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SIZE EFFECT ON BUCKLING OF NON-UNIFORM CIRCULAR PLATE MADE OF FOAM MATERIAL

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Abstract. In the framework of nonlinear micropolar elasticity, the stability of a radially compressed circular plate made of foam is studied. It is assumed that the elastic properties of the plate vary through the thickness. For the case of axisymmetric buckling the linearized equilibrium equations are derived. Using them, the size effect on plate stability is analyzed.

1. Introduction

Due to the increasing number of new structural materials, the understudied problem of stability analysis for bodies with a microstructure becomes important. Examples of such new materials are metal and polymer foams [1, 2], which are widely used in the modern automotive and airspace industries. According to the experimental data, when the average size of the foam cell is not small in comparison to the macroscopic dimensions of the body, the behavior of the latter is noticeably different from the expected in the classical elasticity. To describe such size effects the model of micropolar continuum [3-5], i.e. medium with couple stresses and rotational degrees of freedom, can be used. Within the framework of this model, the microrotation can be viewed as the rotation of nodes between ribs in the foam, and the couple stress can be viewed as a spatial average of the bending and twisting moments transmitted by the foam ribs. Considering the above, in the present paper we studied the stability of a non-uniform circular plate made of foam material using the micropolar continuum model.

2. Initial strain state

For nonlinearly elastic micropolar medium, the set of static equations in the absence of body forces and moments consists of the equilibrium equations for the stresses [6, 7]

$$\mathrm{Div}\mathbf{D} = 0, \qquad \mathrm{Div}\mathbf{G} + \left(\mathbf{C}^{\mathrm{T}} \cdot \mathbf{D}\right)_{\times} = 0, \tag{1}$$

constitutive equations

$$\mathbf{D} = \frac{\partial W(\mathbf{Y},\mathbf{L})}{\partial \mathbf{Y}} \cdot \mathbf{H}, \qquad \mathbf{G} = \frac{\partial W(\mathbf{Y},\mathbf{L})}{\partial \mathbf{L}} \cdot \mathbf{H}, \tag{2}$$

and geometric relations [8]

$$\mathbf{Y} = \mathbf{C} \cdot \mathbf{H}^{\mathrm{T}}, \qquad \mathbf{L} \times \mathbf{I} = -\left(\mathrm{Grad}\mathbf{H}\right) \cdot \mathbf{H}^{\mathrm{T}}, \qquad \mathbf{C} = \mathrm{Grad}\mathbf{R}. \tag{3}$$

Here Grad and Div are the gradient and divergence operators in the Lagrangian coordinates, $\mathbf{D}$ and $\mathbf{G}$ are the Piola-type stress and couple stress tensors, $\mathbf{C}$ is the deformation gradient, $\mathbf{Y}$ is the stretch tensor, $\mathbf{L}$ is the wryness tensor, $W(\mathbf{Y},\mathbf{L})$ is the specific strain energy, $\mathbf{R}$ is the radius vector, $\mathbf{H}$ is the tensor of microrotation, and $\mathbf{I}$ is the identity tensor. The symbol "$_{\times}$" represents the vector invariant of a second-order tensor.

We consider the circular plate of micropolar material, the elastic properties of which vary through the thickness and are described by the the 6-parametric model [6, 9]:

$$W(\mathbf{Y},\mathbf{L}) = \frac{1}{2}\lambda(z)\mathrm{tr}^2(\mathbf{Y}-\mathbf{I}) + \frac{1}{2}\mu(z)\mathrm{tr}(\mathbf{Y}-\mathbf{I})^2 + \frac{1}{2}[\mu(z)+\kappa(z)]\mathrm{tr}\left[(\mathbf{Y}-\mathbf{I})\cdot(\mathbf{Y}-\mathbf{I})^{\mathrm{T}}\right]$$
$$+\frac{1}{2}\gamma_1(z)\mathrm{tr}^2\mathbf{L} + \frac{1}{2}\gamma_2(z)\mathrm{tr}\left(\mathbf{L}\cdot\mathbf{L}^{\mathrm{T}}\right) + \frac{1}{2}\gamma_3(z)\mathrm{tr}\,\mathbf{L}, \quad (4)$$

