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PECULIARITIES OF THE WAVE FIELD LOCALIZATION IN THE FUNCTIONALLY GRADED LAYER

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Abstract. Within the framework of the linearized theory of elasticity, as exemplified by the problem of the shear harmonic oscillations of the pre-stressed functionally graded layer, the influence of the type of inhomogeneity and of the character of the initial stressed state on the distribution of the displacements with depth is investigated. The initially deformed state is assumed to be homogeneous, the inhomogeneity of the initial stresses is caused by the change of the material properties. The transformation of the displacements with depth for different oscillation frequencies of the layer under the conditions of the equal and arbitrary intensity of the change of properties is demonstrated. The possibility to dampen the amplitude of displacements at the definite frequencies by means of changing the initial actions is established.

1. Introduction

A wide application of composite materials in geomechanics, construction industry and high technology equipment production resulted in the necessity of investigating physical, technological and strength properties of new materials depending on the modes and conditions of their operation. It stimulated the extensive experimental, fundamental and applied research. Efficiency of composite materials and coatings is closely connected with their internal state, i.e. with the effects arising in the vicinity of the materials inhomogeneity. The complexity of the dynamic problems arising from the investigation is in the fact that it is impossible to obtain the analytical solution for the semi-bounded media with the properties that change in space or for the structures coated with the similar materials. In scientific papers it is often assumed that all the properties of material change according to one law with one space variable with the same intensity [1-7]. When modeling the functionally graded material, it is often divided into the layered elements in which the material properties are the linear [6] or the quadratic functions of the thickness; and easily differentiable functions and polynomials [2, 5] are used as functional dependencies. As a rule, the change of the material properties in the models is considered either with respect to one «basic» material or with respect to two materials [4, 7]. The assumption of the equivalent change of all the properties of material allows to obtain the analytical solution that is important when estimating the results of a more complex numerical and numerically- analytical modeling; but to investigate the material properties it is effective only in some special cases. In this paper in the framework of the linearized theory of elasticity for the pre-stressed semi-bounded bodies we use the suggested in [8] and improved in [9–12] numerically analytical model of the pre-stressed functionally graded medium. It is based on reducing the solution of the linearized equations of motion, i.e. of the system of the partial differential equations of the second order with the variable coefficients, to the solution of the system of the ordinary differential equations (ODE) of the first order with the boundary and initially boundary conditions with respect to the components of the displacement vector and the normal components of the stress vector. In its turn the solution of the ODE system with the variable coefficients is obtained by Runge-Kutta's method with Merson's modification which allows to control the calculation error.

Using as a reference material Murnaghan model takes into account the elastic constants III order, and thus more adequately describe the influence of the initial actions on the properties of the material even with large initial deformation.

2. The problem formulation

We assume that in the layer $|x_1|, |x_2| \le \infty, 0 \le x_3 \le h$ under the initial mechanical actions the homogeneous initial deformation is induced:

$$\mathbf{R} = \mathbf{r} \cdot \mathbf{\Lambda}, \qquad \mathbf{G} = \mathbf{\Lambda} \cdot \mathbf{\Lambda}^{\mathrm{T}}, \quad \mathbf{\Lambda} = \delta_{ij} v_i \mathbf{r}_i \mathbf{r}_j, \qquad v_i = \text{const.} \tag{1}$$

Here $\mathbf{R}, \mathbf{r}$ are the radius-vectors of the medium point in the initially deformed and natural states respectively, $v_i = 1 + \delta_i$, δ_i are the relative lengthenings of the fibres directed in the natural configuration along the axes a_i, $i = 1, 2, 3$, coinciding with Cartesian coordinates, δ_{ij} is Kronecker delta.

We assume that the layer properties change with depth and that the model of the initially isotropic material with the elastic potential in Murnaghan's form is used as the basic one.

In some domain on the surface layer there is a source of harmonic oscillations. The lower bound of the layer is rigidly coupled. In the context of these assumptions the boundary-value problem in Euler coordinates connected with the initially deformed state (IDS) is formulated by the linearized motion equations [8]

$$\nabla^1 \cdot \mathbf{\Theta} = \rho^1 \ddot{\mathbf{u}} . \tag{2}$$

with the boundary conditions on the surface $O = O_1 + O_2$:

$$O_1: \quad \mathbf{N} \cdot \mathbf{\Theta} = \mathbf{q} e^{-i\omega t}, \tag{3}$$

$$O_2: \quad \mathbf{u} = \mathbf{u}^*, \tag{4}$$

on the lower bound:

$$\mathbf{u} = 0, \qquad x_3 = x_0 . \tag{5}$$

Here ∇^1 is Hamilton's operator in IDS; $\mathbf{N}$ is the vector of the external normal to the medium surface in IDS; $\mathbf{u}$, $\mathbf{q}$ are the vectors of displacements and stresses determined in the Euler's coordinate system; ρ^1 is the density of the medium material in IDS.

