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METHOD OF THE EQUIVALENT BOUNDARY CONDITIONS IN THE UNSTEADY PROBLEM FOR ELASTIC DIFFUSION LAYER

A.V. Zemskov[1*], D.V. Tarlakovskiy[2]

[1]Moscow Aviation Institute (National Research University),

Volokolamskoye shosse 4, A-80, GSP-3, Moscow, 125993, Russia

[2]Institute of Mechanics (Lomonosov Moscow State University),

Michurinsky prospekt 1, Moskow, 119192, Russia

*e-mail: azemskov1975@mail.ru

Abstract. The proposed approach to solving initial-boundary problems is based on integral relations which connect right sides of boundary conditions of different types. One of these solutions is assumed to be found. In this case, integral relations are considered as equations as regard to right sides which are equivalent to other conditions. Quadrature formulas are used to solve these equations.

1. Introduction

Solving many of unsteady problems in continuum mechanics including elastic diffusion problems is associated with serious mathematical challenges. These are due to the need of Laplace transform conversion used to solve problems of this type. Depending on certain types of boundary conditions, solution for these problems may be produced using the Fourier trigonometric series, which significantly simplifies the originals' finding algorithm [1]. The disadvantage of this method is the restricted application area, which is due to the specifics of boundary conditions.

In order to overcome this disadvantage, it is proposed to build relations among boundary conditions of different types. In this case, it will be sufficient to solve a single (benchmark) problem while all other problems will be reduced to it using the relations given below. The proposed method is demonstrated through the example of a one-dimensional unsteady problem for elastic diffusion layer.

2. Algorithm general description

Let there be given a boundary-value problem which will be hereinafter referred to as original problem:

$$L(u) = 0,\ u = u(x,t),\ x \in [a,b],\quad t > 0, \tag{1}$$

$$M_1(u)\big|_{x=a} = f_1(t),\ M_2(u)\big|_{x=b} = f_2(t). \tag{2}$$

where L, M_1 and M_2 are linear differential operators and t is time.

Along with it, a benchmark boundary-value problem is being considered in which the required function satisfies equation (1) and boundary conditions

$$N_1(u)\big|_{x=a} = g_1(t),\ N_2(u)\big|_{x=b} = g_2(t), \tag{3}$$

where N_1 and N_2 are linear differential operators.

Problem solution (1), (3) is presented as follows:

$$u=\int_0^t G_1(x,t,\tau)g_1(\tau)d\tau+\int_0^t G_2(x,t,\tau)g_2(\tau)d\tau\,. \tag{4}$$

Here, G_1 and G_2 are the functions satisfying equation (1) and accordingly the following boundary conditions ($\delta(t)$ - Dirac delta function):

$$N_1(G_1)\big|_{x=a}=\delta(t),\ N_2(G_1)\big|_{x=b}=0,\ N_1(G_2)\big|_{x=a}=0,\ N_2(G_2)\big|_{x=b}=\delta(t).$$

Assuming that functions G_1 and G_2 are known and claiming that g_1 and g_2 satisfy the conditions (2), we arrive at a system of integral equations in regard to g_1 and g_2:

$$\int_0^t M_k\left[G_1(x,t,\tau)\right]g_1(\tau)d\tau+\int_0^t M_k\left[G_2(x,t,\tau)\right]g_2(\tau)d\tau=f_k,\quad k=1,x=a;\quad k=2,x=b.$$

Substitution of solution to this system in (4) leads to the solution of the original problem.

Remark. The proposed algorithm may be generalized to vector functions $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$**,** $\mathbf{u}=\{u_1(\mathbf{x},t),\dots u_m(\mathbf{x},t)\}$**,** $\mathbf{x}=\{x_1,\dots,x_n\}\in\Re^n$, correspondent matrix differential operators $\mathbf{L}$, $\mathbf{M}_k$, $\mathbf{N}_k$ and right sides of boundary conditions in the form of vector functions $\mathbf{g}_k(t)$ $(k=1,2)$.