where $\lambda(z)$ and $\mu(z)$ are functions describing the change in the Lame parameters; $\kappa(z)$, $\gamma_1(z)$, $\gamma_2(z)$ and $\gamma_3(z)$ are micropolar elastic parameters changing with the thickness coordinate. In the case of radial compression of the plate, the radius vector $\mathbf{R}$ and proper orthogonal tensor of microrotation $\mathbf{H}$, which determine the position and rotation of a medium point, are given by the following expressions [7, 10]:

$$R = \alpha r, \qquad \Phi = \varphi, \qquad Z = f(z), \qquad 0 \le r \le r_1, \qquad 0 \le \varphi \le 2\pi, \qquad |z| \le h/2; \quad (5)$$

$$\mathbf{R} = \alpha r\mathbf{e}_R + f(z)\mathbf{e}_Z, \qquad \mathbf{H} = \mathbf{e}_r \otimes \mathbf{e}_R + \mathbf{e}_\varphi \otimes \mathbf{e}_\Phi + \mathbf{e}_z \otimes \mathbf{e}_Z. \quad (6)$$

Here r, φ, z are the cylindrical coordinates in the reference configuration (Lagrangian coordinates), R, Φ, Z are the Eulerian cylindrical coordinates, $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$ and $\{\mathbf{e}_R, \mathbf{e}_\Phi, \mathbf{e}_Z\}$ are the orthonormal vector bases of Lagrangian and Eulerian coordinates, respectively; α is the radial compression ratio, $f(z)$ is the unknown function, which describes the strain in thickness direction of the non-uniform plate; r_1 and h are the radius and the thickness of the undeformed circular plate, respectively.

We assume that there are no external loads on the plate faces $(z = \pm h/2)$, and no vertical displacement on the middle surface $(z = 0)$:

$$\mathbf{e}_z \cdot \mathbf{D}\big|_{z=\pm h/2} = 0, \qquad \mathbf{e}_z \cdot \mathbf{G}\big|_{z=\pm h/2} = 0, \qquad f(0) = 0. \quad (7)$$

The unknown function $f(z)$ is found by solving the boundary-value problem (1), (7) while taking into account the relations (2) – (6). If the pattern of change for the elastic parameters λ, μ, κ is the same, i.e. $\lambda(z) = \lambda_0\xi(z)$, $\mu(z) = \mu_0\xi(z)$ and $\kappa(z) = \kappa_0\xi(z)$, then the expression for the function $f(z)$ takes the form [10]: $f(z) = \alpha_3 z$, $\quad \alpha_3 = 1 + \dfrac{2\lambda_0(1-\alpha)}{\lambda_0 + 2\mu_0 + \kappa_0}$.

3. Perturbed state

Consider a small perturbation of the initial strain state discussed above. The perturbed state of micropolar plate exists under the same external loads and is determined by the next relations:

$$\mathbf{R} \to \mathbf{R} + \eta\mathbf{v}, \qquad \mathbf{H} \to \mathbf{H} - \eta\mathbf{H}\times\boldsymbol{\omega}, \quad \mathbf{v} = v_R\mathbf{e}_R + v_\Phi\mathbf{e}_\Phi + v_Z\mathbf{e}_Z, \qquad \boldsymbol{\omega} = \omega_R\mathbf{e}_R + \omega_\Phi\mathbf{e}_\Phi + \omega_Z\mathbf{e}_Z, \quad (8)$$

where η is a small parameter, $\mathbf{v}$ is the vector of additional displacements, $\boldsymbol{\omega}$ is the linear incremental rotation vector, which characterizes the small rotation of the medium points, measured from the initial strain state.

Linearized equilibrium equations for the micropolar medium have the form [6, 11]:

$$\mathrm{Div}\mathbf{D}^\bullet = 0, \qquad \mathrm{Div}\mathbf{G}^\bullet + \left[\mathrm{Grad}\mathbf{v}^{\mathrm{T}} \cdot \mathbf{D} + \mathbf{C}^{\mathrm{T}} \cdot \mathbf{D}^\bullet\right]_\times = 0, \quad (9)$$

where $\mathbf{D}^\bullet$ and $\mathbf{G}^\bullet$ are the linearized Piola-type stress and couple stress tensors.

