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EXFOLIATION OF THIN PERIODIC ELASTIC COATING DUE TO TRAPPING AND PROPAGATION OF WAVES

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Abstract. An elastic waveguide consists of a straight strip covered with a thin $(h \ll 1)$ periodic coating at one of strip's lateral surfaces. The material of the coating is much softer than the one of the massif but their densities are similar. Under a certain relationship between the physical and geometric parameters of the composite waveguide, an asymptotic analysis as $h \to +0$ demonstrates the effect of plurality of spectral gaps, i.e. stopping zones for elastic waves. Moreover, local perturbations of the waveguide profile can bring into the discrete spectrum eigenvalues either below the essential spectrum, or inside discovered gaps. In other words, matching certain parameters in a periodic composite elastic waveguide provides any prescribed number of open gaps in the spectrum as well as any prescribed number of isolated eigenvalues in these gaps and the corresponding trapped modes. Both, travelling and trapped waves at frequencies in the spectral bands, passing zones, and in the discrete spectrum, respectively, provoke localization and concentration of shear stresses at the interface near points where the coating profile function attends its maxima so that the fracture process can be predicted in the vicinity of these points and realizes as fragmentation of the adhesive and a sparsely distributed exfoliation of the thin light periodic coating.

1. Problem setting

Let ϖ^1 be the straight strip $(-\infty, +\infty) \times (-d, 0)$ and let ϖ^h be the periodic set,

$$\varpi^h = \{(x, y) : x \in (-\infty, +\infty), 0 < y < hH(x)\}, \tag{1}$$

where H is a smooth L-periodic function. By rescaling we reduce the period $L > 0$ to 1 and thus make the Cartesian coordinates x, y and all geometric parameters dimensionless, in particular, $h \in (0,1)$ is further regarded as a small parameter. The elastic planar bodies ϖ^1 and ϖ^h are isotropic homogeneous with the following Lame constants and the material densities:

$$\lambda^1 \geq 0,\ \mu^1 > 0,\ \rho^1 > 0 \text{ and } \lambda^h = h^{2+\delta}\, \lambda^0 \geq 0,\ \mu^h = h^{2+\delta}\, \mu^0 > 0,\ \rho^h = \rho^0 > 0. \tag{2}$$

Here, $\delta > 0$ and the couples μ^1, ρ^1 and μ^0, ρ^0 are compatible while the ratios μ^1/ρ^1 and μ^0/ρ^0 are of order $1 = h^0$. Due to the small factor $h^{2+\delta}$ in (2) the thin layer ϖ^h is much softer than the massif ϖ^1 but their unit weights are similar according to the last relation in (2). The boundaries

$$\Gamma^h = \{(x, y) : x \in (-\infty, +\infty), y = hH(x)\} \tag{3}$$

and $\Gamma^1 = (-\infty, +\infty) \times \{-d\}$, respectively, are traction-free and clamped, namely

$$n_x^h\, \sigma_{xx}^h + n_y^h\, \sigma_{xy}^h = 0,\ n_x^h\, \sigma_{yx}^h + n_y^h\, \sigma_{yy}^h = 0 \qquad \text{on } \Gamma^h, \tag{4}$$

$$u_x^h = u_y^h = 0 \qquad \text{on } \Gamma^1, \tag{5}$$

where $u = (u_x, u_y)$ is the displacement vector,

$$n^h(x,y) = (n_x^h(x,y), n_y^h(x,y)) = (1 + h^2|\partial_x H(x)|^2)^{-1/2}(-h\partial_x H(x), 1) \tag{6}$$

is the unit vector of the outward normal and Cartesian components of the stress tensor σ are given by

$$\sigma_{xx} = (\lambda+\mu)\partial_x u_x + \lambda(\partial_x u_x + \partial_y u_y), \;\; \sigma_{yy} = (\lambda+\mu)\partial_y u_y + \lambda(\partial_x u_x + \partial_y u_y),$$

$$\sigma_{xy} = \sigma_{yx} = \mu(\partial_y u_x + \partial_x u_y), \qquad \partial_x = \partial/\partial x, \partial_y = \partial/\partial y. \tag{7}$$

Restrictions of these elastic fields on ϖ^1 and ϖ^h are supplied with the superscripts 1 and h, respectively.

Harmonic in time oscillations of the composite waveguide $\Pi = \varpi^1 \cup \varpi^h$ are described by the differential equations

$$-\partial_x\sigma_{xx} - \partial_y\sigma_{xy} = \rho\omega^2 u_x, \;\; -\partial_x\sigma_{yx} - \partial_y\sigma_{yy} = \rho\omega^2 u_y \qquad \text{in } \Pi, \tag{8}$$

where $\omega > 0$ is the frequency of oscillations. At the interface $\Gamma^0 = (-\infty, +\infty) \times \{0\}$, the conditions of ideal contact are imposed, namely

$$u_x^1 = u_x^h, \;\; u_y^1 = u_y^h, \qquad \sigma_{xy}^1 = \sigma_{xy}^h, \;\; \sigma_{yy}^1 = \sigma_{yy}^h \qquad \text{on } \Gamma^0. \tag{9}$$

