

ELECTROMAGNETIC FIELD IN MOVING SPACE WITH SPHERICAL ENCLOSURE

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Abstract. The unsteady axisymmetric problem being considered is related to the identification of the components of electromagnetic field in moving space with spherical enclosure filled with homogeneous isotropic conductor. The model used includes Maxwell's equations and generalized Ohm's law. In order to find a solution, a series expansion in Legendre polynomials and Laplace time transform are applied. Integral representations of the components of the electromagnetic field with Green's function kernels have been generated.

1. Introduction

As of today, the problem related to wave propagation in electromagnetoelastic bodies has been mainly considered in regard to stationary problems (see example [1]). When researching non-stationary processes in such bodies (example [2]), the small parameter method is convenient to be used represented by mechanical and electromagnetic field interaction factor. At the same time, a solution is required for the auxiliary problem of identification of the electromagnetic field parameters in accordance with the predetermined motion law. This problem is considered below in respect of a plane with spherical enclosure. It also has its own independent value in terms of, for example, research on the motion of various aircrafts affected by electromagnetic field.

2. Setting up the problem

It is assumed that the space with the spherical enclosure of radius R is filled with homogeneous isotropic conductor and moves in accordance with the predetermined law. Axisymmetric change of the electromagnetic field in it is described using the Maxwell's equations and generalized Ohm's law in the spherical coordinate system r, θ, ϑ ($r \geq 0, 0 \leq \theta \leq \pi, -\pi < \vartheta \leq \pi$):

$$-r^{-1} (rH)_{,r} = \eta_e^2 (\gamma j_\theta + \dot{E}_\theta), \quad r^{-1} (H \sin \theta)_{,\theta} \sin^{-1} \theta = \eta_e^2 (\gamma j_r + \dot{E}_r),$$

$$r^{-1} \left[(rE_\theta)_{,r} - E_{r,\theta} \right] = -\dot{H}, \quad E_{r,r} + r^{-1} (E_{\theta,\theta} + E_\theta \text{ctg} \theta + 2E_r) = \rho_e; \quad (1)$$

$$j_r = E_r + \rho_{e0} \dot{u} / \gamma, \quad j_\theta = E_\theta + \rho_{e0} \dot{v} / \gamma. \quad (2)$$

From now on, dots stand for time derivatives while the variable after the coma in the lower index indicates its derivative. The following dimensionless quantities are also used (if the tracing is the same, the prime corresponds to dimensional analogues):

$$\tau = \frac{c_1 t}{L}, \quad r = \frac{r'}{L}, \quad r_0 = \frac{R}{L}, \quad u = \frac{u'}{L}, \quad v = \frac{v'}{L}, \quad H = \frac{H' \mu_e c_1}{c E_*}, \quad \rho_e = \frac{4\pi \rho_e' L}{\varepsilon E_*},$$

$$E_r = \frac{E_r'}{E_*}, \quad E_\theta = \frac{E_\theta'}{E_*}, \quad j_r = \frac{j_r'}{\sigma E_*}, \quad j_\theta = \frac{j_\theta'}{\sigma E_*}, \quad \eta_e^2 = \frac{\mu_e \varepsilon c_1^2}{c^2}, \quad \gamma = \frac{4\pi \sigma L}{\varepsilon c_1},$$

where t - time; u and v , E_r and E_θ , j_r and j_θ - radial and tangential displacements, components of electric field and current density vectors; H - magnetic field vector nonzero component; ρ_e - surface charge density; c and c_1 - speed of light and tension wave propagation; ε and μ_e - dielectric and magnetic conductivity factors; σ - electric conductivity factor; L and E_* - electric field linear dimension and intensity.

The Ohm's law linearization was performed in regard to initial electromagnetic field (which has a correspondent additional index 0) with the following components:

$$E_{0g} = E_{0\theta} \equiv 0, \quad E_{0r} = E_0(r), \quad H_{0r} = H_{0\theta} = H_{0g} \equiv 0.$$

At the enclosure boundary, the following electric field intensity is specified:

$$E_\theta|_{r=r_0} = e_0(\tau, \theta). \quad (3)$$

All the required functions are bounded while the initial conditions are homogeneous:

$$E_r|_{\tau=0} = \dot{E}_r|_{\tau=0} = E_\theta|_{\tau=0} = \dot{E}_\theta|_{\tau=0} = H|_{\tau=0} = \dot{H}|_{\tau=0} = 0. \quad (4)$$