Linearized boundary conditions on the plate faces are written as follows:

$$\mathbf{e}_z \cdot \mathbf{D}^\bullet\big|_{z=\pm h/2} = 0, \qquad \mathbf{e}_z \cdot \mathbf{G}^\bullet\big|_{z=\pm h/2} = 0. \quad (10)$$

We assume that the constant radial displacement is given at the edge of the circular plate $(r = r_1)$, the azimuthal displacement is absent, and there is no friction during vertical displacement. This leads to the following linearized edge conditions:

$$\mathbf{e}_r \cdot \mathbf{D}^{\bullet} \cdot \mathbf{e}_Z\big|_{r=r_1} = \mathbf{e}_r \cdot \mathbf{G}^{\bullet} \cdot \mathbf{e}_Z\big|_{r=r_1} = 0, \quad \mathbf{v}\cdot\mathbf{e}_R\big|_{r=r_1} = \mathbf{v}\cdot\mathbf{e}_\Phi\big|_{r=r_1} = \boldsymbol{\omega}\cdot\mathbf{e}_R\big|_{r=r_1} = \boldsymbol{\omega}\cdot\mathbf{e}_\Phi\big|_{r=r_1} = 0. \qquad (11)$$

Similar to [10], in the general case of non-axisymmetric buckling the stability analysis of the non-uniform micropolar plate can be reduced to solving the homogeneous boundary-value problem (9) – (11) for the system of six partial differential equations in six unknown functions of two variables r, z. In the case of axisymmetric perturbations, the solution of the boundary-value problem (9) – (11) takes a more general form then in [10]:

$$\begin{gathered} v_R = V_R(z) J_1(\beta r), \quad v_\Phi = V_\Phi(z) J_1(\beta r), \quad v_Z = V_Z(z) J_0(\beta r), \\ \omega_R = \Omega_R(z) J_1(\beta r), \quad \omega_\Phi = \Omega_\Phi(z) J_1(\beta r), \quad \omega_Z = \Omega_Z(z) J_0(\beta r), \; \beta = \zeta_m / r_1, \; J_1(\zeta_m) = 0, \; m = 1,2,... \end{gathered} \qquad (12)$$

Here J_0 and J_1 are the Bessel functions of the first kind.

Given the representation (12), the linearized edge conditions (11) are satisfied automatically. The linearized equilibrium equations (9) are written as follows:

$$\begin{gathered} (\mu+\kappa)V_R'' + (\mu'+\kappa')V_R' - (\lambda+2\mu+\kappa)\beta^2 V_R - (\lambda+\mu)\beta V_Z' - \beta\mu' V_Z + \theta_1\Omega_\Phi' + \theta_1'\Omega_\Phi = 0, \\ (\mu+\kappa)V_\Phi'' + (\mu'+\kappa')V_\Phi' - (\mu+\kappa)\beta^2 V_\Phi - \theta_1\Omega_R' - \theta_1'\Omega_R - \beta\theta_2\Omega_Z = 0, \\ (\lambda+2\mu+\kappa)V_Z'' + (\lambda'+2\mu'+\kappa')V_Z' - (\mu+\kappa)\beta^2 V_Z + (\lambda+\mu)\beta V_R' + \beta\lambda' V_R - \theta_1\beta\Omega_\Phi = 0, \\ \gamma_2\Omega_R'' + \gamma_2'\Omega_R' + \left[(\alpha+f')\theta_1 - (\gamma_1+\gamma_2+\gamma_3)\beta^2\right]\Omega_R - (\gamma_1+\gamma_3)\beta\Omega_Z' - \beta\gamma_3'\Omega_Z + \theta_1 V_\Phi' = 0, \\ \gamma_2\Omega_\Phi'' + \gamma_2'\Omega_\Phi' + \left[(\alpha+f')\theta_1 - \gamma_2\beta^2\right]\Omega_\Phi - \theta_1 V_R' - \beta\theta_1 V_Z = 0, \\ (\gamma_1+\gamma_2+\gamma_3)\Omega_Z'' + (\gamma_1'+\gamma_2'+\gamma_3')\Omega_Z' + \left(2\alpha\theta_2 - \gamma_2\beta^2\right)\Omega_Z + (\gamma_1+\gamma_3)\beta\Omega_R' + \beta\gamma_1'\Omega_R - \beta\theta_2 V_\Phi = 0. \end{gathered} \qquad (13)$$

Here we use the following notation

$$\theta_1 = \lambda(2\alpha + f' - 3) + \mu(\alpha + f' - 2) - \kappa, \qquad \theta_2 = \lambda(2\alpha + f' - 3) + 2\mu(\alpha - 1) - \kappa.$$