The relationship (2) between the physical and geometrical characteristics of the composite Π decouples the transmission conditions (9) in the limit $h \to +0$ so that, for a small $h > 0$, the system

$$-\partial_x\sigma_{xx}^0 - \partial_y\sigma_{xy}^0 = \Lambda^h u_x^h, \quad -\partial_x\sigma_{yx}^0 - \partial_y\sigma_{yy}^0 = \Lambda^h u_y^h \qquad \text{in } \varpi^h \tag{10}$$

inherits the Dirichlet boundary conditions

$$u_x^h = u_y^h = 0 \qquad \text{on } \Gamma^0 \tag{11}$$

in addition to (4) on Γ^h, but the Neumann conditions

$$\sigma_{xy}^1 = \sigma_{yy}^1 = 0 \qquad \text{on } \Gamma^0 \tag{12}$$

together with (5) serve for the other system

$$-\partial_x\sigma_{xx}^1 - \partial_y\sigma_{xy}^1 = \Lambda^1 u_x^1, \quad -\partial_x\sigma_{yx}^1 - \partial_y\sigma_{yy}^1 = \Lambda^1 u_y^1 \qquad \text{in } \varpi^1. \tag{13}$$

Here, components of the rescaled stress tensor σ^0 are given by formulas (7) with the change $\lambda, \mu \mapsto \lambda^0, \mu^0$ and furthermore

$$\Lambda^1 = \rho^1\omega^2, \quad \Lambda^h = h^{-2-\delta}\rho^0\omega^2 \tag{14}$$

are the new notation for the spectral parameter. In what follows we study the independent problems (13), (12), (5) and (10), (11), (4) but finally we complete our asymptotic analysis of the original problem (8), (9), (4), (5) in Section 4 by reinstalling the transmission conditions on Γ^0 with an asymptotically small error that passes our general conclusions on the wave propagation in the isolated waveguide ϖ^h to the composite waveguide Π.

Regarding the problem (13), (12), (5), we recall the Korn inequality, see, e.g., [1],

$$E(u^1; \varpi^1) \geq K^1 \, \|u^1; H^1(\varpi^1)\|^2 \qquad \forall \; u^1 \in H_0^1(\varpi^1; \Gamma^1), \tag{15}$$

where the subspace $H_0^1(\varpi^1;\Gamma^1)$ is extracted by the Dirichlet conditions (5) from the Sobolev space $H^1(\varpi^1;\Gamma^1)$ with the standard norm and $2^{-1}E(u;\Pi)$ is the elastic energy functional,

$$E(u;\Pi)=\int_\Pi\left(2\mu|\partial_x u_x|^2+2\mu|\partial_y u_y|^2+\mu|\partial_y u_x+\partial_x u_y|^2+\lambda|\partial_x u_x+\partial_y u_y|^2\right)dxdy. \quad (16)$$

The Korn constant K^1, of course, does not depend on the small parameter h and the Λ-spectrum S^1 of the problem in the straight elastic strip ϖ^1 constitutes the intact ray $[k^1,+\infty)$ with some cutoff value $k^1\geq K^1$. In this way the interval $(0,K^1)$ is surely free of the spectrum S^1.

The Λ^h-spectrum S^h of the problem (10), (11), (4) in the thin periodic elastic waveguide ϖ^h, see (1), has a quite different structure, namely it consists of the *spectral bands* B_n^h, passing zones for elastic waves, which can be interleaved by the *spectral gaps* G_n^h, stopping zones preventing from propagation of waves at the corresponding frequencies.

The main goal of this paper is to provide an asymptotic description for the low frequency range of the spectrum

$$S^h=\cup_k B_n^h,\quad k=1,2,3,\ldots \quad (17)$$

of the problem (10), (11), (4) in ϖ^h and to adopt this asymptotic result to the spectrum S of the original problem (8), (9), (4), (5) in Π which also has a band-gap structure of type (17). Specifically, we will verify that, for any given $N=1,2,3,\ldots$, one finds $h_N>0$ such that the spectra S^h and S of the above-mentioned problems gain at least N *open*, i.e. *non-empty*, gaps.

Notice that B_n^h and G_n^h are finite and connected subsets, closed and open respectively, in the semi-axis $(0,+\infty)$. If $B_n^h\cap B_{n+1}^h\neq\emptyset$, then the *empty* gap $G_n^h=\emptyset$ is called *closed*. For elasticity problems in periodic waveguides, it is not known yet if a band B_n^h may collapse into a single point which therefore becomes an eigenvalue of infinite multiplicity. The latter is the very reason to call S^h and S the *essential* spectra instead of the *continuous* spectrum.

The asymptotic analysis performed below also allows us to detect any prescribed number of isolated, that is belonging to the discrete spectrum, eigenvalues either below the essential spectrum S^h, or inside any of discovered spectral gaps B_n^h. We mention that plurality of gaps and eigenvalues has been described in the papers [2, 3] and [4, 5] although in different geometries.

