operatorname{ch}\sqrt{z}\operatorname{sh}\left(\gamma zxh^{-1}\right)-$$
$$-\gamma\hat{V}\left(\operatorname{ch}\left(\sqrt{z}xh^{-1}\right)\operatorname{ch}\gamma z-\gamma\sqrt{z}\operatorname{sh}\sqrt{z}\operatorname{sh}\left(\gamma zxh^{-1}\right)-\operatorname{ch}\left(\gamma z(h-x)h^{-1}\right)\right),$$

$$N_\sigma(x,z)=\gamma z\left\lfloor N_\sigma^0(x,z)\operatorname{sh}\gamma z-\operatorname{ch}\left(\gamma zxh^{-1}\right)R\left(z\right)\right\rfloor,\ R_\sigma(z)=\operatorname{sh}\gamma z\ R(z),$$

$$N_\sigma^0(x,z)=\left(1-\gamma^2 z\right)\operatorname{ch}\sqrt{z}\operatorname{ch}\left(\gamma zxh^{-1}\right)+$$
$$+\gamma\hat{V}\left(\gamma\sqrt{z}\left(\operatorname{sh}\sqrt{z}\operatorname{ch}\left(\gamma zxh^{-1}\right)-\operatorname{sh}\left(\sqrt{z}xh^{-1}\right)\operatorname{ch}\gamma z\right)-\operatorname{sh}\left(\gamma z(h-x)h^{-1}\right)\right),$$

where $\Gamma=\left\{z:-i\infty+dt_\kappa,i\infty+dt_\kappa\right\}$ is the integration contour in the complex plane representing narrow line parallel to the imaginary axis and standing aside of it by dt_κ, where *d* is chosen in the way so all of poles of integrands in (6)-(8) would be left to dt_κ.

Dimensionless parameters γ and $\hat{V}$ are defined as

$$\gamma=\frac{\kappa}{ah},\ \hat{V}=\frac{fV\alpha}{K}\cdot\frac{2\mu(1+v)h}{1-2v}.$$

Contour integrals in (6)-(8) exist at the fulfillment of the condition $t>\left(h-x\right)/a$ and due to integrands decay at infinity.

Investigation of integrands properties in (6)-(8) reveals that all of them are meromorphic in the complex plane of integration variable $z=\xi+i\eta$, i.e. they have only poles as their isolated singular points, which are derived by denominators of the integrands: $R(z)$, $R_\sigma(z)$.

3. Zeros of $R(z)$ in the complex plane

To calculate contour integrals in (6)-(8) using complex analysis techniques, we need to know zeros of $R(z)$ and their properties in the complex plane. Zeros of $R(z)$ in (6)-(8) are determined from equation

$$R(z)=\left(1-\gamma^2 z\right)\operatorname{ch}\sqrt{z}\operatorname{sh}\gamma z-\gamma\hat{V}\left(\operatorname{ch}\sqrt{z}\operatorname{ch}\gamma z-\gamma\sqrt{z}\operatorname{sh}\sqrt{z}\operatorname{sh}\gamma z-1\right)=0 \tag{9}$$

which is identical to the corresponding characteristic equation in [1].

We determined [6] that equation (7) has zeros $z=0$ и $z=\gamma^{-2}$, which are not poles of integrand functions, and two infinite sets of zeros, which are poles of integrands: $z_k=z_k(\hat{V})$ k = 0,1,2… and $z_n^\pm=z_n^\pm(\hat{V})$ n = 1,2…. With increasing of $\hat{V}$, $z_n^\pm=z_n^\pm(\hat{V})$ go from the imaginary axis to the complex plane, each drawing an unfinished ellipse, whose primary axis is parallel to the real axis, and $\operatorname{Re}(z_n^\pm(\hat{V}))>0$ at $\hat{V}>0$. Sets $z_k=z_k\left(\hat{V}\right)$ k = 1,2… are placed as pieces of negative part of the real axis, and only $z_0(\hat{V})$ at $\hat{V}=2$ situates in the origin, and with increasing of $\hat{V}$ passes to positive part of the real axis and tends to the infinity.

