ABOUT THE SPECIFICS OF IDENTIFICATION THERMOMECHANICAL CHARACTERISTICS OF FUNCTIONALLY GRADED HOLLOW CYLINDER

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Abstract. The coefficient inverse problem for functionally graded hollow cylinder is considered. The direct problem for a cylinder solved based on the joint use of the Laplace transform and the method of adjustment. The solution of the inverse problem is reduced to an iterative procedure, at each step, which is necessary for solving Fredholm integral equation of the 1st kind. Concrete examples of the reconstruction of thermomechanical characteristics are given.

1. Introduction
Many components of various machines, mechanisms and structures are in the shape of a hollow cylinder. To analyze the strength of these structural elements is often necessary to solve the problems associated with finding fields of temperature and stresses in the model of linear thermoelasticity. For thermo-mechanical calculations require knowledge of the physical properties of the body. Usually thermomechanical calculations performed for homogeneous or layered bodies. However, in recent years as alternative layered materials are functionally graded materials [1]. FGM - inhomogeneous materials created to optimize designs. But due to the multistage production of these materials in the final product may contain deviations from established norms. .In the case of inhomogeneous bodies, direct measurements of thermomechanical characteristics are impossible, because they represent some function of the coordinates. There is thus a need to develop effective technology quality control FGM after fabrication based on the solution of inverse problems of thermoelasticity.

The coefficient inverse problems (CIP) - understudied section of mathematical physics. Usually in practice the study of inverse problems of thermoelasticity spend for layered [2] or weakly-inhomogeneous bodies [3]. Generally, CIP reduced to the solution of the corresponding extremal problems of gradient methods [4]. The use of gradient methods of minimization requires significant computer time and has several other deficiencies. The search for alternative methods of solving continues.

In some works [5-7] was proposed approach for the solution of coefficient inverse problems of thermoelasticity for an inhomogeneous rod. For the solution of inverse problems built an iterative process at each stage of which is solved Fredholm integral equation of the 1st kind. В данной работе этот подход распространен на решение задачи для цилиндра. In this paper, this approach is extended to the solution of the problem for the cylinder.

2. Direct problem of thermoelasticity for a functionally graded hollow cylinder
Consider the problem of radial oscillation of functionally graded hollow cylinder under the action of a uniformly distributed load applied on the outer surface $r = b$. The inner surface of

the cylinder $r = a$ is insulated. However, we must distinguish two ways of excitation of the oscillations, thermal and mechanical.

Initial-boundary value problem of the radial oscillations of a cylinder under applied on the outer boundary $r = b$ force $\sigma_{rr}(b,t) = p_0 \varphi(t)$ has the form:

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r} - \gamma\theta \,, \tag{1}$$

$$\sigma_{\phi\phi} = \lambda\frac{\partial u}{\partial r} + (\lambda + 2\mu)\frac{u}{r} - \gamma\theta \,, \tag{2}$$

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = \rho\frac{\partial^2 u}{\partial t^2} \,, \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(k(r)r\frac{\partial\theta}{\partial r}) = c_\varepsilon(r)\frac{\partial\theta}{\partial t} + T_0\gamma(r)(\frac{\partial^2 u}{\partial r\partial t} + \frac{1}{r}\frac{\partial u}{\partial t}) \,, \tag{4}$$

$$\theta(r,0) = u(r,0) = \frac{\partial u}{\partial t}(r,0) = 0 \,, \tag{5}$$

$$\frac{\partial\theta}{\partial r}(a,t) = 0 \,, \quad \frac{\partial\theta}{\partial r}(b,t) = 0 \,, \tag{6}$$

$$\sigma_{rr}(a,t) = 0 \,, \quad \sigma_{rr}(b,t) = p_0\varphi(t) \,. \tag{7}$$

We shall move over to (1)-(7) to the dimensionless parameters and functions, indicating: $z = \dfrac{r-a}{b-a}$, $z_0 = \dfrac{a}{b-a}$, $s(r) = \lambda + 2\mu$, $\overline{s}(z) = \dfrac{s(r)}{s_0}$, $\overline{\lambda}(z) = \dfrac{\lambda(r)}{s_0}$, $\overline{k}(z) = \dfrac{k(r)}{k_0}$,