Propagating elastic waves in the thin and soft coating ϖ^h at low frequencies are quite expectable phenomenon because, even for the straight boundary Γ^h with the constant profile function $H(x)=H^0$, the continuous spectrum of the problem (10), (11), (4) with the spectral parameter Λ^h from (14) is the intact ray $[\mu\pi^2(2H_0)^{-2}h^\delta,+\infty)$ covering $S^1=[k^1,+\infty)$ with a plethora when the parameter $h>0$ is small. However, new effects of the periodic boundary (3), first of all, reveal a family of wide stopping zones $B_n^h\neq\emptyset$ and, second, detect a substantial concentration of stresses in the $c\sqrt{h}$-neighborhood of the *maximum* points of the width function $H(x)$. The latter helps to predict the fracture process caused by vibrations, namely exfoliation and further fragmentation of the coating and/or adhesive, but contradicts the well-known proverb "The thread breaks where it is the weakest".

2. The model problem in the periodicity cell

The Gelfand transform in the framework of the Floquet-Bloch theory, cf. [6 - 8] and [9; Ch. 3, 5], associates with the problem (19), (11), (4) in the infinite periodic waveguide (1) a spectral problem in the finite periodicity cell

$$\varpi_{cell}^h=\{(x,y)\in\varpi^h:\ |x|<1/2\}=\{(x,y):|x|<1/2,\ 0<y<hH(x)\}; \quad (18)$$

recall that period L has been reduced to 1. This model problem in the crooked trapezoid (18) depends on the Gelfand dual variable $\theta\in[-\pi,\pi]$, that is the Floquet parameter, and reads as follows:

$$-\partial_x\sigma_{xx}^{\theta}-\partial_y\sigma_{xy}^{\theta}=h^{-2}M^{\theta h}u_x^{\theta}\,,\quad -\partial_x\sigma_{yx}^{\theta}-\partial_y\sigma_{yy}^{\theta}=h^{-2}M^{\theta h}u_y^{\theta}\qquad \text{in } \varpi_{cell}^{h}, \tag{19}$$

$$n_x^h\sigma_{xx}^{\theta}+n_y^h\sigma_{xy}^{\theta}=0,\quad n_x^h\sigma_{yx}^{\theta}+n_y^h\sigma_{yy}^{\theta}=0 \qquad \text{on } \Gamma_{cell}^{h}, \tag{20}$$

$$u_x^h=u_y^h=0 \qquad \text{on } \Gamma_{cell}^{0}, \tag{21}$$

$$u^{\theta}\left(\tfrac{1}{2},y\right)=e^{i\theta}u^{\theta}\left(-\tfrac{1}{2},y\right),\ \partial_x u^{\theta}\left(\tfrac{1}{2},y\right)=e^{i\theta}\partial_x u^{\theta}\left(-\tfrac{1}{2},y\right)\ \text{for } y\in\left(0,hH\left(\tfrac{1}{2}\right)\right). \tag{22}$$

The quasi-periodicity conditions (22) relate traces of a vector eigenfunction u^{θ} on the vertical sides $\{(x,y)\in\ \partial\varpi_{cell}^{h}\,:\ y=\pm 1/2\}$ of the trapezoid ϖ_{cell}^{h} and Γ_{cell}^{h} , Γ_{cell}^{0} denote its bases while

$$M^{\theta h}=h^{-\delta}\rho^0\omega^2 \tag{23}$$

is the new notation for the rescaled second spectral parameter in (14). Since the domain (18) is bounded, the spectrum of the formally self-adjoint problem (19) - (22) is discrete and constitutes the monotone unbounded positive sequence

$$0<\ M_1^{\theta h}\le\ M_2^{\theta h}\le\ M_3^{\theta h}\ \le\ \dots\ \le\ M_m^{\theta h}\ \le\ \dots\ \to\ +\infty, \tag{24}$$

where multiplicity of eigenvalues is taken into account.

If, for a certain $\theta\in[-\pi,\pi]$, the model problem (19) - (22) gets an eigenpair $\{M_k^{\theta h},u_k^{\theta}\}$, then due to the quasi-periodicity condition (22) the Floquet wave U^{θ} determined by

$$U^{\theta}(x+j,y)=e^{i\theta j}\,u^{\theta}(x,y),\ (x,y)\in\varpi^h,\ \ j=0,\pm 1\,,\pm 2\,,\pm 3\,,\ \cdots \tag{25}$$

is continuous and smooth in the infinite waveguide and satisfies the problem (10), (11), (4) with $\Lambda^h=h^2M^{\theta h}$. The existence of the Floquet wave proves the loss of the Fredholm property by the operator of the problem in ϖ^h and, therefore, places the point $h^2M^{\theta h}$ into the essential spectrum (17). At the same time in the case $\Lambda^h\notin S^h$ the problem (19) - (22) with $M^{\theta h}=h^{-2}\Lambda_k^{\theta \mathrm{h}}$ is uniquely solvable for any $\theta\in[-\pi,\pi]$ and, hence, the direct and inverse Gelfand transforms applied consequently, assure the unique solvability of the inhomogeneous problem (10), (11), (4) in the Sobolev spaces scale, cf. [6] and [9; Section 3.4]. Thereafter we conclude that

$$B_k^h=\{h^2M_k^{\theta h}:\ \theta\in[-\pi,\pi]\},\quad k=1,2,3, \tag{26}$$

in formula (17). Since entries in the eigenvalue sequence (24) are obviously 2π-periodic and continuous in $\theta\in[-\pi,\pi]$, the spectral bands (26) are finite, closed and connected, indeed.