The linearized boundary conditions (10) take the form:

$$\begin{gathered} (\mu+\kappa)V_R' - \mu\beta V_Z + \theta_1\Omega_\Phi = 0, \quad (\mu+\kappa)V_\Phi' - \theta_1\Omega_R = 0, \quad \beta\lambda V_R + (\lambda+2\mu+\kappa)V_Z' = 0, \\ \gamma_2\Omega_R' - \gamma_3\beta\Omega_Z = 0, \quad \Omega_\Phi' = 0, \quad \gamma_1\beta\Omega_R + (\gamma_1+\gamma_2+\gamma_3)\Omega_Z' = 0; \quad z = \pm h/2. \end{gathered} \qquad (14)$$

As a result, the axisymmetric buckling analysis for the non-uniform circular plate is reduced to solving a linear homogeneous boundary-value problem (13), (14) for the system of six ordinary differential equations.

4. Symmetric plate

If the functions describing the change in the elastic parameters of the plate through the thickness are even functions, i.e. $\lambda(z) = \lambda(-z)$, $\mu(z) = \mu(-z)$, $\kappa(z) = \kappa(-z)$, $\gamma_1(z) = \gamma_1(-z)$, $\gamma_2(z) = \gamma_2(-z)$, $\gamma_3(z) = \gamma_3(-z)$, then the non-uniform circular plate is symmetric about the middle surface $z = 0$. It is easy to show that in this case the boundary-value problem (13), (14) has two independent sets of solutions [10]. The *First set* consists of solutions for which the plate deflection is an odd function of z (symmetric buckling modes):

$$\begin{gathered} V_R(z) = V_R(-z), \quad V_\Phi(z) = V_\Phi(-z), \quad V_Z(z) = -V_Z(-z), \\ \Omega_R(z) = -\Omega_R(-z), \quad \Omega_\Phi(z) = -\Omega_\Phi(-z), \quad \Omega_Z(z) = \Omega_Z(-z). \end{gathered}$$

For the *Second set* of solutions the deflection is an even function of z (bending buckling modes):

$$\begin{gathered} V_R(z) = -V_R(-z), \quad V_\Phi(z) = -V_\Phi(-z), \quad V_Z(z) = V_Z(-z), \\ \Omega_R(z) = \Omega_R(-z), \quad \Omega_\Phi(z) = \Omega_\Phi(-z), \quad \Omega_Z(z) = -\Omega_Z(-z). \end{gathered}$$

Due to this, it is sufficient to consider only the half of the symmetric plate $(0 \le z \le h/2)$ to study its stability. The boundary conditions at the middle surface follow from the evenness and oddness of the unknown functions:

a) for the symmetric buckling:

$$V_R'(0) = V_\Phi'(0) = V_Z(0) = 0, \qquad \Omega_R(0) = \Omega_\Phi(0) = \Omega_Z'(0) = 0; \tag{15}$$

b) for the bending buckling:

$$V_R(0) = V_\Phi(0) = V_Z'(0) = 0, \qquad \Omega_R'(0) = \Omega_\Phi'(0) = \Omega_Z(0) = 0. \tag{16}$$

Thus, the stability analysis for the symmetric non-uniform micropolar plate is reduced to solving two linear homogeneous boundary-value problems (Problem (13) – (15) and Problem (13), (14), (16)) for the system of six ordinary differential equations.

5. Size effect on plate buckling

In the present paper, the axisymmetric buckling analysis has been carried out for the non-uniform circular plate made of dense polyurethane foam. The micropolar elastic parameters for this material have been previously determined by Lakes [9].

We assumed that the pattern of change in the elastic parameters λ, μ, κ through the plate thickness is the same, while the micropolar elastic parameters $\gamma_1, \gamma_2, \gamma_3$ are constant:

$$\lambda(z) = \lambda_0 \xi(z), \quad \mu(z) = \mu_0 \xi(z), \quad \kappa(z) = \kappa_0 \xi(z), \quad \lambda_0 = 797.3 \text{ MPA}, \quad \mu_0 = 99.67 \text{ MPA},$$

$$\kappa_0 = 8.67 \text{ MPA}, \quad \gamma_1 = -26.65 \text{ N}, \quad \gamma_2 = 45.3 \text{ N}, \quad \gamma_3 = 34.65 \text{ N}.$$

