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APPLICATION OF UNSTRUCTURED APPROXIMATING MESHES IN DISCRETE-CONTINUAL FINITE ELEMENT METHOD OF STRUCTURAL ANALYSIS

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Abstract. This paper is devoted to actual problems of application of various approximating meshes in discrete-continual finite element method of analysis of structures with regular (constant or piecewise constant) physical and geometrical parameters in one dimension (direction). Unstructured approximating meshes (with the use of discrete-continual finite elements with quadrilateral or triangular cross-sections) are under consideration. Besides, basic introduction to discrete-continual finite element method is presented as well.

1. Introduction

Development, research, verification and implementation of correct mathematical models and methods of structural mechanics are the most important aspects of ensuring safety of buildings and complexes. The analysis and design of structures normally require accurate computer-intensive calculations using numerical methods. The field of application of semianalytical (discrete-continual) methods, that are now becoming available for computer realization, comprises structures with regular (in particular, constant or piecewise constant) physical and geometrical parameters in some dimension (so-called "basic" direction (dimension)). Considering problems remain continual along "basic" direction while along other directions discrete-continual finite element method (DCFEM), proposed by P.A. Akimov and A.B. Zolotov [1], presupposes finite element approximation. Solution of corresponding resultant multipoint boundary problems (MBP) of the first-order system of ordinary differential equations with piecewise constant coefficients is the most time-consuming stage of the computing, especially if we take into account the limitation in performance of personal computers, contemporary software and necessity to obtain correct solution in a reasonable time. Analytical solution of such MBP is apparently preferable in all aspects for qualitative analysis of calculation data. It allows investigator to consider boundary effects when some components of solution are rapidly varying functions. Due to the abrupt decrease inside of mesh elements in many cases, their rate of change cannot be adequately considered by conventional numerical methods while analytics enables study. Another feature of DCFEM is the absence of limitations on lengths of structures. Hence, it appears that in this context DCFEM is peculiarly relevant. Generally, semi-analytical formulation are contemporary mathematical models which becoming available for computer realization.

Meshes topologically equivalent to rectangular [1] have been used mainly in previous versions of DCFEM for approximation of cross-sections of structures. Nevertheless this limitation can be removed after implementation of unstructured mesh (for instance without void discrete-continual finite elements in structures with embrasures) [2]. As is known triangular finite element has some advantages, including simple algorithms of creating corresponding meshes for arbitrary domains. Application of discrete-continual finite elements with triangular cross-section is also under consideration in the distinctive paper [3].

2. Unstructured Meshes from Discrete-Continual Finite Elements with Quadrilateral Cross-Section

Arbitrary numbering of nodes can introduced within DCFEM. Let us each discrete-continual finite element has list of four nodes. Corresponding data about such mesh is stored as in matrix of $4N_e$ size (N_e is the number of elements) with lists of nodes for each element.

Besides, modification of some formulas of approximation is required (two indexes of nodes were used in numbering algorithms in previous versions of DCFEM). In order to simplify these formulas transformation of cross-section of discrete-continual finite element to square (Fig. 1a)