We use a linearized tensor of stresses which in case of the initially isotropic elastic material is of the form [8] as tensor $\mathbf{\Theta}$:

$$\Theta = \mathbf{T} \cdot \nabla_1 \mathbf{u} + 4J^{-1} \left\lfloor -\psi_0 \boldsymbol{\varepsilon}(\mathbf{u}) + \psi_2 \mathbf{F} \cdot \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{F} + \sum_{k=0}^{2} \sum_{m=0}^{2} V_{km} \mathbf{F}^k \mathbf{F}^m \cdot\cdot \boldsymbol{\varepsilon}(\mathbf{u}) \right\rfloor . \tag{6}$$

$\mathbf{T}$ in (6) is the tensor of initial stresses in IDS. It is determined by the formula

$$\mathbf{T} = 2J^{-1} \left(\psi_0 \mathbf{I} + \psi_1 \mathbf{F} + \psi_2 \mathbf{F}^2 \right). \tag{7}$$

$J = v_1 v_2 v_3$ is the metric factor; $\mathbf{F}$ is the measure of Finger's deformation; $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linear tensor of the deformation of the perturbation state; $\mathbf{I}$ is the unit tensor. $\mathrm{I}_k = I_k(\mathbf{F})$ are the invariants of Finger's measure. The form of coefficients ψ_k, V_{km} is given in [8]. As the elastic potential, we use the elastic potential in Murnaghan's form:

$$\chi = \frac{1}{4}\left[(-3\lambda - 2\mu + \frac{9}{2}l + \frac{n}{2})\mathrm{I}_1 + \frac{1}{2}(\lambda + 2\mu - 3l - 2m)\mathrm{I}_1^2 + (-2\mu + 3m - \frac{n}{2})\mathrm{I}_2 - m\mathrm{I}_1\mathrm{I}_2 + \frac{1}{6}(l + 2m)\mathrm{I}_1^3 + \frac{n}{2}(\mathrm{I}_3 - 1)\right]. \tag{8}$$

In the context of assumptions with respect to (7), (8) the representation of the components of tensor $\mathbf{\Theta}$ is in the form:

$$\Theta_{lk} = \theta_{lksp}\frac{\partial u_s}{\partial x_p}, \quad \theta_{lksp} = \delta_{ls}\delta_{kp}s_{lk}^{(1)} + \delta_{ks}\delta_{lp}v_l^2 s_{lk}^{(2)} + \delta_{lk}\delta_{sp}s_{ls}^{(3)}. \tag{9}$$

$$s_{lk}^{(1)} = \frac{2}{J}\left[-\psi_0 + \psi_2 v_l^2 v_k^2\right], \quad s_{lk}^{(2)} = \frac{2}{J}\left[\psi_1 + \psi_2\left(v_l^2 + v_k^2\right)\right],$$
$$s_{lk}^{(3)} = \frac{4}{J}\sum_{M=0}^{2}\sum_{N=0}^{2} V_{MN} v_l^{2M} v_k^{2N}. \tag{10}$$

Let the layer make harmonic oscillations under the action of the shear loading $q(x_1)e^{-i\omega t}$ distributed in the region $|x_1| < a, \quad x_3 = h$, the change of the physical parameters of the layer is determined by formulae:

$$\rho(x_3) = \rho_0 f_\rho(x_3), \quad \mu(x_3) = \mu_0 f_\mu(x_3), \quad \lambda(x_3) = \lambda_0 f_\lambda(x_3),$$
$$l(x_3) = l_0 f_l(x_3), \quad m(x_3) = m_0 f_m(x_3), \quad n(x_3) = n_0 f_n(x_3). \tag{11}$$

In this case in Fourier's transforms the boundary-value problem (2)–(5) with respect to (9), (10) is of the form:

$$-\left(\alpha^2\theta_{1221} - \rho\omega^2\right)U_2 + \theta_{3223}U_2'' + \theta_{3223}'U_2' = 0 \tag{12}$$

with the boundary conditions:

$$\Theta_{32}^{\Lambda} = \theta_{3223}U_2' = Q_{|x_3=h}, \quad U_2 = 0_{|x_3=0}. \tag{13}$$

Here Θ_{32}^{Λ}, U_2, Q are Fourier's transforms of the components of the stress tensor, of the displacement vector, of the preset loading, α is the transformation parameter.

