Materials Physics and Mechanics 28 (2016) 57-61 Received: June 10, 2015

ON PLASTIC FLOW OF SOLIDS FOR STRESS STATES CORRESPONDING TO AN EDGE OF THE COULOMB-TRESCA PRISM

V.A. Kovalev[1], Y.N. Radayev[2*]

[1]Moscow City Government University of Management, Ul. Sretenka 28, Moscow, 107045, Russia

[2]Intsitute for Problems in Mechanics of RAS, Prosp. Vernadskogo 101, Moscow, 119526, Russia

*e-mail: radayev@ipmnet.ru

Abstract. Plastic flow states corresponding to an edge of the Coulomb-Tresca prism in the Haigh-Westergaard three-dimensional space of principal stresses are considered. Constitutive equations are formulated by the generalized associated plastic flow rule due to Koiter. These equations impose the minimal kinematical constraints on plastic strains increments and as it is elucidated are equivalent to three-dimensional equations of the mathematical plasticity proposed by Ishlinskii in 1946. It is then shown that obtained constitutive equations can be formulated as a tensor permutability equation for the stress tensor and the plastic strains tensor increment. A new explicit form of the plastic flow rule for stress states corresponding to an edge of the Coulomb-Tresca prism is obtained and discussed.

1. Introduction

The Coulomb-Tresca yield criterion holds a unique position in mathematical theory of plasticity as the both two-dimensional and then the first three-dimensional theories of perfect plastic flow of metals have been created by this criterion by Saint-Venant and Levy.

A special position of the Coulomb-Tresca yield criterion and the associated plastic flow rule were fully recognized by D.D. Ivlev, who developed a mathematical theory of plasticity on an edge of the Coulomb-Tresca prism in a number of researches started from 1959. In this connection he relied on an early researches of A.Y. Ishlinskii, who proposed the original theory of plasticity in the most general three-dimensional formulation and took into account the necessity of satisfying the two yield conditions as it takes place in the case of "full plasticity", the condition of incompressibility of plastic flow, and the original tensor commutativity condition for the stress tensor and the tensor of increment of plastic deformations. However the generalized associated flow rule by Koiter was used in further discussions instead of the tensor commutativity condition. Nevertheless, it can be demonstrated that the tensor commutativity condition follows from the generalized associated flow rule and vice versa. Therefore, the Ishlinskii theory of plasticity is fully conforms to modern viewpoints on plastic flow of metals. It should be noted that analytical relations for stresses and increments of displacements in three-dimensional formulations of the Ishlinskii-Ivlev theory of plasticity form regular systems of partial differential equations of *hyperbolic* analytical type.

In the present paper a three-dimensional constitutive equation for perfect plastic solids is obtained by using the trinomial tensor formula by V.V. Novozhilov. The plastic flow of solid corresponds to an edge of the Coulomb-Tresca prism. It is shown that the constitutive tensor equation is essentially nonlinear and cannot be resolved with respect to incremental tensor of plastic deformations.

2. Equation of generalized associated flow rule on on edge of Coulomb-Tresca prism

Associated flow rule is a fundamental constitutive rule of mechanics of plastic flow of metals. It states (see [1-3]) that the vector representing in the principal stress space the incremental tensor of plastic deformations $d\boldsymbol{\varepsilon}^P$ is orthogonal to regular yield surface

$$f(\boldsymbol{\sigma})=0 \tag{1}$$

in a given stress state $\boldsymbol{\sigma}$:

$$d\boldsymbol{\varepsilon}^P=\frac{\partial f}{\partial \boldsymbol{\sigma}}d\lambda\,, \tag{2}$$

wherein $d\lambda$ – undetermined multiplier which is positive while active plastic flow is observed ($f=0,\quad df=0$).

The yield surface (1) of an isotropic solid is a finite relation between the principal stresses $\sigma_1,\sigma_2,\sigma_3$:

$$f(\sigma_1,\sigma_2,\sigma_3)=0\,. \tag{3}$$

The associated flow rule of an isotropic solid (2) in the principal axes of stress tensor has the following form:

$$d\varepsilon_j^P=\frac{\partial f}{\partial \sigma_j}d\lambda\,. \tag{4}$$

The associated flow rule in singular points of the yield surface like edges and conical points, which are analytically defined as intersections of a number of the regular yield surfaces $f_\kappa(\boldsymbol{\sigma})=0$ has the form [4]

$$d\boldsymbol{\varepsilon}^P=\sum_\alpha\frac{\partial f_\alpha}{\partial \boldsymbol{\sigma}}d\lambda_\alpha\,. \tag{5}$$

