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THE ANALYTICAL SOLUTIONS OF THE BOUNDARY-VALUE PROBLEMS BY THE METHOD OF FACTORIZATION

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Abstract. The differential factorization method developed by previously [1,2] is applied to boundary-value problems for the analytical method [3]. There problems are convenient for comparing different methods of their solution and demonstrate great potentialities of the differential factorization method. It is important to note that these problems include cases involving the main types of roots (simple and double zeros) of the determinants of characteristic equations of the corresponding differential equations.

1. Introduction

Let us assume that domain Ω occupied by an isotropic linearly deformable body in convex and its boundary $\partial\Omega$ id smooth. In the case of nonconvex boundaries, there are two ways to solve the boundary-value problem: either to pass on the generalized factorization or to subdivide the domain into block structures and investigate boundary-value problems in partial convex domains by means of simple factorization [1.2]. Note that the latter procedure implies that rectangular Cartesian coordinated are introduced at the tangent bundle of the boundary $\partial\Omega$ which is also used below. Consider the homogeneous differential Lamė equations in the conventional forms [1]

$$(\lambda+\mu) grad\ div \mathbf{u} + \mu\Delta\mathbf{u} - \delta\mathbf{u} = 0, \quad \mathbf{u} = \{u_1, u_2, u_3\}. \tag{1}$$

Here, $\delta = -\rho\omega^2$ in vibration problems and $\delta = \rho p^2$ in nonstationary problems, where ω is a vibration frequency, p is the Laplace transform parameter, and ρ is the density of the material. In the boundary-value problem, certain boundary conditions to be set, which will be discussed below.

After applying a three-dimensional Fourier transform with operator $\mathbf{F_3} = \mathbf{F_3}(\alpha_1, \alpha_2, \alpha_3)$ [1] over all the coordinates x_1, x_2, and x_2; substituting the $-i\alpha_k$ parameters of the Fourier transform for the corresponding derivatives; and multiplying by -1, the above system of equations takes the following form:

$$\mathbf{KU} = \int\int_{\partial\Omega} \boldsymbol{\omega}, \quad \mathbf{U} = \{\mathrm{U}_1, \mathrm{U}_2, \mathrm{U}_3\}, \ \mathbf{U} = \mathbf{F_3}(\alpha_1, \alpha_2, \alpha_3)\mathbf{u}, \tag{2}$$

$$\mathbf{K} = \begin{Vmatrix} (\lambda+2\mu)\alpha_1^2 + \mu\alpha_2^2 + \mu\alpha_3^2 + \delta & (\lambda+\mu)\alpha_1\alpha_2 & (\lambda+\mu)\alpha_1\alpha_3 \\ (\lambda+\mu)\alpha_1\alpha_2 & \mu\alpha_1^2 + (\lambda+2\mu)\alpha_2^2 + \mu\alpha_3^2 + \delta & (\lambda+\mu)\alpha_2\alpha_3 \\ (\lambda+\mu)\alpha_1\alpha_3 & (\lambda+\mu)\alpha_2\alpha_3 & \mu\alpha_1^2 + \mu\alpha_2^2 + (\lambda+2\mu)\alpha_3^2 + \delta \end{Vmatrix}.$$

Let us consider a tangent of the boundary $\partial\Omega$ and introduce a local rectangular Cartesian coordinate system $\mathbf{x}^\nu$ such that the x_1^ν, x_2^ν axes lie in the tangent plane and the x_3^ν axis as aligned with the outward normal to boundary. The Fourier transport parameters corresponding to them are denoted as $\boldsymbol{\alpha}^\nu$. Formulas for the passage from one local system to another are given by the well-known transformation relationships:

$$\mathbf{x}^\nu = \mathbf{c}_\nu^\tau \mathbf{x}^\tau + \mathbf{x}_0^\tau, \qquad \boldsymbol{\alpha}^\nu = \mathbf{c}_\nu^\tau \boldsymbol{\alpha}^\tau. \tag{3}$$

Where $\mathbf{x}_0^\tau$ are the coordinates of the origin of the new coordinate system in the initial one.

Using similar expressions, let us pass to new unknown quantities denied the following formulas:

$$\mathbf{u}^\nu = \mathbf{c}_\nu^\tau \mathbf{u}^\tau. \tag{4}$$

Lemma. On the passage to the new local coordinate system, the images and preimages of the Fourtier transforms in Eqs. (1) are transformed according to formulas(3) and (4).

The Lemma is proved by direct substitution of the transform into (1), after which differential equations (1) should be written in each local coordinate system $\mathbf{x}^\nu$ with $\mathbf{u}^\nu = \{u_1^\nu, u_2^\nu, u_3^\nu\}$.