Our immediate objective becomes to study asymptotics of eigenpairs, that is (eigenvalue, vector eigenfunction), of the model problem in the periodicity cell ϖ_{cell}^{h}.

3. Asymptotic analysis

We assume that the profile function H in (18) has the only point $x=0$ of the global maximum in the segment $[-1/2,1/2]\ni x$ and, moreover,

$$H(x)=H_0-Ax^2+O(|x|^3),\quad H_0=H(0),\quad A=-2^{-1}\partial_x^2H(0)>0. \tag{27}$$

Although many aspects of the vector problem in elasticity under consideration differ from the scalar Dirichlet problem for the Laplace operator in the thin domain (18) studied in detail by many authors, see [10 - 16] etc., primary asymptotic ansätze are kept for the problem (19) - (22), in particular, the nonuniform dilation of Cartesian coordinates

$$x \mapsto X = h^{-1/2}x, \qquad y \mapsto Y = h^{-1}y. \tag{28}$$

Performing the latter coordinate change in the equations (19) and (20), we extract the main asymptotic parts of differential operators and compose the following one-dimensional limit problem on vertical cross-sections of the trapezoid (18)

$$-\mu\partial_Y^2 W_x(Y;x) = M(x)W_x(Y;x),\ \ Y \in \big(0, H(x)\big),\ \ W_x(0;x) = 0, \partial_Y W_x(H(x);x) = 0, \tag{29}$$

$$-(\lambda + 2\mu)\partial_Y^2 W_y(Y;x) = M(x)\, W_y(Y;x),\ \ Y \in \big(0, H(x)\big),$$

$$W_y(0;x) = 0,\ (\lambda + 2\mu)\partial_Y W_y(H(x);x) = 0. \tag{30}$$

Here, the variable x is regarded as a parameter. According to (27) the first (minimal) eigenvalue of the family of parameter-dependent problems (29), (30) is nothing but

$$M_0 := M(0) = \frac{\mu\pi^2}{4H_0^2}, \tag{31}$$

which generates the following vector eigenfunction of the problem (29), (3) at $x = 0$:

$$W^0(Y) = \big(W_x^0(Y), W_y^0(Y)\big) = \Big(\sin\frac{\pi Y}{2H_0}, 0\Big). \tag{32}$$

The above calculation proposes the asymptotic ansätze

$$M^\theta = M_0 + hm + \cdots, \tag{33}$$

$$u^\theta(x,y) = U^0(H_0H(x)^{-1}Y, X) + h^{1/2}U^1(H_0H(x)^{-1}Y, X) + hU^2(H_0H(x)^{-1}Y, X) + \cdots, \tag{34}$$

where ellipses stand for higher-order terms, inessential in our asymptotic analysis, and the main term in (34) takes the form

$$U^0(Y, X) = w(X)W^0(Y). \tag{35}$$

The scalar function w in (35) and the number m in (33) are to be determined and it will be of very importance that $w(X)$ decays at a high rate when $X \to \pm\infty$ so that all terms on the right-hand side of (34) are localized in the vicinity of the point $x = 0$. Notice that, owing to the modified argument Y of functions in (34), the vector $U^0(H_0H(x)^{-1}Y, X)$ satisfies in main the boundary conditions (11) and (4) near the point $x = 0$. Furthermore,

$$-\partial_x\sigma_{xx}^\theta(x,y) - \partial_y\sigma_{xy}^\theta(x,y) - h^{-2}M^{\theta h}u_x^\theta(x,y) =$$

$$= \left(\frac{\mu\pi^2}{4h^2H(x)^2} - \frac{\mu\pi^2}{4h^2H_0^2}\right) U_x^0\left(\frac{H_0Y}{H(x)}, X\right) + \cdots = \frac{\mu\pi^2}{4h^2H_0^2}\ \frac{H_0^2 - H(x)^2}{H(x)^2}\ U_x^0\left(\frac{H_0Y}{H(x)}, X\right) + \cdots =$$

$$= \frac{\mu\pi^2}{4h^2H_0^2} Ax^2\, U_x^0\left(\frac{H_0Y}{H(x)}, X\right) + \cdots = h^{-1}\frac{\mu\pi^2}{4h^2H_0^2} AX^2\, U_x^0\left(\frac{H_0Y}{H(x)}, X\right) + \cdots, \tag{36}$$

$$-\partial_x\sigma_{yx}^\theta(x,y) - \partial_y\sigma_{yy}^\theta(x,y) - h^{-2}M^{\theta h}u_x^\theta(x,y) = \cdots.$$

In other words, all discrepancies have been made small by our choice of the expressions (31) and (35).