3. Setting up the first boundary-value problem of elastic diffusion

Let there be a homogeneous layer bounded by surfaces $x_3=0$ and $x_3=L$ ($Ox_1x_2x_3$ - the Cartesian coordinate system). One-dimensional physical and mechanical processes in the medium are described using elastic bounded diffusion model [1, 2]:

$$\ddot{u}=u''-\alpha\eta',\quad \dot{\eta}=D\eta''-\Lambda u''';\tag{5}$$

$$\eta\big|_{x=0}=f_{11}^1(\tau),\quad u\big|_{x=0}=f_{21}^1(\tau),\quad \eta\big|_{x=1}=f_{12}^1(\tau),\quad u\big|_{x=1}=f_{22}^1(\tau)\ ;\tag{6}$$

$$u\big|_{\tau=0}=\dot{u}\big|_{\tau=0}=\eta\big|_{\tau=0}=0\,,\tag{7}$$

where the prime and the dots mean the spatial derivative x and time derivatives τ accordingly.

From now on, the following dimensionless values in (5) – (7) are used (where the same tracing is used, they are marked with an asterisk which is omitted in other equations):

$$x=\frac{x_3}{L},\quad u=\frac{u_3}{L},\quad \tau=\frac{ct}{L},\quad \eta^*=\frac{\eta}{n_0},\quad c^2=\frac{C_{3333}}{\rho},\quad \alpha=\frac{n_0\alpha_{33}}{C_{3333}},\ D=\frac{D_{33}g}{cL},$$

$$\Lambda=\frac{\Lambda_{3333}}{n_0cL},\quad \Lambda_{3333}=\frac{n_0\alpha_{33}D_{33}}{RT_0},\ f_{1k}^{1*}(\tau)=\frac{f_{1k}^1(\tau)}{n_0},\quad f_{2k}^{1*}(\tau)=\frac{f_{2k}^1(\tau)}{L}\quad (k=1,2),$$

where t - time; u_3- translation along axis Ox_3; T_0 - initial temperature; $\eta=n-n_0$ - concentration increment; n_0 and n - initial and current concentration of substance; C_{ijkl} - elastic constants' tensor components; ρ - medium density; α_{ij} - coefficients determined by the type of the crystal lattice (the relative volume change due to diffusion); R - universal gas constant; D_{ij} - self-diffusion coefficients' tensor components .

4. Transition to equivalent boundary conditions

In order to solve problem (5) - (7), an auxiliary (benchmark) problem is being considered which is defined by equations (5), initial conditions (7) and boundary conditions (right sides of the second and third equations coincide here and in (6)):

$$\left(\Lambda u'' - D\eta'\right)\Big|_{x=0} = f_{11}^{2}(\tau), \quad u\big|_{x=0} = f_{21}^{1}(\tau), \quad \left(\Lambda u'' - D\eta'\right)\Big|_{x=1} = f_{12}^{2}(\tau), \quad u\big|_{x=1} = f_{22}^{1}(\tau), \qquad (8)$$

Its solution is found in work [1] and is given in the integral form by (the asteriks means time convolution):

$$u = \sum_{l=1}^{2}\left(G_{21l} * f_{1l}^{2} + G_{22l} * f_{2l}^{1}\right) \quad \eta = \sum_{l=1}^{2}\left(G_{11l} * f_{1l}^{2} + G_{12l} * f_{2l}^{1}\right) \qquad (9)$$

Here, $G_{1kl} = \eta, G_{2kl} = u$ - the Green's function in problem (5), (7), (8), i.e. solutions of four problems (k,l - their numbers) which include equations (5), initial conditions (7) and the following boundary conditions:

$$\left(\Lambda G''_{2kl} - DG'_{1kl}\right)\Big|_{x=0} = \delta_{1k}\delta_{1l}\delta(\tau), \quad G_{2kl}\big|_{x=0} = \delta_{2k}\delta_{1l}\delta(\tau),$$
$$\left(\Lambda G''_{2kl} - DG'_{1kl}\right)\Big|_{x=1} = \delta_{1k}\delta_{2l}\delta(\tau), \quad G_{2kl}\big|_{x=1} = \delta_{2k}\delta_{2l}\delta(\tau),$$

where $\delta(\tau)$ - Dirac delta function; δ_{ik} - Kronecker symbol, $i = 1, 2$.