4. Analysis of the obtained solution

After solving contour integrals (6)-(8) in a complex plane for functions $T(x,t)$, $u(x,t)$, and $\sigma(x,t)$ we obtain formulas in the form of series involving poles of integrands, which can be easily calculated

$$T(x,t)=\frac{1-\nu}{1+\nu}\cdot\frac{\gamma\hat{V}}{\alpha}\cdot\frac{\Delta_0}{h}\sum_{k=1}^{2}S_k(x,\tilde{t})H\left((-1)^k\left(\tilde{t}-t_{\varepsilon\kappa}\right)\right),\ 0\le x\le h,\quad t>0 \tag{10}$$

$$u(x,t)=-\Delta_0\sum_{k=1}^{2}S_k(x,\tilde{t})H\left((-1)^k\left(\tilde{t}-t_{\varepsilon\kappa}\right)\right),\ 0\le x\le h,\quad t>0 \tag{11}$$

$$\sigma(x,t)=-\frac{2\mu(1-\nu)}{1-2\nu}\frac{\Delta_0}{h}\left[\sum_{k=1}^{\infty}t_a\left(\dot{\Delta}\left(\omega_n^-(x,t)\right)+\dot{\Delta}\left(\omega_n^+(x,t)\right)\right)+\right.$$
$$\left.+\sum_{k=1}^{\infty}\left(S_k(x,\tilde{t})+S_k^0(x,\tilde{t})\right)H\left((-1)^k\left(\tilde{t}-t_{\varepsilon\kappa}\right)\right)\right],\quad 0\le x\le h,\quad t>0, \tag{12}$$

where $S_k(x,t)$ $(k=1,2)$ are given by formulas

$$S_1(x,t)=s_1(x,t)+G(x,t),\ S_2(x,t)=s_2(x,t)+G(x,t)-2G(x,t-t_{\varepsilon\kappa}), \tag{13}$$

$$G(x,t)=2\,\mathrm{Re}\sum_{k=1}^{\infty}\theta(z_k^+)K'(z_k^+,x,t)+\sum_{k=0}^{\infty}\theta(z_k)K'(z_k,x,t), \tag{14}$$

$$s_1(x,t)=K(t_\kappa\varepsilon,x,t)-K(0,x,0),\ s_2(x,t)=K(0,x,0),\ K(z,x,t)=\frac{N(x,z)}{R(z)}\exp(zt),$$

$$K'(z,x,t)=\frac{N(x,z)}{R'(z)}\exp(zt),\ K(0,x,0)=\lim_{z\to 0}\frac{N(x,z)}{R(z)},\ \theta(z)=\frac{t_\kappa\varepsilon}{z(z-t_\kappa\varepsilon)},$$

$$S_1^0(x,t)=2\,\mathrm{Re}\sum_{k=1}^{\infty}\theta(z_{0k}^+)K'(z_{0k}^+,x,t),\ S_2^0(x,t)=2S_1^0(x,t-t_{\varepsilon\kappa})-S_1^0(x,t), \tag{15}$$

where $R'(z)$ is derivative by z, $H(t)$ is the Heaviside function, $z_{0k}^+=z_k^+(0)$. In (12), $\dot{\Delta}(t)$ stands for function from (1) divided by Δ_0: $\dot{\Delta}(t)=\varepsilon e^{\varepsilon t}H(t_\varepsilon-t)H(t)$, where $\varepsilon=v_0/\Delta_0$; $N(x,z)$ stands for $N_T, N_u, N_\sigma(x,z)$ from (6)-(8) for (10)-(12), respectively.

Function $G(x,t)$ (14) in formulas (13) for $S_k(x,t)$ $(k=1,2)$ contains two infinite series involving poles of integrand. First of them involves poles $z_n^\pm$ n = 1,2… with positive real part $S_k(x,t)$ and for fixed values of $\tilde{t}$ $(0<\tilde{t}<\infty)$ is convergent, while at unbounded growth of $\tilde{t}$ $(\tilde{t}\to\infty)$ is not convergent, and solution containing this series is unstable at $\hat{V}>0$. Other infinite series involves poles z_k $k=0,1,2,...$ and for fixed values of $\tilde{t}$ converges like a geometric series at any γ and $\hat{V}$, but at $\hat{V}>2$, as $z_0(\hat{V})>0$, sum of the series at $\tilde{t}\to\infty$ grows without limit, and solution containing this series is unstable. Consequently, solutions of the problem considered are thermoelastodynamically unstable at any sliding speed $V>0$.

