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NUMERICAL MODELLING OF DEFORMATION OF HYPERELASTIC INCOMPRESSIBLE SOLIDS

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Abstract. In this paper the model of investigation of large deformations of solids for incompressible elastic materials is considered. The constitutive equations are derived from the potential of elastic deformation. Calculation algorithm is based on the linearized equation of virtual work, defined to actual state. To account incompressibility a penalty method is applied. Numerical implementation is based on the finite element method. The deformation of a square plate with round neckline is provided.

1. Kinematics. Variation formulation

The next tensors are used for describing of kinematics of a continuum: the deformation gradient $(\boldsymbol{F})$, the left Cauchy–Green tensor $(\boldsymbol{B})=(\boldsymbol{F})\cdot(\boldsymbol{F})^T$, the spatial gradient of rate tensor $(\boldsymbol{h})=(\dot{\boldsymbol{F}})\cdot(\boldsymbol{F}^{-1})$, the deformation rate tensor [1-7]:

$$(\boldsymbol{d})=\frac{1}{2}\left[(\boldsymbol{h})+(\boldsymbol{h})^T\right]=\frac{1}{2}\left[(\dot{\boldsymbol{F}})\cdot(\boldsymbol{F}^{-1})+(\boldsymbol{F}^{-1})^T\cdot(\dot{\boldsymbol{F}})^T\right].$$

Using as a reference configuration – current, and as a resolving equation take the variational equation of principle of virtual work in terms of the virtual velocity, which is follows as:

$$\int_{\Omega}(\boldsymbol{\sigma})\cdot\cdot(\delta\boldsymbol{d})d\Omega=\int_{S^{\sigma}}\vec{\boldsymbol{t}}_n\delta\vec{\mathbf{v}}ds+\int_{\Omega}\vec{\boldsymbol{f}}\delta\vec{\mathbf{v}}d\Omega, \tag{1}$$

where $(\boldsymbol{\sigma})$ is the Cauchy stress tensor, $\vec{\boldsymbol{v}}$ is a velocity vector; Ω is the current volume; S^{σ} is the surface on which the force $\vec{\boldsymbol{t}}_n$ is applied, $\vec{\boldsymbol{f}}$ is the body force vector.

2. Constitutive equations

The constitutive equations are obtained using the strain energy density W in the reference configuration. An isotropic material is considered and in this case the strain energy density is defined as [2]: $W=W(I_{1\mathbf{B}},I_{2\mathbf{B}},I_{3\mathbf{B}})$, where $I_{i\mathbf{B}}$ – is the corresponding invariants of tensor $(\boldsymbol{B})$.

Then the Cauchy stress tensor is used for defining of stress state of a solid is defined as $(\boldsymbol{\sigma})=\frac{2}{J}(\boldsymbol{B})\cdot\left(\frac{\partial W}{\partial\boldsymbol{B}}\right)$, and expression for the Cauchy–Euler stress rate tensor

$$(\dot{\boldsymbol{\sigma}})=2\left\{\frac{1}{J}(\dot{\boldsymbol{B}})\cdot\left(\frac{\partial W}{\partial\boldsymbol{B}}\right)+\frac{1}{J}\left[(\boldsymbol{B})\cdot\left(\frac{\partial^2 W}{\partial\boldsymbol{B}^2}\right)\right]\cdot\cdot(\dot{\boldsymbol{B}})-\frac{1}{J}(\boldsymbol{B})\cdot\left(\frac{\partial W}{\partial\boldsymbol{B}}\right)I_{1\mathbf{d}}\right\},$$

that can be converted to

$(\dot{\boldsymbol{\sigma}}) = (\mathbf{c})\cdot\cdot(\boldsymbol{d}) + (\boldsymbol{h})\cdot(\boldsymbol{\sigma}) + (\boldsymbol{\sigma})\cdot(\boldsymbol{h})^T - (\boldsymbol{\sigma}) I_{1d}$ or $\left(\boldsymbol{\sigma}^{Tr}\right) = (\boldsymbol{c})\cdot\cdot(\boldsymbol{d})$,

where $(\mathbf{c}) = \frac{4}{J}(\boldsymbol{B})\cdot\left(\frac{\partial^2 W}{\partial\boldsymbol{B}\partial\boldsymbol{B}}\right)\cdot(\boldsymbol{B})$ – elasticity tensor, J – a changing of volume, $\left(\boldsymbol{\sigma}^{Tr}\right) = (\dot{\boldsymbol{\sigma}}) + (\boldsymbol{h})\cdot(\boldsymbol{\sigma}) + (\boldsymbol{\sigma})\cdot(\boldsymbol{h})^T - I_{1d}(\boldsymbol{\sigma})$ – Truesdell derivative.

For an isotropic material having a low compressibility, in constitutive equations separated in a changing volume deformations. To do this, we introduce the measure deformations that are not accompanied by a change in volume. For these measures $I_{3B} = 1$, and the strain energy density is represented in the form of a sum of two terms, the first of which depends on the changes in volume, and the second – on the invariants of the measures imposed deformations: $W = W_0(J) + \hat{W}(I_{1B}, I_{2B})$.

The penalty method is applied for solving of problems with incompressibility [8]. There are two ways. In the first way eliminated the pressure as an independent variable, is to consider the material as being nearly incompressible, whereby a large value of the bulk modulus effectively prevents significant volumetric changes. A second is to perturb the Lagrangian functional by the addition of a further “penalty” term that enables the pressure to eventually be artificially associated with the deformation, thereby again eliminating the pressure variable. It will transpire that these two methods lead to identical equations. The elasticity tensor $(\mathbf{c})$ is then added to such term $(\mathbf{c}_k) = J\frac{dp}{dJ}(\boldsymbol{I})\cdot(\boldsymbol{I}) = kJ(\boldsymbol{I})\cdot(\boldsymbol{I})$, where $p = k(J-1)$, k is the penalty number and acting of the bulk modulus, wherein, for incompressible material $k \to \infty$.