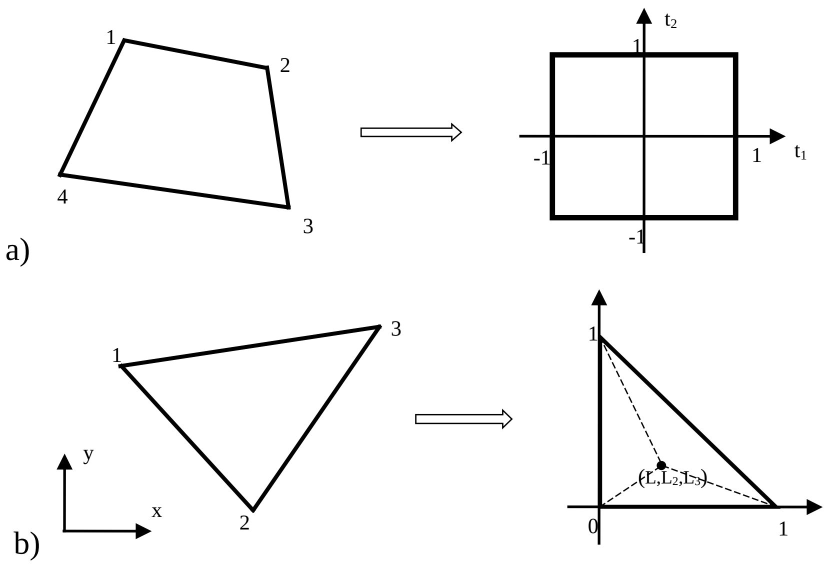

Fig. 1. Transformation of quadrilateral cross-section of discrete-continual finite element: a) quadrilateral cross-section; b) triangular cross-section.

Thus, we have $t_1 \in [-1, 1]$, $t_2 \in [-1, 1]$, where t_1 , t_2 are corresponding local coordinates.

Transformation of coordinates take place according to the formula:

$$\bar{\mathbf{x}} = 0.25 \cdot [(1-t_1)(1+t_2)\bar{\mathbf{x}}_1 + (1+t_1)(1+t_2)\bar{\mathbf{x}}_2 + (1+t_1)(1-t_2)\bar{\mathbf{x}}_3 + (1-t_1)(1-t_2)\bar{\mathbf{x}}_4], \quad (1)$$

where $\bar{\mathbf{x}}_k$ is global coordinate vector of k-th node (nodal line) of considering discrete-continual finite element.

Let φ be arbitrary function. Therefore, we have the following differentiation formulas:

$$\frac{\partial \varphi}{\partial x_k} = \sum_{s=1}^{2} \frac{\partial \varphi}{\partial t_s} \frac{\partial t_s}{\partial x_k}, \quad s = 1, 2 \ . \quad (2)$$

Derivatives of the local variables can be obtained from Jacobian of this transformation. Jacoby matrix has the form:

$$J(x_1,x_2) = \frac{D\bar{\mathbf{t}}}{D\bar{\mathbf{x}}} = \begin{bmatrix} \partial t_1/\partial x_1 & \partial t_1/\partial x_2 \\ \partial t_2/\partial x_1 & \partial t_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}. \tag{3}$$

Jacobian can be calculated after inverse of matrix $\beta = D\bar{x}/D\bar{t}$,

$$\frac{D\bar{\mathbf{x}}}{D\bar{\mathbf{t}}} = \begin{bmatrix} \partial x_1/\partial t_1 & \partial x_1/\partial t_2 \\ \partial x_2/\partial t_1 & \partial x_2/\partial t_2 \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}. \tag{4}$$

Derivative of global variables should be obtained by formula:

$$\frac{\partial \bar{\mathbf{x}}}{\partial t_s} = \frac{1}{4}[(\bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_1 + \bar{\mathbf{x}}_4 - \bar{\mathrm{x}}_3)t_{3-s} + (-1)^s(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_3) + \bar{\mathbf{x}}_2 - \bar{\mathbf{x}}_4]. \tag{5}$$

Integration of corresponding functions is required for construction of local stiffness matrices of discrete-continual finite element. Gauss method can be used in particular. Points of integration and their coordinates ($\xi = 1/\sqrt{3}$) are showed on Fig. 2a, weight of each point is equal to one. The following integration formula can be used:

$$\iint_S f(x,y)dS = \sum_{k=1}^{4} \det(J(x_k,y_k)) \cdot f(x_k,y_k) \ . \tag{6}$$

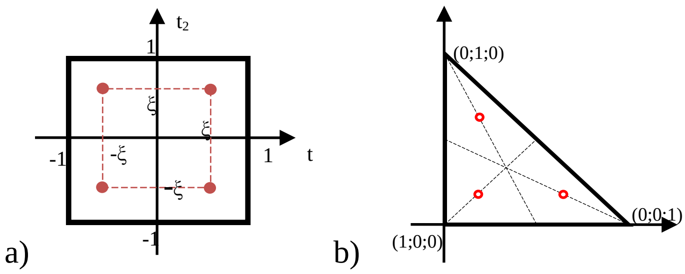

Fig. 2. Illustration of the Gauss method. Points of integration:
a) quadrilateral cross-section; b) triangular cross-section.