6. Numerical analysis of the obtained solutions

For detailed study of thermoelastodynamically unstable solution of the problem considered we use formulas (10)-(12). Parameter values is taken as follows: $\mu = 44\times 10^9$ N/m^2,

ν = 0.25, α = 1.04 × 10^{-5} K^{-1}, a = 4 × 10^3 m/s, κ = 8 × 10^{-4} m^2/s, K = 62.8 W×m/K (corresponding to the properties of cast iron), f = 0.15, $\Delta_0 = 0.1h$, $v_0 = 0.01$ m/s, $h = 2\times10^{-3}$ m, V = 1.58 m/s, $t_\varepsilon = 6.93\cdot10^{-3}$s, and in this case dimensionless parameters γ and $\hat{V}$ have following values: γ = 1.0 · 10^{-4} and $\hat{V}$ = 0.25. Figure 1 shows plots of displacements $u(x,t)$ distribution by depth of the coating $(0 \le x \le h)$, calculated using formula (11) at different values of x, numbered from 1 to 5, where 1 corresponds to x = 0, 2 – 0,25h, 3 – 0,5h, 4 – 0,75h, 5 – h (contact interface). We can see in Fig. 1 evolution of amplitude of displacements $u(x,t)$ in the unstable solution, increasing with speed of a geometric series and oscillating on eigenfrequency $\mathrm{Im}(z_1^+)$, becoming physically unrealistic at some time. Note that at x=0,5h it happens slightly earlier than for other x.

Figure 2 demonstrates plots of temperature $T(x,t)$ distribution in the coating for different values of x, numbered from 1 to 5, as in Fig. 1. Most important features of temperature $T(x,t)$ evolution are shown in Fig. 2*b*, 2*c* in a wider time scale: Fig. 2*b* shows temperature $T(h,t)$ (on contact interface) at time $t > 3t_\varepsilon$, demonstrating temperature plot oscillating on the eigenfrequency $\mathrm{Im}(z_1^+)$; Fig. 2*c* shows amplitude of oscillations being increased 50 times than in Fig. 2*b*.

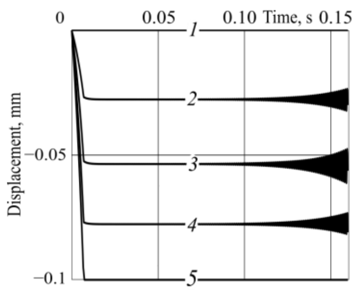

Fig. 1. Displacements $u(x,t)$ distribution by depth of the coating.

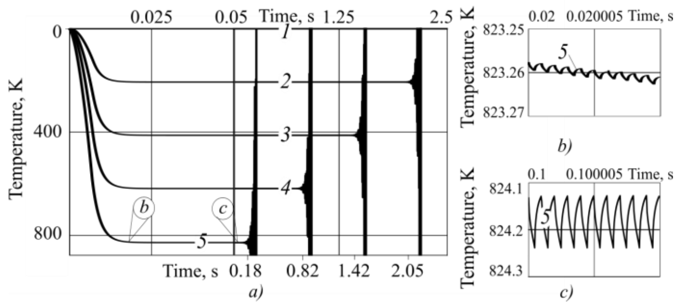

Fig. 2. Temperature $T(x,t)$ distribution by depth of the coating, with approximate time when the solution becomes physically unrealistic marked on the bottom axis.

Further (Fig. 2*a*) we can see exponential growth of amplitude of $T(h,t)$ plot oscillations, reaching tens of K, that corresponds to physically unrealistic behavior for $x = h$ at $t > 0{,}18$ s, for x = 0,75h – at $t > 0{,}82$ s, for x = 0,5h – at $t > 1{,}42$ s, for x = 0,25h – at $t > 2{,}05$ s. Such dependency of physically unrealistic behavior with respect to x appeared to be specific just for temperature, which is determined from parabolic differential equation.

7. Conclusions

Thermoelastodynamic instability of frictional sliding contact during motion of a rigid half-plane over the surface of an elastic coating on a rigid foundation takes place at any speed of its movement. It should be noted that behavior of displacements and stresses in the obtained solution becomes physically unrealistic earlier than for temperature (Figs. 1, 2), which is important for developing of diagnostics techniques for identifying thermoelastodynamic instability of sliding contact.

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