It should be noted that the components of tensor $\mathbf{\Theta}$ (2), (3), (6), the tensor of initial stresses $\mathbf{T}$ (7), the function of the specific potential energy χ (8) are the smooth functions of the coordinate x_3. The change of the characteristics is determined both by the functional dependence (11), and by the character of the applied initial stresses. According to [8–12], the formulations of the boundary-value problems as well as the solution are given in the dimensionless parameters.

3. The boundary-value problem solution

The solution of the boundary-value problem on the shear oscillations of the pre-stressed functionally gradient layer is of the form [12]:

$$u_2(x_1, x_3) = \frac{1}{2\pi}\int_{-a}^{a} k_{22}(x_1 - \xi, x_3, \omega)\, q(\xi)\, d\xi, \tag{14}$$

$$k_{22}(s,x_3,\omega)=\int_\Gamma K_{22}(\alpha,x_3,\omega)e^{-i\alpha s}d\alpha,\; K_{22}(\alpha,x_3,\omega)=y_{21}(\alpha,x_3,\omega)\left[y_{11}(\alpha,h,\omega)\right]^{-1}, \quad (15)$$

here $y_{jk}(\alpha,x_3,\omega)$ are the linear independent solutions of Cauchy's problem with the initial conditions $y_{jk}(\alpha,0,\omega)=\delta_{jk}$ for the equation:

$$\mathbf{Y}'=\mathbf{M}(\alpha,x_3)\mathbf{Y}, \qquad \mathbf{M}(\alpha,x_3)=\begin{pmatrix} 0 & \theta_{1221}\alpha^2-\rho\omega^2 \\ (\theta_{3223})^{-1} & 0 \end{pmatrix}. \quad (16)$$

4. Numerical results

The shear harmonic oscillations of the functionally gradient elastic layer with a rigid inclusion (Fig. 1a) and with a soft inclusion (Fig. 1b) caused by the action of the point source are considered. As a basic material we use the transversal isotropic material with parameters [8]: $\rho = 7.748\cdot10^3$ kg/m^3, $\mu = 0.804\cdot10^{11}$ N/m^2, $\lambda = 1.1\cdot10^{11}$ N/m^2, $l = -3.25\cdot10^{11}$ N/m^2, $m = -6.32\cdot10^{11}$ N/m^2, $n = -8.04\cdot10^{11}$ N/m^2. The results of the numerical investigations are presented in dimensionless parameters.

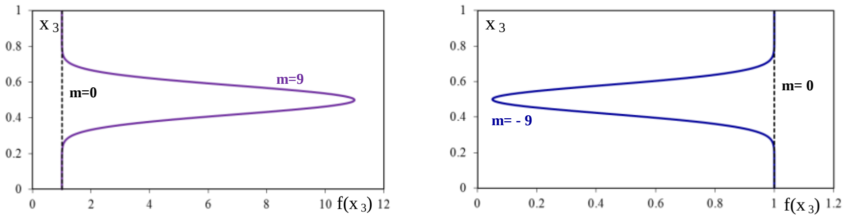

Fig. 1. Functional dependences of the change of properties.

In Figures 2 a) – d) and 3 a) – d) the transformation of the function of the distribution of the displacement amplitude $|u_2(0,x_3)|$ and of the function of the relative change of the displacement amplitude $\Delta=\left\|u_2^0(0,x_3)\right|-\left|u_2^\sigma(0,x_3)\right\|$ in IDS in depth for the dimensionless frequencies ω = 0.6, 4.9 respectively is given.

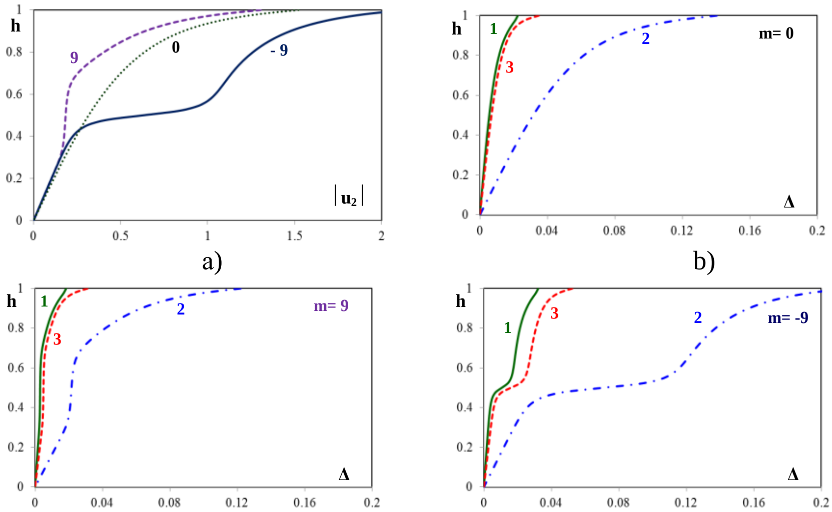

Fig. 2. The influence of the uniaxial IDS on the displacement amplitude for the layer with different types of inclusions. $\Delta\rho=\Delta f$, ω = 0.6.