We again insert (33), (34) into (21), collect coefficients of $h^{-3/2}$ and arrive at the ordinary differential equations

$$-\mu\partial_Y^2 U_x^1(Y,X) - M_0U_x^1(Y,X) = 0, \qquad Y \in (0, H_0),$$

$$-(\lambda + 2\mu)\partial_Y^2 U_y^1(Y,X) - M_0U_y^1(Y,X) = F_y^1(Y,X) := \tag{37}$$

$:= (\lambda+\mu)\partial_Y W_x^0(Y)\partial_X w(X), \quad Y \in (0, H_0).$

We close these equations with the boundary conditions obtained in a similar way

$$U_x^1(0,X)=0, \quad \mu\partial_Y U_x^1(H_0,X)=0,$$
$$U_y^1(0,X)=0, \ (\lambda+2\mu)\partial_Y U_y^1(H_0,X)=G_y^1(Y;x):=-\lambda\partial_Y W_x^0(H_0)\partial_X w(X)\,. \tag{38}$$

According to (32) and (35), we have

$$F_y^1(Y,X)=(\lambda+\mu)\frac{\pi}{2H_0^2}\cos\frac{\pi Y}{2H_0}\,\partial_X w(X), \quad G_y^1(Y;x)=-\lambda\partial_X w(X)$$

and, therefore, a particular solution $U^1(Y,X)$ of the problem (37), (38) gets the following components:

$$U_x^1(Y,X)=0,$$
$$U_y^1(Y,X)=\frac{2H_0}{\pi}\left(\cos\frac{\pi Y}{2H_0}-\cos\frac{\pi\alpha Y}{2H_0}+\left(\cos\frac{\pi\alpha}{2}\right)^{-1}\left(2\alpha-\sin\frac{\pi\alpha}{2}\right)\sin\frac{\pi\alpha Y}{2H_0}\right)\partial_X w(X), \tag{39}$$

where

$$\alpha=\sqrt{\frac{\mu}{\lambda+2\mu}}\ , \ 0\le\alpha<\frac{1}{\sqrt{2}}. \tag{40}$$

Notice that the second term in (34) is obtained from (39) by the replacement $H_0 \mapsto H(x)$.

Let us find out the third term in the asymptotic ansatz (34). Recalling that the problem for the component U_y^2 is uniquely solvable (cf., the equations for U_y^1 in (37) and (38)), we refine the first relation in (36) as follows:

$$-\partial_x\sigma_{xx}^\theta(x,y)-\partial_y\sigma_{xy}^\theta(x,y)-h^{-2}M^{\theta h}u_x^\theta(x,y)=$$
$$=\frac{\mu\pi^2}{2H_0^2}\,A\,X^2 w(X)\sin\frac{\pi Y}{2H(x)}-\frac{1}{h}\left(\mu\partial_Y^2 U_x^2\left(\frac{H_0 Y}{H(x)},X\right)+M_0 U_x^2\left(\frac{H_0 Y}{H(x)},X\right)\right)-$$
$$-\frac{1}{h}(\lambda+\mu)\partial_Y U_x^1\left(\frac{H_0 Y}{H(x)},X\right)-\frac{1}{h}\big((\lambda+2\mu)\partial_X^2 w(X)+mw(X)\big)\sin\frac{\pi Y}{2H_0}+\cdots.$$

Moreover, analyzing the first boundary condition in (4) and using formulas (3), (6) yield

$$n_x^h\sigma_{xx}^\theta\big(x,hH(x)\big)+n_y^h\sigma_{xy}^\theta\big(x,hH(x)\big)=$$
$$=\frac{1}{h}\mu\frac{H_0}{H(x)}\partial_X\big(U_x^0(H_0,X)+hU_x^2(H_0,X)\big)+\mu\,\partial_X U_y^1(H_0,X)+\cdots$$
$$=\mu\left(\partial_Y U_x^0(H_0,X)+\partial_X U_y^1(H_0,X)\right)+\cdots.$$

In the upshot, we compose the one-dimensional problem

$$-\mu\partial_Y^2 U_x^2(Y,X)-M_0 U_x^2(Y,X)=F_x^2(Y,X), \qquad Y\in(0,H_0),$$
$$U_x^2(0,X)=0, \ \mu\partial_Y U_x^2(H_0,X)=G_x^2(X) \tag{41}$$

with the right-hand sides

$$F_x^2(Y,X)=\left((\lambda+2\mu)\partial_X^2 w(X)+mw(X)+\frac{\mu\pi^2}{2H_0^3}AX^2 w(X)\right)\sin\frac{\pi Y}{2H_0}+$$
$$+(\lambda+\mu)\partial_Y\partial_X U_y^1(Y,X), \qquad G_x^2(X)=-\mu\,\partial_Y U_y^1(H_0,X). \tag{42}$$

Since M_0 is a simple eigenvalue of the problem (29), (30) at $x = 0$, the Fredholm alternative provides the only compatibility condition in the problem (41), namely

$$\int_0^{H_0} F_x^2(Y, X)\ \sin\frac{\pi Y}{2H_0}\ dY + G_x^2(X) = 0. \tag{43}$$