3. Incremental method

The algorithm of investigation is based on incremental method [3], founded on the linearized equation (1):

$$\int_{\Omega_k} \{\left({}^k\boldsymbol{d}\right)\cdot\cdot\left({}^k\boldsymbol{c}\right)\cdot\cdot(\delta\boldsymbol{d}) + \frac{1}{2}\left({}^k\boldsymbol{\sigma}\right)\cdot\cdot\left[(\delta\boldsymbol{h})^T\cdot\left({}^k\boldsymbol{h}\right) + \left({}^k\boldsymbol{h}\right)^T\cdot(\delta\boldsymbol{h})\right] -$$

$$-\frac{{}^k\dot{J}}{{}^kJ}{}^k\vec{\boldsymbol{f}}\cdot\delta\vec{\mathbf{v}}\}d\Omega_k + \int_{S_k^\sigma}\left\{{}^k\vec{\boldsymbol{t}}_n\cdot\left({}^k\boldsymbol{h}\right)^T - {}^k\vec{\boldsymbol{t}}_n\frac{{}^k\dot{J}}{{}^kJ}\right\}\delta\vec{\mathbf{v}}dS_k =$$

$$= \int_{S_k^\sigma}{}^k\vec{\boldsymbol{t}}_n\cdot\delta\vec{\mathbf{v}}dS_k + \int_{\Omega_k}{}^k\vec{\boldsymbol{f}}\cdot\delta\vec{\mathbf{v}}d\Omega_k - \frac{1}{\Delta t}\left\{\int_{\Omega_k}\left({}^k\boldsymbol{\sigma}\right)\cdot\cdot\left(\delta^k\boldsymbol{d}\right)d\Omega_k - \int_{S_k^\sigma}{}^k\vec{\boldsymbol{t}}_n\cdot\delta\vec{\mathbf{v}}dS_k - \int_{\Omega_k}{}^k\vec{\boldsymbol{f}}\cdot\delta\vec{\mathbf{v}}d\Omega_k\right\}.$$

The resulting equation is linear relative to the velocity ${}^k\vec{\boldsymbol{v}}$. Since in researched processes acceleration is not to consider, under the time you can understand any monotonically increasing parameter defining the variation of the load. From this point of view, it is appropriate to take the time derivative as the ratio of the increment in the corresponding values. For example, ${}^k\vec{\boldsymbol{v}} = \frac{\Delta^k\vec{\boldsymbol{u}}}{\Delta t}$. Since the argument t is arbitrary, allowable to take $\Delta t = 1$. As a result, we get the resolving equation for the displacement increments $\Delta^k\vec{\boldsymbol{u}}$.

4. Numerical example

The numerical implementation is based on the finite element method. An 8-node brick element is used [4]. As an example the strain energy density of the neo-Hookean material is considered:

$$W = \frac{\mu}{2}\left(tr\left(\boldsymbol{B}\right) - 3\right),$$

where μ is shear modulus.

Elasticity tensor **c** is written as a sum of three terms:

$$(\mathbf{c}) = 2\mu J^{-\frac{5}{3}}\left(-\frac{1}{3}(\boldsymbol{B})\cdot(\boldsymbol{I}) + \frac{1}{9}I_{1\boldsymbol{B}}\cdot(\boldsymbol{I})\cdot(\boldsymbol{I}) + \frac{1}{3}(\boldsymbol{C}_{II})\cdot I_{1\boldsymbol{B}} - \frac{1}{3}(\boldsymbol{I})\cdot(\boldsymbol{B})\right) + pJ\left((\boldsymbol{I})\cdot(\boldsymbol{I}) - 2(\boldsymbol{C}_{II})\right) + (\mathbf{c}_k),$$

where $(\boldsymbol{C}_{II}) = (\mathbf{e}_i\mathbf{e}_j\mathbf{e}_i\mathbf{e}_j)$ is tensor of the fourth rank, defined in [2, 5], $(\boldsymbol{I})$ is an unit tensor of the second rank. The Cauchy stress tensor then takes the following form:

$$(\boldsymbol{\sigma}) = \mu J^{-\frac{5}{3}}\left((\boldsymbol{B}) - \frac{1}{3}(\boldsymbol{I})\cdot I_{1\boldsymbol{B}}\right) + p(\boldsymbol{I}).$$

The plane stress of the strip 6.5×6.5×0.079 cm^3 with a hole 0.5 cm in diameter is considered. The shear modulus is equal to 54.04 kg/cm^2. Given the two planes of symmetry, eight-noded polyline elements are used to describe a quarter of the problem. The strip was stretched in the horizontal direction in six time (Fig. 1).

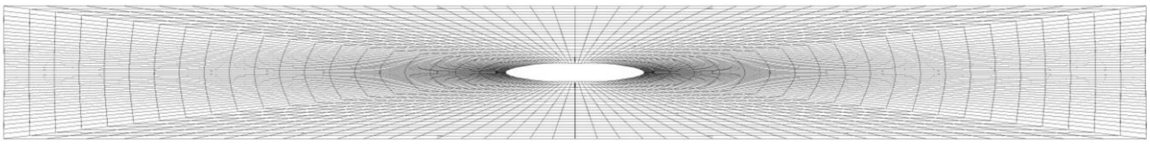

Fig. 1. Final mesh.

5. Conclusion

Thus, in this paper we construct a method of numerical investigation of solids that physical relationships are specified using the strain energy density. The constitutive equations and resolving equation are obtained. The problem of plane deformation of the strip is solved. The calculation results show efficiency of this method of investigation of nonlinear problems.

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