Undetermined coefficients $d\lambda_\alpha$ herein are constrained by the following conditions:

$$\begin{cases} d\lambda_\alpha=0 & (f_\alpha<0),\\ d\lambda_\alpha>0 & (f_\alpha=0,\quad df_\alpha=0),\\ d\lambda_\alpha=0 & (f_\alpha=0,\quad df_\alpha<0).\end{cases}$$

Consider the equation of associated flow rule (5) for yield surfaces, which is determined by the Coulomb-Tresca yield criterion. This criterion relates the transition to a plastic state with the attaining of critical value of the maximum of tangent stress. If the principal shear stresses are defined as

$$\tau_1=\frac{\sigma_2-\sigma_3}{2},\quad \tau_2=\frac{\sigma_3-\sigma_1}{2},\quad \tau_3=\frac{\sigma_1-\sigma_2}{2}\,. \tag{6}$$

Then the yield surface can be determined by the following equation:

$$[\tau_1^2-k^2][\tau_2^2-k^2][\tau_3^2-k^2]=0\,, \tag{7}$$

where k – is the shear yield limit, which defines the surface of hexagonal prism (the

Coulomb-Tresca prism) in the space of principal stresses (the Haigh-Westergaard stress space). For the edge of the prism determined by equations $\tau_1 = -k, \tau_2 = k, \tau_3 = 0$, in the *common* principal axes of stress tensor $\boldsymbol{\sigma}$ and incremental tensor of plastic deformations $d\boldsymbol{\varepsilon}^P$ the following equations can be obtained ($d\varepsilon_j^P$ – eigenvalues of $d\boldsymbol{\varepsilon}^P$):

$$d\varepsilon_1^P = -d\lambda_2, \quad d\varepsilon_2^P = -d\lambda_1, \quad d\varepsilon_3^P = d\lambda_1 + d\lambda_2 . \tag{8}$$

Therefore, the associated flow rule on the edge of Coulomb-Tresca prism is equivalent to the condition of incompressibility of plastic flow and a possibility to co-orient in the space the principal trihedrons of tensors $\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}^P$ at least in one way. According to $\tau_3 = 0$ only one principal stress axis is uniquely determined. This axis corresponds to the third principal stress. Therefore, the principal trihedron of $d\boldsymbol{\varepsilon}^P$ tensor will be principal for $\boldsymbol{\sigma}$ as well. Inversed proposition is *not valid*. Hence, the associated flow rule on the edge of Coulomb-Tresca prism applies minimum of constraints on the kinematics of plastic flow maximally “releasing” it.

3. Tensor commutativity equations for stresses and increments of plastic deformation

The mathematical theory of plasticity in its complete three-dimensional formulation was proposed by A.Y. Ishlinskii in 1946. Complete system of equations of the Ishlinskii theory [5] includes two yield criterions:

$$f_1(\sigma_1, \sigma_2, \sigma_3) = 0, \quad f_2(\sigma_1, \sigma_2, \sigma_3) = 0 , \tag{9}$$

the equation of incompressibility of perfectly plastic flow

$$d\varepsilon_1^P + d\varepsilon_2^P + d\varepsilon_3^P = 0 , \tag{10}$$

and the condition of $\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}^P$ tensors coaxiality which was accepted as:

$$\begin{aligned}
\sigma_{11}d\varepsilon_{12}^P + \sigma_{12}d\varepsilon_{22}^P + \sigma_{13}d\varepsilon_{23}^P &= \sigma_{21}d\varepsilon_{11}^P + \sigma_{22}d\varepsilon_{12}^P + \sigma_{23}d\varepsilon_{13}^P, \\
\sigma_{21}d\varepsilon_{31}^P + \sigma_{22}d\varepsilon_{32}^P + \sigma_{23}d\varepsilon_{33}^P &= \sigma_{31}d\varepsilon_{21}^P + \sigma_{32}d\varepsilon_{22}^P + \sigma_{33}d\varepsilon_{23}^P, \\
\sigma_{31}d\varepsilon_{11}^P + \sigma_{32}d\varepsilon_{12}^P + \sigma_{33}d\varepsilon_{13}^P &= \sigma_{11}d\varepsilon_{31}^P + \sigma_{12}d\varepsilon_{32}^P + \sigma_{13}d\varepsilon_{33}^P.
\end{aligned} \tag{11}$$

As one can see, the last equations essentially are the conditions of tensor commutativity for $\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}^P$:

$$\boldsymbol{\sigma} \cdot (d\boldsymbol{\varepsilon}^P) = (d\boldsymbol{\varepsilon}^P) \cdot \boldsymbol{\sigma} . \tag{12}$$

The consequence of (12) is the tensor symmetry of $\boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^P$, which means that $(\boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^P)^{\mathrm{T}} = \boldsymbol{\sigma} \cdot d\boldsymbol{\varepsilon}^P$.