Assuming now that the solution for the benchmark problem satisfies the equalities $\eta(0,\tau) = f_{11}^{1}(\tau)$, $\eta(1,\tau) = f_{12}^{1}(\tau)$ and considering that $f_{21}^{2}(\tau) = f_{21}^{1}(\tau)$, $f_{22}^{2}(\tau) = f_{22}^{1}(\tau)$, we arrive at the Volterra equations of convolution type in regard to the functions $f_{11}^{2}(\tau)$ and $f_{12}^{2}(\tau)$:

$$\begin{aligned} G_{111}(0,\tau) * f_{11}^{2}(\tau) - G_{111}(1,\tau) * f_{12}^{2}(\tau) = \phi_1(\tau), \\ G_{111}(1,\tau) * f_{11}^{2}(\tau) - G_{111}(0,\tau) * f_{12}^{2}(\tau) = \phi_2(\tau), \end{aligned} \qquad (10)$$

where

$$\phi_1(\tau) = f_{11}^{1}(\tau) - G_{121}(0,\tau) * f_{21}^{1}(\tau) + G_{121}(1,\tau) * f_{22}^{1}(\tau),$$
$$\phi_2(\tau) = f_{12}^{1}(\tau) - G_{121}(1,\tau) * f_{21}^{1}(\tau) + G_{121}(0,\tau) * f_{22}^{1}(\tau).$$

The Green's function symmetry specified in [1] is taken into account here.

The main difficulty when solving system (10) is that the functions $G_{111}(0,\tau)$ and $G_{111}(1,\tau)$ have singularity if $\tau = 0$. In order to research the nature of the singularities, we proceed as follows. In the Laplace transform domain, the Green's function $G_{111}(x,\tau)$ is given by [1]:

$$G_{111}^{L}(x,s) = \frac{1}{s} + \sum_{n=1}^{\infty} G_{111n}^{L}(s)\cos\lambda_n x, \quad \lambda_n = \pi n,$$

where

$$G_{111n}^{L} = \frac{2\left(s^2 + \lambda_n^2\right)}{P\left(\lambda_n^2, s\right)}, \quad P\left(\lambda_n^2, s\right) = \left(s^2 + \lambda_n^2\right)\left(s + D\lambda_n^2\right) - \alpha\Lambda\lambda_n^4. \qquad (11)$$

The following transformation is made in formula (11) for function G_{111n}^{L}:

$$G_{111n}^{L}=\frac{2}{s+D\lambda_n^2}+\frac{2\alpha\Lambda\lambda_n^4}{Q\left(\lambda_n^2,s\right)},\quad Q\left(\lambda_n^2,s\right)=\left(s+D\lambda_n^2\right)P\left(\lambda_n^2,s\right).$$

Original function $G_{111}^{L}\left(x,s\right)$ will have then the following representation:

$$G_{111}\left(x,\tau\right)=\vartheta_3\left(\frac{x}{2},e^{-D\pi^2\tau}\right)+2\alpha\Lambda\sum_{n=1}^{\infty}G_{111n}\left(\tau\right)\cos\lambda_n x,\ \vartheta_3\left(x,q\right)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2\pi n x\,, \tag{12}$$