In this case all the properties of the medium change according to one law with equal intensity $\Delta\rho = \Delta f$, $\Delta f = \Delta\mu = \Delta\lambda = \Delta l = \Delta m = \Delta n$. Numbers in fig. a) denote the curves corresponding to the functional dependences in fig. 1. Numbers 1,2,3 in fig. b) – d) correspond to uniaxial extensions 1x1, 1x2, 1x3 (v_i=1.02).

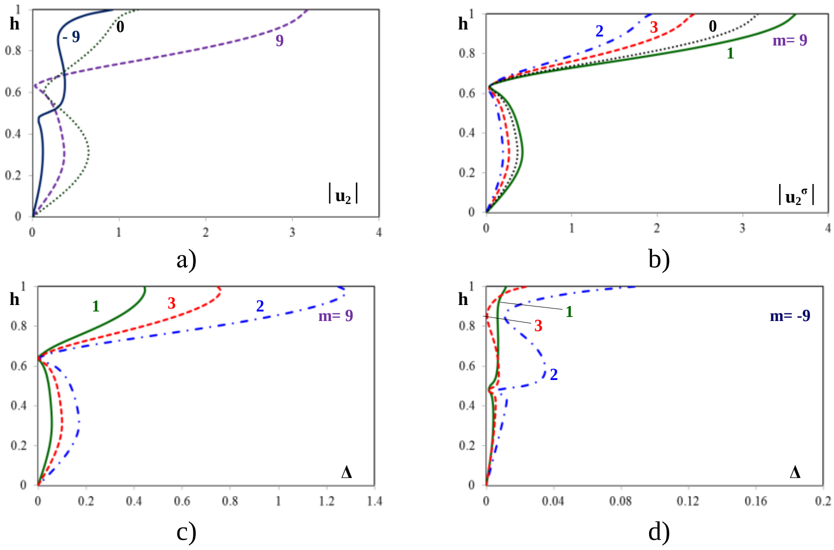

Fig. 3. The influence of the uniaxial IDS on the displacement amplitude for the layer with the different types of inclusions. $\Delta\rho = \Delta f$, ω = 4.9.

Figure 4 a) – d) illustrates the influence of the intensity of the density change on $|u_2(0,x_3)|$ and Δ for the rigid inclusion (a)) and for the soft inclusion (b)).

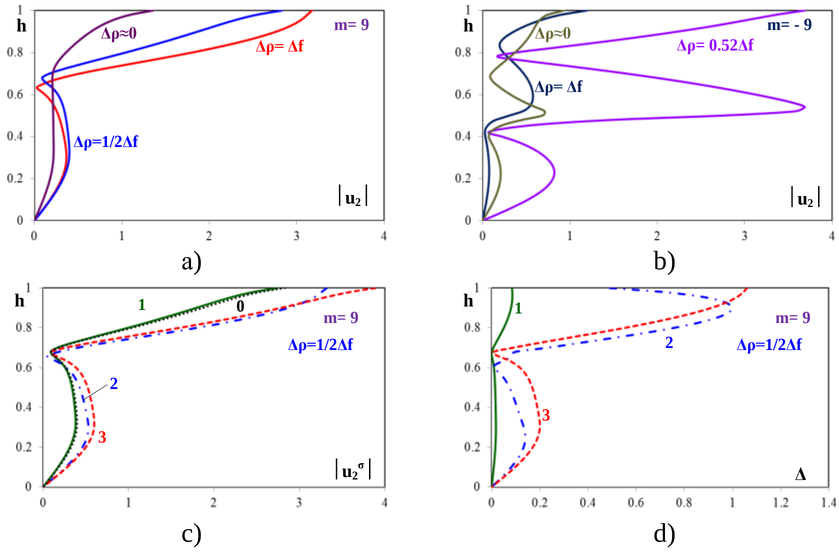

Fig. 4. The influence of the density change intensity on the displacement amplitude at different dimensionless frequencies without taking the pre-stresses into account a) - ω = 4.9, b) - ω = 7; c), d) at uniaxial IDS, ω = 4.9.

It follows from the figures that the presence of the rigid inclusion in a layer leads to the localization of displacements at the surface of the layer. The amplitude of oscillations can increase when it approaches the resonance frequency. In the case of the soft inclusion, the localization of displacements is not only at the surface of the layer but also in the middle part of it. Changing IDS, it is possible to dampen or to increase the amplitude of oscillations on the surface of the layer or inside it.

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