Several laws describing the change in the elastic properties have been considered:

$$\text{EXP1: } \xi(z) = \left[9999 + \exp\left(\frac{2\ln\left(10^4 [c-1] + 1\right)}{h} |z| \right) \right] \cdot 10^{-4}, \quad \text{SIN1: } \xi(z) = (c-1)\sin\left(\frac{\pi |z|}{h} - \frac{\pi}{2} \right) + c,$$

$$\text{EXP2: } \xi(z) = \left[9 + \exp\left(\frac{2\ln\left(10 [c-1] + 1\right)}{h} |z| \right) \right] \cdot 10^{-1}, \quad \text{SIN2: } \xi(z) = \frac{c-1}{2}\sin\left(\frac{2\pi |z|}{h} - \frac{\pi}{2} \right) + \frac{c+1}{2},$$

where $c = \xi(\pm h/2) > 1$ is the parameter characterizing the relative stiffness of the plate faces (in comparison with the middle surface of the plate): the values of the elastic parameters λ, μ, κ at $z = \pm h/2$ are c times higher than their values at $z = 0$.

For convenience, the following dimensionless parameters were introduced: the relative radial compression $\delta = 1 - \alpha$, the diameter-to-thickness ratio $d^* = 2r_1/h$, and the relative thickness $h^* = h/l_b$. The characteristic length for bending l_b is the engineering constant of micropolar material [9], and in the case of dense polyurethane foam $l_b = 0.33$ mm.

By numerical solution of the linearized boundary-value problems (13) – (15) and (13), (14), (16) we find the spectra of critical values of the relative radial compression δ, corresponding to the symmetric and bending buckling modes of the non-uniform micropolar plate. By analyzing these spectra the critical radial compression δ_c is obtained. In the present paper we studied the stability of relatively thin plates $(d^* \le 20)$. According to the results, in this case the bending buckling occurs at the lowest loads, and the critical radial compression δ_c corresponds to the first bending mode $(m = 1)$.

To study the size effect on buckling of foam plates, we have carried out the stability analysis for plates having the same diameter-to-thickness ratio, but different overall size. In the classical elasticity, such plates become unstable under the same strains. But, according to the results, the situation is different for the micropolar continuum. The Figure 1 shows the dependences of the critical radial compression δ_c on the plate size for different laws describing the change in the elastic properties of material. The relative thickness h^* has been used as the size (scale) parameter of the plates at fixed diameter-to-thickness ratio $(d^* = 20)$.

It is evident from figure that plates become more stable with a decrease in size. This size effect is very significant for small plates $(h^* \le 5)$, but negligible for large ones $(h^* > 10)$. All of this suggests a stabilizing effect of the couple stresses which becomes negligible if the macroscopic dimensions of the plate are much larger than the characteristic length scale of the material microstructure (i.e. than the average size of the foam cell).

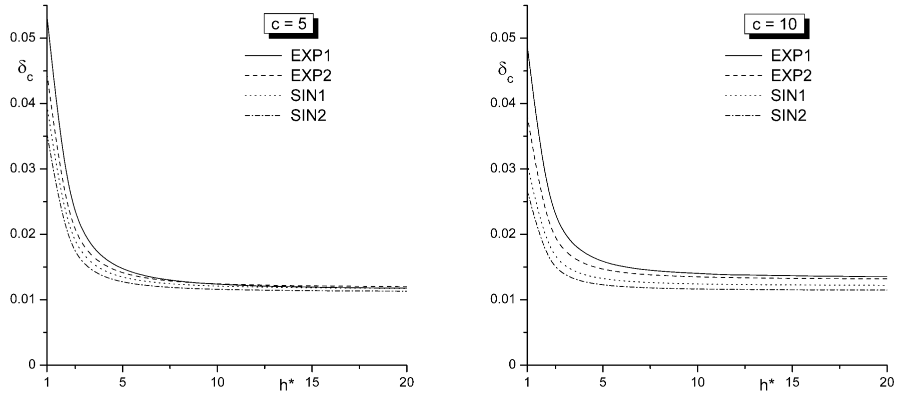

Fig. 1. Size effect on stability of non-uniform circular plates.

6. Conclusion

In the framework of nonlinear micropolar elasticity, we studied the stability of non-uniform circular foam plate subjected to radial compression. For a plate symmetric about the middle surface, the stability analysis was reduced to solving two linear homogeneous boundary-value problems for the half-plate. The axisymmetric buckling analysis has been carried out for the plate made of dense polyurethane foam. As a result, it was found that the plate becomes more stable with a decrease in size. This size effect is only significant for small plates.

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