Equations (1), (2) lead to the following equation further used as the basic:

$$\eta_e^2 (\ddot{H} + \gamma \dot{H}) = \Delta \dot{H} - r^{-2} \dot{H} \sin^2 \theta + r^{-1} \eta_e^2 \left[(r \rho_{e0} \ddot{v})_{,r} - \rho_{e0} \ddot{u}_{,\theta} \right],$$

$$\Delta H = r^{-2} \left[(r^2 H_{,r})_{,r} + \sin^{-1} \theta (H_{,\theta} \sin \theta)_{,\theta} \right] \quad (5)$$

and the following charge density ratio:

$$\dot{\rho}_e + \gamma \rho_e = -r^{-2} (r^2 \rho_{e0} \dot{u})_{,r} - r^{-1} \sin^{-1} \theta (\rho_{e0} \dot{v} \sin \theta)_{,\theta}. \quad (6)$$

3. Integral solutions

In order to solve initial boundary value problem (3)-(5) with consideration of (1), functions E_r , ρ_e , j_r , u and E_θ , H , j_θ , v , e_0 are expanded in Legendre $P_n(x)$ and Gegenbauer $C_{n-1}^{3/2}(x)$ polynomials accordingly (two series are given for example):

$$E_r(r, \theta, \tau) = \sum_{n=0}^{\infty} E_{rn}(r, \tau) P_n(\cos \theta), \quad H(r, \theta, \tau) = \sin \theta \sum_{n=1}^{\infty} H_n(r, \tau) C_{n-1}^{3/2}(\cos \theta). \quad (7)$$

This leads to the following equations in regard to bounded factors $H_n(r, \tau)$ ($n \geq 1$):

$$\eta_e^2 (\ddot{H}_n + \gamma \dot{H}_n) = \Delta_n \dot{H}_n + \eta_e^2 l_H (\ddot{u}_n, \ddot{v}_n), \quad \Delta_n H = r^{-2} \left[(r^2 H_{,r})_{,r} - m H \right],$$

$$m = n(n+1), \quad l_H(u, v) = r^{-1} \left[(r \rho_{e0} v)_{,r} + \rho_{e0} u \right]. \quad (8)$$

4. Green's functions

The solutions for boundary-value problems (18) and (19) have the following view ($h(x)$ - Heaviside function [3]):

$$G_{Hn}^L(r, \xi, s) = \xi^2 \left[\tilde{G}_{Hn}^L(r, \xi, s) h(\xi - r) + \tilde{G}_{Hn}^L(\xi, r, s) h(r - \xi) \right], \quad (22)$$

$$\tilde{G}_{Hn}^L(r, \xi, s) = \eta_e s_e S_{en}(\eta_e r_0 s_e, \eta_e r s_e) Z_{1n}(\eta_e \xi s_e) Y_{3n}^{-1}(\eta_e r_0 s_e);$$

$$G_{Hn0}^L(r, s) = -\eta_e^{-1} s_e^{-1} Z_{1n}(\eta_e r s_e) Y_{3n}^{-1}(\eta_e r_0 s_e), \quad (23)$$

where

$$Z_{1n}(z) = z^{-1/2} K_{n+1/2}(z), \quad Z_{2n}(z) = z^{-1/2} I_{n+1/2}(z), \quad S_{en}(x, y) = Z_{1n}(y) Y_{4n}(x) - Y_{3n}(x) Z_{2n}(y),$$

$$Y_{3n}(z) = -z^{-3/2} \left[(n+1) K_{n+1/2}(z) - z K_{n+3/2}(z) \right], \quad Y_{4n}(z) = -z^{-3/2} \left[(n+1) I_{n+1/2}(z) + z I_{n+3/2}(z) \right].$$

Here, $K_{n+1/2}(z)$ and $I_{n+1/2}(z)$ are modified Bessel functions.

When building formulas (22) and (23), the Bessel function expressions through elementary functions [4] were used from which follow the below listed equalities for $Z_{1n}(z)$, $Z_{2n}(z)$ and $Y_{3n}(z)$, $Y_{4n}(z)$:

$$Z_{1n}(z) = z^{-n-1} R_{n0}(z) e^{-z} \sqrt{\pi/2}, \quad Z_{2n}(z) = (-1)^n z^{-n-1} (2\pi)^{-1/2} \left[R_{n0}(-z) e^z - R_{n0}(z) e^{-z} \right],$$

$$Y_{3n}(z) = z^{-n-2} R_{n3}(z) e^{-z} \sqrt{\pi/2}, \quad Y_{4n}(z) = (-1)^n z^{-n-2} (2\pi)^{-1/2} \left[R_{n3}(-z) e^z - R_{n3}(z) e^{-z} \right],$$

where

$$R_{n0}(z) = \sum_{k=0}^n A_{nk} z^{n-k}, \quad A_{nk} = \frac{(n+k)!}{2^k (n-k)! k!}, \quad R_{n3}(z) = R_{n1}(z) - R_{n0}(z), \quad R_{n1}(z) = R_{n+1,0}(z) - n R_{n0}(z).$$