2. Exterior forms

In functional equations (2), the vector of exterior forms ω has the following components [1]:

$$\omega_{sk} = R_{sk} dx_1 \Lambda dx_2 + Q_{sk} dx_1 \Lambda dx_3 + P_{sk} dx_2 \Lambda dx_3,$$

$$\boldsymbol{\omega} = \{\omega_{s1}, \omega_{s2}, \omega_{s3}\}, \tag{5}$$

where subscript s indicates the group of exterior forms and k s the number of the row of the Lame equations. Transformation of the components $\mathbf{R}_3 = \{R_{31}, R_{32}, R_{33}\}$ of the obtained vector of the exterior form yields the following representation:

$$R_{31} = [\sigma_{13} - i\mu\alpha_3 u_1 - i\lambda\alpha_1 u_3] e^{i\langle \alpha x\rangle}; \quad R_{32} = [\sigma_{23} - i\mu\alpha_3 u_2 - i\lambda\alpha_2 u_3] e^{i\langle \alpha x\rangle};$$

$$R_{33} = [\sigma_{33} - i(\lambda + 2\mu)\alpha_3 u_3 - i\mu(\alpha_1 u_1 + \alpha_2 u_2)] e^{i\langle \alpha x\rangle}. \tag{6}$$

Taking into account that an element of the tangent bundle is described by the oriented area $dx_1 \Lambda dx_2$, we conclude that unknown quantities at the boundary can be set in terms of various combinations, using either stresses, or displacement, or mixed conditions.

Thus, for an isotropic body, the functional equations of the boundary-value problem under consideration in one of the local coordinate systems can be presented in the following form:

$$\mathbf{K}(\boldsymbol{\alpha}^\nu)\mathbf{U}^\nu = \iint_{\partial\Omega} \boldsymbol{\omega}^\nu = \sum_\tau \iint_{\partial\Omega} \varepsilon_\tau \boldsymbol{\omega}^\nu(\xi^\tau, \boldsymbol{\alpha}^\nu), \tag{7}$$

where ε_τ is the partition of unity [4].

3. Factorization

For application of the differential factorization method to construction of the pseudodifferential equations for a matrix function, we use the approach developed in (6). As a result, the factorizing matrix functions take the following form:

$$\det \mathrm{L} = 2\alpha_1\sqrt{\tau_2^2-\upsilon^2}\left\lfloor \upsilon^2\sqrt{\tau_1^2-\upsilon^2}\,\sqrt{\tau_2^2-\upsilon^2}+s^2\right\rfloor = \Delta_2,$$

$$\alpha_{310} = -i\sigma_1 = \sqrt{\tau_1^2-\upsilon^2},\quad \alpha_{320} = -i\sigma_2 = \sqrt{\tau_2^2-\upsilon^2},\quad \operatorname{Im}\alpha_{3n0} \le 0,\quad n = 1,2.$$

Multiplying system (10) by the matrix-function L^{-1} on the left and applying the two-dimensional inverse Fourier transform, we get the following representation:

$$\mathrm{F}_2^{-1}\mathrm{K}_0\mathrm{F}_2\mathrm{t}=\mathrm{u}, \tag{11}$$

$$\mathrm{K}_0 = -\frac{1}{2\mu}\begin{Vmatrix} \alpha_1^2M+\alpha_2^2N & \alpha_1\alpha_2(M-N) & i\alpha_1 P \\ \alpha_1\alpha_2(M-N) & \alpha_1^2N+\alpha_2^2M & i\alpha_2 P \\ -i\alpha_1 P & -i\alpha_2 P & R \end{Vmatrix}, \tag{12}$$

$$\mathrm{M}(\upsilon) = \frac{-0{,}5\tau_2^2\sigma_2}{\upsilon^2\Delta_0},\quad \mathrm{N}(\upsilon) = \frac{2}{\upsilon^2\sigma_2},\quad \mathrm{P}(\upsilon) = \frac{\upsilon^2-0{,}5\tau_2^2-\sigma_1\sigma_2}{\Delta_0},\quad R(\upsilon) = \frac{-0{,}5\tau_2^2\sigma_1}{\Delta_0},$$

$$\Delta_0 = (\upsilon^2-0{,}5\tau_2^2)^2-\upsilon^2\sigma_1\sigma_2.$$

Using similar formulas, one can calculate the Leray reside forms in the right-hand side of functional equations (7) for remaining τ after the change of variables $\boldsymbol{\alpha}^\tau = \mathbf{c}_\tau^\nu\boldsymbol{\alpha}^\nu$.

An analysis of expression (11) shows that the obtained formulas coincide with those for the case where the body id half-space. However, it should be borne in mind that there is significant distinction consisting in the fact that functions u, t are defined in the neighborhood of the local coordinate systems generated by the tangents bundle of the boundary. Taking into account that the unity partition leads to coverage of the boundary by disjoint neighborhood, we conclude that the set of given and unknown functions for the system of pseudodifferential equations under consideration will contain functions defined in the neighborhoods of the local coordinate systems.