$\overline{c}(z) = \dfrac{c_\varepsilon(r)}{c_0}$, $\overline{\gamma}(z) = \dfrac{\gamma(r)}{\gamma_0}$, $\overline{\rho}(z) = \dfrac{\rho(r)}{\rho_0}$, $v = \sqrt{\dfrac{s_0}{\rho_0}}$, $t_1 = \dfrac{(b-a)}{v}$, $\tau = \dfrac{t}{t_1}$, $W = \dfrac{\gamma_0\theta}{s_0}$,

$U = \dfrac{u}{(b-a)}$, $\Omega_z = \dfrac{\sigma_{rr}}{s_0}$, $\Omega_\phi = \dfrac{\sigma_{\phi\phi}}{s_0}$, $\delta = \dfrac{\gamma_0^2 T_0}{c_0 s_0}$, $p^* = \dfrac{p_0}{s_0}$, $k_0 = \max\limits_{r\in[a,b]} k(r)$, $c_0 = \max\limits_{r\in[a,b]} c_\varepsilon(r)$,

$\gamma_0 = \max\limits_{r\in[a,b]} \gamma(r)$, $\rho_0 = \max\limits_{r\in[a,b]} \rho(r)$, $\lambda_0 = \max\limits_{r\in[a,b]} \lambda(r)$, $s_0 = \max\limits_{r\in[a,b]} s(r)$.

After dimensionless boundary value problem (1) - (7) becomes:

$$\Omega_z = \overline{s}\frac{\partial U}{\partial z} + \overline{\lambda}\frac{U}{z + z_0} - \overline{\gamma}W \,, \tag{8}$$

$$\Omega_\phi = \overline{\lambda}\frac{\partial U}{\partial z} + \overline{s}\frac{U}{z + z_0} - \overline{\gamma}W \,, \tag{9}$$

$$\frac{\partial\Omega_z}{\partial z} + \frac{\Omega_z - \Omega_\phi}{z + z_0} = \overline{\rho}\frac{\partial^2 U}{\partial\tau^2} \,, \tag{10}$$

$$\frac{1}{z + z_0}\frac{\partial}{\partial z}(\overline{k}(z)(z + z_0)\frac{\partial W}{\partial z}) = \overline{c}(z)\frac{\partial W}{\partial\tau} + \delta\overline{\gamma}(z)(\frac{\partial^2 U}{\partial z\partial\tau} + \frac{1}{z + z_0}\frac{\partial U}{\partial\tau}) \,, \tag{11}$$

$$W(z,0) = U(z,0) = \frac{\partial U}{\partial\tau}(z,0) = 0 \,, \tag{12}$$

$$\frac{\partial W}{\partial z}(0,\tau) = 0 \,, \quad \frac{\partial W}{\partial z}(1,\tau) = 0 \,, \tag{13}$$

$$\Omega_z(0,\tau) = 0, \ \Omega_z(1,\tau) = p^*\varphi(\tau). \tag{14}$$

The direct problem of thermoelasticity is to determine the functions $U(z,\tau)$ and $W(z,\tau)$ from (8)-(14) with known thermomechanical characteristics $\bar{c}(z)$, $\bar{k}(z)$, $\bar{\gamma}(z)$, $\bar{\rho}(z)$, $\bar{s}(z)$, and $\bar{\lambda}(z)$. In this paper the direct problem (8) - (14) after the application of the Laplace transform is solved by the method of adjustment. To find the original of the solutions, the Durbin method is applied [8].

3. The inverse problem of thermoelasticity for a functionally graded hollow cylinder

In the inverse problem requires defining one of the thermomechanical characteristics of the cylinder under certain others. In the case of thermal loading of the cylinder, the additional information is the temperature value at its exterior:

$$W(1,\tau) = f(\tau), \tau \in [a,b], \tag{15}$$

and in the case of mechanical loading - the displacement on its outside surface:

$$U(1,\tau) = g(\tau), \ \tau \in [c,d]. \tag{16}$$

Coefficient inverse problem of thermoelasticity refers to non-linear problems, which are solved, usually by constructing an iterative process, each step of which is solved by a linear operator equation.