In view of the evident formula $\int_0^{H_0} \left(\sin\frac{\pi Y}{2H_0}\right)^2\ dY = \frac{H_0}{2}$ the representations (39) and (42) reduce the relation (43) to the ordinary differential equation

$$-b\ \partial_X^2 w(X) + aX^2 w(X) = mw(X), \quad X \in (-\infty, +\infty), \tag{44}$$

with the coefficients

$$a = \frac{\mu\pi^2}{2H_0^3} A, \quad b = \lambda + 2\mu + B\ , \tag{45}$$

where, according to (42) and (39),

$$B = (\lambda + \mu)\frac{2}{H_0}\int_0^{H_0}\ \sin\frac{\pi Y}{2H_0}\Big(-\sin\frac{\pi Y}{2H_0} + \ \alpha \sin\frac{\pi\alpha Y}{2H_0}\ +$$

$$+\alpha\left(\cos\frac{\pi\alpha}{2}\right)^{-1}\left(2\alpha - \sin\frac{\pi\alpha}{2}\right)\cos\frac{\pi\alpha Y}{2H_0}\Big)\ dY$$

$$+\frac{4\mu}{\pi}\left(\cos\frac{\pi\alpha}{2} - \left(\cos\frac{\pi\alpha}{2}\right)^{-1}\left(2\alpha - \sin\frac{\pi\alpha}{2}\right)\sin\frac{\pi\alpha}{2}\right).$$

A calculation demonstrates that

$$b = \mu + \frac{4\mu}{\pi}\left(\cos\frac{\pi\alpha}{2}\right)^{-1}\left(4 - (4\alpha^2 + 1)\ \frac{1}{\alpha}\ \ \sin\frac{\pi\alpha}{2}\right). \tag{46}$$

Under the restrictions on α and A in (40) and (27), the coefficients a and b in (45) and (46) are positive. Thus, the spectrum of the differential equation (44), named the *harmonic oscillator*, cf. [17], is discrete and consists of the following eigenvalues:

$$m_n = \sqrt{ab}\left(n - \frac{1}{2}\right), \qquad n = 1,2,3, \cdots \tag{47}$$

Each of the eigenvalues is simple. The corresponding eigenfunctions $w_n(X)$ decay as $O\big(\exp\big(-X^2\sqrt{a/4b}\big)\big)$ when $X \to \pm\infty$.

Thus, all ingredients of the ansätze (33) and (34) have been defined properly and, moreover, the terms (35), (39) and so on in (34) become of exponentially small order $e^{-\delta x^2/h}$, $\delta > 0$, at a distance from the point $x = 0$. In other words, we do not need to care especially on the quasi-periodicity conditions (22) which can bring only very small corrections to the asymptotic expansions.

4. Justification of asymptotics

Although the vector nature of the elasticity problem gives certain peculiarities to all the asymptotic structures examined in Section 3, a justification scheme remains the same as in the scalar case, see [11, 14], while the most elaborated version is presented in [15, 16] where piecewise smooth profiles H in (1) and (3) are accepted even in multi-dimensional case. We here omit cumbersome but routine calculations which in a standard way lead to asymptotically sharp error estimates and, in particular, replace the ellipsis in (33) by $O(h^0)$. As for vector eigenfunctions, weighted Sobolev [15] and Hölder [16] estimates of asymptotic remainders in (34) are available, too, so that the localization effect announced in Section 1 can be fairly observed and asymptotic formulas for the stress field of our usage in Section 6 may be obtained directly by differentiating terms on the right-hand side of (34).

We emphasize that the exponential decay of eigenfunctions of the harmonic oscillator equation (44) supports the inequalities

$$\left|M_n^{\theta h} - M_n^{0h}\right| \leq c_n e^{-\delta/h}, \quad \delta > 0, \ \ n = 1,2,3,\cdots. \tag{48}$$

The latter means that, owing to formulas (23), (14) and (26), the band B_k^h of the Λ-spectrum of the problem (10), (11), (4) lays in the very vicinity of the point $h^{-2}M_0 + h^{-1}m_k$, see (31) and (47), and has length $O\left(h^{-2}e^{-\delta/h}\right)$, $\delta > 0$.

In total, we conclude that, $N = 1,2,3,\cdots$ given, there exists $h_N > 0$ such that, for $h \in (0, h_N)$, the gap G_k^h of width $O\left(h^{-1}(m_{k+1} - m_k)\right)$ is open between the bands B_k^h and B_{k+1}^h with any $k = 1, \cdots, N$. In other words, the elastic waveguide (1) may gain arbitrarily many open spectral gaps if its physical and geometrical parameters are chosen and related properly.

The above conclusions made for the detached periodic waveguide ϖ^h are understandably adopted by the composite waveguide $\Pi = \varpi^1 \cup \varpi^h$. A supporting argument for this inference is two-fold. First, taking into account scaling in (14), we see that the spectral parameter $\Lambda = \rho\omega^2$ in the problem (8), (9), (4), (5) is directly associated with the parameter $\Lambda^h = h^\delta M^h$ in the problem (10), (11), (4). Hence, the bands (26) convert into the segments

$$\left\{\Lambda: h^{-2-\delta}\Lambda \in B_k^h\right\}, \tag{49}$$

which, for a small h, belong to the low frequency range $(0, K^1)$ of the spectrum of the second limit problem (12), (13), (5). This problem with $\Lambda^h \in (0, K^1)$ is uniquely solvable in the Sobolev space $H^1(\varpi^1)$ according to the Korn inequality (15).