where

$G_{111n}\left(\tau\right)=e^{\gamma_n\tau}\left(A_{n1}\cos\beta_n\tau-A_{n2}\sin\beta_n\tau\right)+A_{n3}e^{s_{3n}\tau}+A_{n4}e^{-D\lambda_n^2\tau}$; s_{1n}, s_{2n} are complex and s_{3n} is the real root of polynomial $P\left(\lambda_n^2,s\right)$; $\gamma_n=\operatorname{Re}s_{1n}<0$; $\beta_n=\operatorname{Im}s_{1n}$; $s_{2n}=\overline{s}_{1n}$, $s_{3n}<0$; $\vartheta_3\left(x,q\right)$ - Jacobi theta-function [3]; coefficients $A_{nq},\ q=\overline{1,4}$ are included in the following formulas (where the prime means the derivative s):

$$A_{n1}=2\operatorname{Re}\frac{\lambda_n^4}{Q'\left(\lambda_n^2,s_{1n}\right)},\ A_{n2}=2\operatorname{Im}\frac{\lambda_n^4}{Q'\left(\lambda_n^2,s_{1n}\right)},\ A_{n3}=\frac{\lambda_n^4}{Q'\left(\lambda_n^2,s_{3n}\right)},\ A_{n4}=\frac{\lambda_n^4}{Q'\left(\lambda_n^2,-D\lambda_n^2\right)}.$$

In work [1], approximate values of the polynomial root were generated $P\left(\lambda_n^2,s\right)$. Based on these equalities, a conclusion may be made that common term $G_{111n}\left(\tau\right)$ has order $O\left(n^{-2}\right)$ and correspondent series in (12) which converge absolutely $\forall x$ if $\tau\geq 0$. Function $G_{111}\left(1,\tau\right)$ is bounded if $\tau>0$ while function $G_{111}\left(0,\tau\right)$ in the neighbourhood of $\tau=0$ has order integrable singularity $1/2$ [3]. In this case, in accordance with [4], each equation (10) is multiplied by $d\tau/\left(\xi-\tau\right)^{1/2}$ and integrated between 0 and ξ. Thus, we get:

$$\begin{aligned}&\int_0^{\xi}K^1\left(\xi-t\right)f_{11}^2\left(t\right)dt-\int_0^{\xi}K^2\left(\xi-t\right)f_{12}^2\left(t\right)dt=\Phi^1\left(\xi\right),\\&\int_0^{\xi}K^2\left(\xi-t\right)f_{11}^2\left(t\right)dt-\int_0^{\xi}K^1\left(\xi-t\right)f_{12}^2\left(t\right)dt=\Phi^2\left(\xi\right).\end{aligned} \tag{13}$$

where

$$K^1\left(\zeta\right)=\int_0^{\zeta}\frac{G_{111}\left(0,\zeta-\tau\right)d\tau}{\tau^{1/2}},\ K^2\left(\zeta\right)=\int_0^{\zeta}\frac{G_{111}\left(1,\zeta-\tau\right)d\tau}{\tau^{1/2}},\ \Phi^k\left(\xi\right)=\int_0^{\xi}\frac{\phi_k\left(\tau\right)d\tau}{\left(\xi-\tau\right)^{1/2}}, \tag{14}$$

while kernels $K^k\left(\zeta\right)$ do not have singularities in zero and $\Phi_k\left(0\right)=0,\ k=1,2$.

When solving equations (13), quadrature formulas are good to be used. Range $\left[0,T\right]$ of time variable τ is split into N intervals by dots $\tau_i=ih$ with even pitch $h=T/N$ and mesh functions $y_i^k=f_{1k}^2\left(\tau_i\right),\ K_i^k=K^k\left(\tau_i\right)$ are introduced.