4. Integral equations

For further investigation, let us write the system of pseudodifferential equations (11) constructed after calculating the Leray residue forms as follows:

$$\iint_{\partial\Omega_\nu} \boldsymbol{\omega}_0^\nu(\xi^\nu,\alpha_1^\nu,\alpha_2^\nu,\alpha_{3r-}^\nu(\alpha_1^\nu,\alpha_2^\nu)) +$$

$$+\sum_\tau{}' \iint_{\partial\Omega_\tau} \boldsymbol{\omega}_0^\tau(\xi^\tau,\alpha_1^\nu,\alpha_2^\nu,\alpha_{3r-}^\nu(\alpha_1^\nu,\alpha_2^\nu)) = 0,\ \nu = 1,\ 2,\ ...,\ T. \tag{13}$$

Here $\boldsymbol{\omega}_0^\nu$, $\boldsymbol{\omega}_0^\tau$ are no longer the exterior forms; these quantities are given by expressions obtained after multiplying equations by the factorizing matrix functions and calculating the Leray residue forms.

As a result, applying the inverse Fourtier transform $\mathbf{F}_2^{-1}(x_1^\nu,x_2^\nu)$ with respect to parameters $\alpha_1^\nu,\alpha_2^\nu$, we arrive at the following system of integral equations:

$$\iint_{\partial\Omega_\nu} \mathbf{k}^\nu(x_1^\nu-\xi_1^\nu,x_2^\nu-\xi_2^\nu)\mathbf{t}^\nu(\xi_1^\nu,\xi_2^\nu)d\xi_1^\nu d\xi_2^\nu +$$

$$+\sum_{\tau=1}^{T}{}' \iint_{\partial\Omega_\tau} \mathbf{k}^{\nu\tau}(x_1^\nu,\ \xi_1^\tau,\ x_2^\nu,\ \xi_2^\tau)\mathbf{t}^\tau(\xi_1^\tau,\ \xi_2^\tau)d\xi_1^\tau d\xi_2^\tau = \mathbf{u}^\nu(x_1^\nu,\ x_2^\nu) +$$

$$+\sum_{\tau=1}^{T}{}' \iint_{\partial\Omega_\tau} \mathbf{b}^{\nu\tau}(x_1^\nu,\, \xi_1^\tau,\, x_2^\nu,\, \xi_2^\tau)\,\mathbf{u}^\tau(\xi_1^\tau,\, \xi_2^\tau)\,d\xi_1^\tau\,\xi_2^\tau, \quad x_1^\nu,\, x_2^\nu \in \partial\boldsymbol{\Omega}_\nu;\ 1 \le \nu \le T; \qquad (14)$$

$$\mathbf{k}^\nu(x_1^\nu,\, x_2^\nu) = \mathbf{F}_2^{-1}\mathbf{K}^\nu(\alpha_1{}^\nu,\, \alpha_2{}^\nu); \quad \mathbf{k}^{\nu\tau}(x_1^\nu,\, \xi_1^\tau,\, x_2^\nu,\, \xi_2^\tau) = \mathbf{F}_2^{-1}\mathbf{K}^{\nu\tau}(\alpha_1{}^\nu,\, \alpha_2{}^\nu)\exp i < \mathbf{c}_\tau^\nu\boldsymbol{\alpha}^\nu,\, \boldsymbol{\xi}^\tau >;$$

$$\mathbf{b}^{\nu\tau}(x_1^\nu,\, \xi_1^\tau,\, x_2^\nu,\, \xi_2^\tau) = \mathbf{F}_2^{-1}\mathbf{B}^{\nu\tau}(\alpha_1{}^\nu,\, \alpha_2{}^\nu)\exp i < \mathbf{c}_\tau^\nu\boldsymbol{\alpha}^\nu,\, \boldsymbol{\xi}^\tau >;$$

where T is the number of local coordinate systems for the tangent bundle of the boundary. Similarly, one can derive the system of integral equations for a boundary-value problem with present stresses. The following theorem is valid.

5. Conclusion

Constructed integral equations can serve to calculate of the all boundary conditions in the block element, as suggested in [3]. We particularly note that found solution of the boundary-value problem in the space of slowly increasing generalized functions $\mathbf{H}_S$ consists of the classical component and the generalized functions. If the initial boundary-value problems are formulated for sufficiently smooth boundary conditions, for example, providing that the solutions belong to the Sobolev's spaces, the classical component coincides with this solution. The generalized component appears only as a result of the differentiation on the normal to the boundary $\partial\Omega$ of the step-function carrier [5].

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