Second, in view of our assumption (2) the stress transmission conditions in (8) turn into $\sigma_{xy}^1 = h^{2+\delta}\sigma_{xy}^0$, $\sigma_{yy}^1 = h^{2+\delta}\sigma_{yy}^0$ on Γ^0.
According to the presence of the small parameter, the operator of the problem in $\Pi = \varpi^1 \cup \varpi^h$ can be regarded as a small perturbation of the coupled operators of both the limit problems in $\varpi^1 \times \varpi^h$. In this way, general results of the theory of self-adjoint operators in Hilbert space, see, e.g., [18], assure the proximity of the spectra of the concerned operators in $\varpi^1 \cup \varpi^h$ and $\varpi^1 \times \varpi^h$.

In the next section we will investigate a local perturbation of the purely periodic waveguide (1) and construct comprehensible approximations of an eigenvalue outside the essential spectrum (17) and a vector eigenfunction with an exponential decay. The above-mentioned theory again helps to justify these formal asymptotic expansions and indicate points of the discrete spectrum in the vicinity of the constructed approximate eigenvalues of the problem (10), (11), (4). A similar argument readily applies to the composite waveguide Π.

5. Trapped elastic waves

Let the periodic profile H in (1) enjoy a local perturbation in one cell, namely

$$H^{\approx}(x) = \begin{cases} H^*(x), & |x| < 1/2, \\ H(x), & |x| > 1/2, \end{cases} \tag{50}$$

where for simplicity we assume that $H^{\approx}$ is still a smooth function and the new profile function H^* satisfies

$$H^*(x) = H_0^* - A^* x^2 + O(|x|^3), \ H_0^* = H^*(0) > 0, \ A^* = -\frac{1}{2}\,\partial_x^2 H^*(0) > 0,$$

$$H_0^* > H^*(x) \text{ for } |x| \in \left(0, \tfrac{1}{2}\right). \tag{51}$$

We introduce the perturbed waveguide

$$\varpi^{h\approx} = \{(x,y): x \in (-\infty,+\infty), 0 < y < H^{\approx}(x)\} \tag{52}$$

and consider the whole problem (10), (11), (4) in the infinite domain $\varpi^{h\approx}$, that is without the Floquet parameter as was in (19) - (22). We repeat the asymptotic analysis from Section 3 and derive the asymptotic structure

$$M_k^{h*} = \mu\pi^2 \ (2H_0^*)^{-1} \ + h\, m_k^* , \tag{53}$$

$$u_{(k)}^{h*}(x,y) = \left(\sin\left(\frac{\pi y}{2h}\, H^*(x)^{-1}\right) w_k^*\left(h^{-1/2}x\right),\ 0\right) \tag{54}$$

for approximate eigenvalue and eigenfunction. Here, m_k^* is the eigenvalue (47) of the ordinary differential equation (44) with the coefficient $b^* = b$ in (46) and

$$a^* = \frac{1}{2}\,\mu\pi^2 A^* \ (H_0^*)^{-3} . \tag{55}$$

The corresponding eigenfunction is denoted by $w_k^{\approx}$ and decays exponentially as $X \to \pm\infty$ so that we may localize the multiplier $w_k^*\left(h^{-1/2}x\right)$ in (54) onto the perturbed periodicity cell

$$\varpi_{cell}^{h*} = \{(x,y): |x| \in 1/2, 0 < y < H^*(x)\} \tag{56}$$

by multiplying with a smooth cut-off function χ such that $\chi(x) = 1$ for $|x| < 1/6$ and $\chi(x) = 0$ for $|x| > 1/3$.
Thereafter an application of the theory of self-adjoint operators in Hilbert space, cf. [18], becomes possible, indeed.

Let us consider different cases. First, we suppose that

$$H_0^* > H_0. \tag{57}$$

Then the point

$$\Lambda_k^{h*} = h^{-2}M_k^{h*} = \frac{\mu\pi^2}{4h^2} \ (H_0^*)^{-2} \ + \frac{1}{h}\, m_k^* \tag{58}$$

stays below the essential spectrum (17), (26) for a small $h < h(k)$. As was explained in Section 4, the perturbation operator theory ensures the existence of an isolated eigenvalue $\Lambda_k^{h\approx}$ in the discrete spectrum of the problem (10), (11), (4) in the waveguide (52) as well as the relationship

$$\left|\Lambda_k^{h*} - \Lambda_k^{h\approx}\right| \le c_k h \quad \text{for} \quad h \in \left(0, h(k)\right),\ k = 1,2,3,\cdots . \tag{59}$$

As a result, we conclude that, for any $N = 1,2,3,\cdots$, there exists $h_N > 0$ such that the perturbation (50) of the waveguide profile H gives rise to N eigenvalues in the discrete spectrum below the cutoff point of the essential spectrum.

In the case

$$\mathrm{H}_0^* = H_0 \qquad \text{but} \qquad A^* \neq A \tag{60}$$

we can choose some spectral gap

$$G_n^h \subset (h^{-2}M_0 + h^{-1}m_n - C_n, h^{-2}M_0 + h^{-1}m_n + C_n) \tag{61}$$

in the essential spectrum S^h of the waveguide (52). Varying the parameter A^* in (51), that is the curvature of the profile H^* at its maximum point, we may confirm that

$$m_k^* \in \ (m_n, m_{n+1}). \tag{62}$$

Then the point (58) falls into the gap (61) and, therefore, the above-mentioned argument delivers an eigenvalue $\Lambda_k^{h\approx}$ of the problem (10), (11), (4) in $\varpi^{h\approx}$ as well as the estimate (59) which demonstrates that $\Lambda_k^{h\approx} \in G_n^h$ for a small h.