In this case, Green's functions in (22) and (23) obtain the following view:

$$\tilde{G}_{Hn}^L(r, \xi, s) = \sum_{k=1}^2 \tilde{G}_{Hn}^{(0k)L}(r, \xi, s) e^{-\tau_{e0k}(r, \xi) s_e}, \quad G_{Hn0}^L(r, s) = r_0^{n+2} r^{-n-1} G_{Hn0}^{(02)L}(r, s) e^{-\tau_{e02}(r, r_0) s_e},$$

where

$$\tilde{G}_{Hn}^{(01)L}(r, \xi, s) = (-1)^{n+1} R_{n0}(\eta_e \xi s_e) R_{n0}(-\eta_e r s_e) D_n^{-1}(r \xi, \eta_e s_e),$$

$$\tilde{G}_{Hn}^{(02)L}(r, \xi, s) = (-1)^n R_{n0}(\eta_e \xi s_e) R_{n0}(\eta_e r s_e) R_{n3}(-\eta_e r_0 s_e) D_n^{-1}(r \xi, \eta_e s_e),$$

$$G_{Hn0}^{(02)L}(r, s) = -R_{n0}(\eta_e r s_e) R_{n3}^{-1}(\eta_e r_0 s_e),$$

$$D_n(x, y) = 2x^{n+1} y^{2n+1}, \quad \tau_{e01}(r, \xi) = \eta_e (\xi - r), \quad \tau_{e02}(r, \xi) = \eta_e (r + \xi - 2r_0).$$

Their originals may be analytically found using operator calculus theorems. However, since for real mediums $\eta_e \ll 1$, serious challenges arise during calculations associated with the small parameter. That is why quasistatic analogues of Green's functions with $\eta_e = 0$ will be further used.

5. Quasistatic solution

In this version, equations in boundary problems (18) and (19) are simplified and have the following fundamental system of solutions: $r^{-(n+1)}$, r^n . Then, similar to p. 3, we come to the following formulas for Green's functions:

$$G_{Hn}^L(r, \xi, s) = G_{Hn}^c(r, \xi) = \xi^2 \left[\tilde{G}_{Hn}^c(r, \xi) h(\xi - r) + \tilde{G}_{Hn}^c(\xi, r) h(r - \xi) \right],$$

$$\tilde{G}_{Hn}^c(r, \xi) = -\frac{(n+1)r_0^{2n+1} + nr^{2n+1}}{n(2n+1)r^{n+1}\xi^{n+1}}, \quad G_{Hn0}^L(r, s) = G_{Hn0}^c(r) = -\frac{r_0^{n+2}}{nr^{n+1}}. \quad (24)$$

Then, formula (17) is also simplified by substituting kernels $G_{Hn}^L(r, \xi, s)$ and $G_{Hn0}^L(r, s)$ with functions $G_{Hn}^c(r, \xi)$ and $G_{Hn0}^c(r)$. Its original (20) is substituted with the following equality:

$$H_n(r, \tau) = -\eta_e^2 \int_{r_0}^{\infty} G_{Hn}^c(r, \xi) l_H \left[\dot{u}_n(\xi, \tau), \dot{v}_n(\xi, \tau) \right] d\xi - \eta_e^2 G_{Hn0}^c(r) h_0 \left[v_n(r_0, \tau), e_{0n}(\tau) \right]. \quad (25)$$

The algorithm is followed by the summation of series presented as (7) and the identification of the current density component vector using formulas (2).

6. Example

Let's assume that the space material is aluminium ($\eta_e = 0,111 \cdot 10^{-4}$; $\gamma = 5,06$) while the other initial values are represented as: $r_0 = 1$, $\rho_{e0}(r) = 2ar^{-3/2}$, $u = \tau_+^2/2 \cos \theta$, $v = -\tau_+^2/2 \sin \theta$, $e_0 \equiv 0$.

Then, the electromagnetic field components calculated using formulas (7), (21), (24) and (25) are determined as follows:

$$H(r, \theta, \tau) = \eta_e^2 \omega r^{-2} \left[r_0^3 + \frac{2a}{3} (2r^{3/2} + r_0^{3/2}) \right] \tau h(\tau) \sin \theta,$$

$$\rho_e(r, \theta, \tau) = 3\gamma^{-2} a \omega r^{-5/2} (\gamma \tau - 1 + e^{-\gamma \tau}) h(\tau) \cos \theta.$$

7. Conclusions

An analytical solution of the axisymmetric non-stationary problem is presented related to the identification of the components of the electromagnetic field in the space filled with homogeneous isotropic conductor with spherical enclosure. Its movement law and field distribution at the cavity boundary are predetermined. It is shown that the kernels of the integral representations built may be substituted with their quasistatistical analogues, which significantly simplifies the evaluation of the correspondent integrals. The solution obtained may serve as the basis for the research of problems with more complex geometry.

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