With $\tau=\tau_i$, each of the integrals in (14) is approximately substituted with the sum corresponding to the midpoint quadrature rule:

$$\int_0^{\tau_i}K^l\left(\tau_i-t\right)f_{1k}^2\left(t\right)dt\approx hS_{i-1/2}^{lk}+hK_{1/2}^{l}y_{i-1/2}^{k},\quad S_{i-1/2}^{lk}=\sum_{j=1}^{i-1}K_{i-j+1/2}^{l}y_{j-1/2}^{k}\ \left(k,l=1,2\right).$$

where the nodes are included in the following formulas:

$$\tau_{i-1/2} = \frac{\tau_{i-1} + \tau_i}{2} = h\left(i - \frac{1}{2}\right), \quad \tau_{i-j+1/2} = \tau_i - \tau_{j-1/2} = h\left(i - j + \frac{1}{2}\right), \quad i = \overline{1, N},$$

As a result, we get a recurrent sequence of simultaneous linear algebraic equations ($i \geq 1$):

$$\mathbf{A}\mathbf{y}_{i-1/2} = \mathbf{b}_{i-1/2}, \qquad (15)$$

where

$$\mathbf{A} = \begin{pmatrix} K_{1/2}^1 & -K_{1/2}^2 \\ -K_{1/2}^2 & K_{1/2}^1 \end{pmatrix}, \ \mathbf{y}_{i-1/2} = \begin{pmatrix} y_{i-1/2}^1 \\ y_{i-1/2}^2 \end{pmatrix}, \ \mathbf{b}_{i-1/2} = \begin{pmatrix} b_{i-1/2}^1 \\ b_{i-1/2}^2 \end{pmatrix},$$

$$b_{i-1/2}^1 = \Phi^1\left(\tau_{i-1/2}\right)/h - S_{i-1/2}^{11} + S_{i-1/2}^{22}, \quad b_1^2 = -\Phi^2\left(\tau_{i-1/2}\right)/h + S_{i-1/2}^{21} - S_{i-1/2}^{12}, \ i \geq 1.$$

The solution for equation (15) takes the following form:

$$y_{i-1/2}^1 = \frac{K_{1/2}^1 b_{i-1/2}^1 + K_{1/2}^2 b_{i-1/2}^2}{\left(K_{1/2}^1\right)^2 - \left(K_{1/2}^1\right)^2}, \quad y_{i-1/2}^2 = \frac{K_{1/2}^2 b_{i-1/2}^1 + K_{1/2}^1 b_{i-1/2}^2}{\left(K_{1/2}^1\right)^2 - \left(K_{1/2}^1\right)^2}.$$

By using this solution in (9), we get solution for initial problem (5) - (7).

5. Example

As an example, problem (5) - (7) is being considered with boundary conditions in the following form:

$$\eta\big|_{x=0} = f_{11}^1(\tau) = H(\tau), \quad u\big|_{x=0} = 0, \quad \eta\big|_{x=1} = 0, \quad u\big|_{x=1} = 0,$$

where $H(\tau)$ - Heaviside step function.

The medium considered is aluminium having the following characteristics:

$$C_{3333} = 1.26 \cdot 10^{11} \frac{N}{m^2}, \quad T_0 = 773\,K, \quad \rho = 2700 \frac{kg}{m^3}, \quad D_{33} = 6.71 \cdot 10^{-6} \frac{m^2}{c}, \quad L = 1\,m.$$

Figure 1 shows numerical calculations for displacements in distinct points of the layer from formulas (9) where the number of division points is $N = 100$ and the number of terms in a Fourier series is $M = 100$.

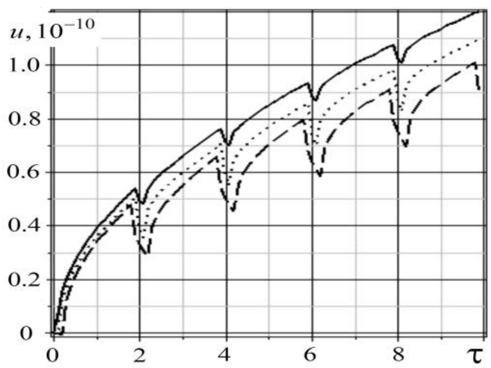

Fig. 1. Time-displacement dependence: $x = 0.05$ - solid graph, $x = 0.12$ - dotted line, $x = 0.18$ - dashed line.

It should be noted that if the subinterval is half reduced while the number of terms in a Fourier series is doubled the diagrams coincide.

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