The described approach, however, is able to detect for sure only one eigenvalue in the chosen gap (61). At the same time, we observe that this approach originally is based on local asymptotic constructions while the justification scheme can distinguish between approximate vector eigenfunctions with disjoint supports (recall the localization by means of the cut-off function χ). Thus, it is possible to replicate the same profile perturbation on other intervals $(q-1/2, q+1/2) \ni x$ and distribute a finite family of the copies along the waveguide (1). In this way we create any given number of isolated eigenvalues inside any fixed gap G_n^h discovered in Section 3.

6. Conclusions on concentration of stresses and fracture

The Hölder estimates of asymptotic remainders in the representation (34) of a displacement field $u_{(k)}^{\theta}(x,y)$, i.e., a vector eigenfunction of the problem (19) – (22) with a particular parameter $\theta \in [-\pi, +\pi]$, allow us to compute stress field $\Sigma_{(k)}^{\theta}(x,y)$ for the Floquet wave $U_{(k)}^{\theta}(x,y)$, defined by formulas (7) and (25) respectively. Namely, a direct differentiation of the expressions (35), (32) and (39) in the ansatz (34) and taking (2) into account yield

$$\Sigma_{(k)xy}^{\theta 0}(x,y) = \Sigma_{(k)yx}^{\theta 0}(x,y) = \frac{1}{h}\mu^0 e^{i\theta j}\frac{\pi}{2H(x)}\cos\left(\frac{\pi y}{2H(x)}\right) w\left(\frac{x-j}{\sqrt{h}}\right) + \cdots, \tag{63}$$

$$\Sigma_{(k)xx}^{\theta 0}(x,y) = \cdots\ ,\ \Sigma_{(k)yy}^{\theta 0}(x,y) = \cdots\ ,\ (x,y) \in \varpi^h, |x-j| < 1/2,\ j = 1,2,3,\cdots \text{ and}$$

$$\Sigma_{(k)}^{\theta h}(x,y) = h^{2+\delta}\Sigma_{(k)}^{\theta 0}(x,y),$$

where ellipses stand for quantities $O\left(h^{-1/2}\exp(-\delta|x-j|^2/h)\right)$. Hence, the wave $K_k^h U_{(k)}^{\theta h}(x,y) = K_k^h U_{(k)}^{\theta}(x,y)$ produces in the waveguide ϖ^h the stress field $\Sigma_{(k)}^{\theta h}(x,y)$, which is localized in the $c_j\sqrt{h}$-neighborhoods of the maximum points $x = j = 0, \pm 1, \pm 2, \pm 3, \cdots$ of the profile function H in (1) due to the exponential decay of the eigenfunction w_k of the differential equation (44). Furthermore, the shear stresses $K_k^h \Sigma_{(k)xy}^{\theta h}(x,y)$ get their maximum values near the maximum points of the width $H(x)$ at the lower base Γ^0, that is at y=0, of the periodic waveguide ϖ^h which according to (11) is in ideal contact with the absolutely rigid half-plane $\{(x,y): y < 0\}$. At the same time, the tearing stresses $K_k^h \Sigma_{(k)xx}^{\theta h}(x,y)$ and $K_k^h \Sigma_{(k)yy}^{\theta h}(x,y)$ obtain much lower order in the small parameter h.

In these circumstances the fracture process commences in the vicinity of the points $(j,0) \in \Gamma^0$, realizes as fragmentation of either the adhesive film, or a thin frontier layer of the material in ϖ^h, and may lead to a periodic family of micro cracks with rough surfaces. According to the max-min principle, cf., [18; Theorem 10.2.2], such cracks shift all the spectral bands downwards so that the given frequency may fall into a new spectral gap. Besides, the wave may stop and, therefore, localize and accelerate the fracture process that finally infringes the periodic structure of the waveguide and creates a new series of eigenfrequencies and trapped elastic modes.

In the case of the profile perturbation (50), (51) the trapped wave $u_k^h \in H^1\left(\varpi^{h\approx}\right)$ at the eigenfrequency $\omega^{h*} = \sqrt{\rho^{-1}\Lambda^{h*}}$ detected in Section 5, aggravates the stress concentration effect and, hence, at the outset of the fracture process the exfoliation localizes at the bottom of the perturbed cell (56) only.

An elongated crack of length $l \gg h$ at the interface Γ^0 crucially rearranges the whole spectrum because a part of the waveguide becomes a beam of relatively small thickness h/l with traction-free bases. Appropriately, the above-introduced asymptotic ansätze must be replaced by new ones borrowed from the theory of thin elastic structures, see, e.g., the book [19] and the paper [20], where the procedure of dimension reduction for curved and periodic beams, plates and rods are formulated in detail. As a result, the ultralow range of the discrete spectrum becomes saturated quite much with two series of eigenvalues, which are again assured by the max-min principle and originate in transversal and longitudinal oscillations of the beam that is an area between the crack and the exterior surface Γ^h.

All the above observations, of course, hold true for the composite waveguide Π under the conditions (2).

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