The procedure of recovery of thermomechanical coefficients consists of two stages. At the first stage is determined by the initial approximation in the class of bounded positive linear functions, based on the minimization of the residual functional, which in the case of thermal loading of the cylinder has the form:

$$J_1 = \int_a^b (f(\tau) - W(1,\tau))^2 \, d\tau, \tag{17}$$

and in the case of mechanical loading its view would be:

$$J_2 = \int_c^d (g(\tau) - U(1,\tau))^2 \, d\tau. \tag{18}$$

In the second stage clarifies the law changes of restore function based on the method of linearization in a neighborhood of some known state. Proceed as [5, 6] were obtained Fredholm integral equations of the 1st kind for finding the corrections restore functions. So, to find the correction of density is obtained:

$$\int_0^1 \delta\bar{\rho}^{(n-1)} R(z,\tau) dz = p^*(g(\tau) - U^{(n-1)}(1,\tau)), \ \tau \in [c,d]. \tag{19}$$

Here the kernel $R(z,\tau) = \int_0^\tau \dfrac{\partial U^{(n-1)}(z,\tau_1)}{\partial \tau} \dfrac{\partial U^{(n-1)}(z,\tau - \tau_1)}{\partial \tau} d\tau_1$.

When sampling the equation (19) is realized parameterization of the amendments in the form $\bar{\rho}^{(n)} = \sum_{i=1}^m \rho_i^{(n)} \chi_i$, $\chi_i = z^{i-1}$. When the regularization of the solution of integral equations of the form (19) the method of Tikhonov is used.

Based on integral equations of the form (19) builds an iterative process to refine the reconstructed functions according to the scheme:

$$\overline{a}^{(n)}(z) = \overline{a}^{(n-1)}(z) + \delta \overline{a}^{(n-1)}(z).\tag{20}$$

The output of the iterative process was carried out for the achievement of functional residual threshold value 10^{-6}.

4. The results of a computational experiment

The calculations assumed: $\varphi(\tau) = H(\tau)$, $p^* = 0.1$, $\delta = 0.05$, $m = 4$, $z_0 = 1$. It is found that the displacement measurement is most useful on the interval $[c,d] = [0,1]$ and 5 observation points inside it, and the temperature measurement на интервале and 5 observation points inside it, and the temperature measurement in the interval at the interval $[a,b] = [0,0.5]$ and 4 observation points inside it. Thermomechanical characteristics are recovered with good accuracy. The error of the reconstruction of monotone functions does not exceed 4 %, and non-monotonic 10 %. To achieve the threshold value in the functional requires no more than 8 iterations.

Fig. 1 shows the result of the reconstruction of the law changes the density of the cylinder $\overline{\rho} = 3 - \ln(1 + 7z)$. The initial approximation $\overline{\rho}_0 = 2.9 - 1.8\,z$. To achieve the threshold value in the functional (18) took 5 iterations. The reconstruction error on the last iteration does not exceed 3 %.
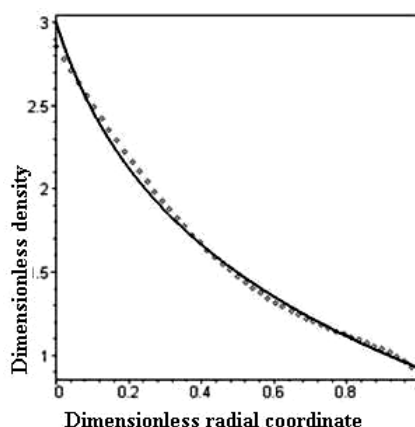
Fig. 1. Reconstruction of the law of density: the solid line is the graph of the exact functions; points - graph restored function.

5. Conclusions

An approach to the determination of the thermomechanical characteristics of functionally graded hollow cylinder is presented. A method for the solution of the direct problem of the radial oscillations of a cylinder is developed. To solve the inverse problem provides a method of constructing operator relations on the basis of the linearization method. Вычислительные эксперименты показали эффективность данного подхода по реконструкции гладких законов неоднородности. Computational experiments show the effectiveness of this approach for reconstruction of smooth laws of heterogeneity.

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