3. Unstructured Meshes from Discrete-Continual Finite Elements with Triangular Cross-Section

Arbitrary numbering of nodes can introduced within DCFEM. Then each discrete-continual finite element has list of four nodes. Corresponding data about such mesh is stored as in matrix of $4N_e$ size (N_e is the number of elements) with lists of nodes for each element. We can certainly introduce discrete-continual finite element with triangular cross-section. In this connection the number cof the fourth node of this element should be equal to zero. In other words this is the “flag” that discrete-continual finite element has triangular cross-section. Local coordinate system (L-coordinates) can be used as well. Transformation of cross-section of discrete-continual finite element is shown at Fig. 1b.

Transformation of coordinates take place according to the formula:

$$\bar{\mathbf{x}} = L_1\bar{\mathbf{x}}_1 + L_2\bar{\mathbf{x}}_2 + L_3\bar{\mathbf{x}}_3 = L_1\bar{\mathbf{x}}_1 + L_2\bar{\mathbf{x}}_2 + (1 - L_2 - L_1)\bar{\mathbf{x}}_3 , \tag{7}$$

where $\bar{\mathbf{x}}_k$ is global coordinate vector of k-th node (nodal line) of considering discrete-continual finite element.

In order to calculate L-coordinates from global coordinates, we should use formula:

$$L_k = \alpha_{k,1}x + \alpha_{k,2}y + \alpha_{k,3} \ , \tag{8}$$

where corresponding matrix of coefficients $\mathbf{A}$ can be obtained from:

$$\mathbf{A} = \mathbf{D}^{-1}; \quad \mathbf{D} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix}. \tag{9}$$

Jacobian of this transform is matrix $\mathbf{D}$. As a result of the changing variables, the calculation of the derivative can be produced with the use of differentiation formulas:

$$\frac{\partial \varphi}{\partial x_k} = \sum_{s=1}^{2} \frac{\partial \varphi}{\partial L_s} \frac{\partial L_s}{\partial x_k} = (\varphi_1 - \varphi_3)\alpha_{k1} + (\varphi_2 - \varphi_3)\alpha_{k2}, \quad s = 1,2 \ , \tag{10}$$

where $\varphi_k = \varphi(\overline{\mathbf{x}}_k)$.

Integration of corresponding functions is required for construction of local stiffness matrices of discrete-continual finite element. Clone of Gauss method can be used in particular. Location of points of integration is shown at Fig. 2b. L-coordinates of these points are presented at Table 1.

Table 1. L-coordinates of integration points.

No.	L_1 -coordinate	L_2 -coordinate	ω
1	1/6	1/6	1/6
2	2/3	1/6	1/6
3	1/6	2/3	1/6

Three L-coordinates depend of each other and satisfy equation

$$L_1 + L_2 + L_3 = 1 \ . \tag{11}$$

Weight of each point equal 1/6. The following integration formula can be used:

$$\iint_S f(x,y)dS = \frac{1}{6}\sum_{k=1}^{3} \det(J(L_1^k, L_2^k, L_3^k)) \cdot f(L_1^k, L_2^k, L_3^k) \ . \tag{12}$$

4. Program Implementation and Verification

Version of DCFEM, considering in the distinctive paper, has been realized in software DCFEM3D. Programming environment is Microsoft Visual Studio 2013 and Intel Parallel Studio XE 2015 (Intel Visual Fortran Composer XE 2015). Mathematical library Intel MKL is used for solving of eigenvalue problem and linear equations. The program has been verified on a representative set of three-dimensional problems of structural analysis.

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References

[1] P.A. Akimov // *Applied Mechanics and Materials* **204-208** (2012) 4502.
[2] P.A. Akimov, O.A. Negrozov // *Procedia Engineering* **111** (2015) 8.
[3] P.A. Akimov, O.A. Negrozov // *Procedia Engineering* **111** (2015) 14.