It can be shown that the Ishlinskii equation of tensor commutativity (12) is the most compact expression for that fact that tensors $\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}^P$ have at least one co-oriented in the space orthonormal principal trihedron. Therefore, the Ishlinskii mathematical theory of plasticity (9)-(11) is equivalent to the associated flow rule on the edge of Coulomb-Tresca prism.

Given interpretation of the tensor commutativity equation (12) enables us to obtain a new tensor equation which relates the stresses with increments of plastic deformations on the edge of Coulomb-Tresca prism. At first, the deviator of the stress tensor **s** and directing tensors should be introduced

$$\tilde{\mathbf{s}} = \frac{\mathbf{s}}{|\mathbf{s}|}, \quad \widetilde{d\boldsymbol{\varepsilon}^P} = \frac{d\boldsymbol{\varepsilon}^P}{|d\boldsymbol{\varepsilon}^P|} , \tag{13}$$

where

$$|\mathbf{s}| = \frac{1}{\sqrt{6}}\sqrt{(s_1 - s_2)^2 + (s_1 - s_3)^2 + (s_2 - s_3)^2},$$

$$\left|d\boldsymbol{\varepsilon}^P\right| = \frac{1}{\sqrt{6}}\sqrt{(d\varepsilon_1^P - d\varepsilon_2^P)^2 + (d\varepsilon_1^P - d\varepsilon_3^P)^2 + (d\varepsilon_2^P - d\varepsilon_3^P)^2}.$$

For the further discussion the Novozhilov trinomial formula needs to be used [6], which is a nonlinear algebraic equation for tensors $\tilde{\mathbf{s}}$ и $d\tilde{\boldsymbol{\varepsilon}}^P$ under the condition that the principal axes of $d\boldsymbol{\varepsilon}^P$ are at the same time the principal axes of the $\mathbf{s}$ as it takes place in the case of flow on the edge of Coulomb-Tresca prism:

$$\tilde{\mathbf{s}} = \frac{1}{\cos 3\psi}\left\{\cos(2\psi + \vartheta)\, d\tilde{\boldsymbol{\varepsilon}}^P + \sqrt{3}\sin(\psi - \vartheta)\left[d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P - \frac{2}{3}\mathbf{I}\right]\right\}. \tag{14}$$

It is known that trinomial tensor equations of the form (14) cannot be inverted, in other words, the equation for $d\tilde{\boldsymbol{\varepsilon}}^P$ cannot be obtained. The angle variables ϑ, ψ in (14) are the form angles of stress and flow

$$\vartheta = \operatorname{arctg}\frac{2s_2 - s_1 - s_3}{\sqrt{3}(s_1 - s_3)}, \qquad \psi = \operatorname{arctg}\frac{2d\varepsilon_2^P - d\varepsilon_1^P - d\varepsilon_3^P}{\sqrt{3}(d\varepsilon_1^P - d\varepsilon_3^P)}.$$

It should be noticed that the phase ϑ can be determined as angle of inclination of the vector representing in the Haigh-Westergaard space the deviator $\mathbf{s}$ and lying in the deviatoric plane to the corresponding shear axis. The latter is orthogonal to projection on the deviatoric plane of the second coordinate axis of the Haigh-Westergaard space. The phase ϑ is constant for stress states on the edge of Coulomb-Tresca prism. In particular, if the plastic flow is on the edge

$$\sigma_1 = \sigma_2 = \sigma_3 + 2k,$$

then the phase ϑ equals $\frac{\pi}{6}$. Besides, the following equation is valid: $\tilde{\mathbf{s}} = \frac{\sqrt{3}}{2k}\mathbf{s}$.

One can also obtain

$$\mathbf{s} = \frac{2k}{\sqrt{3}\cos 3\psi}\left\{\cos(2\psi + \vartheta)\frac{\sqrt{2}\, d\tilde{\boldsymbol{\varepsilon}}^P}{\sqrt{\operatorname{tr}(d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P)}} + 2\sqrt{3}\sin(\psi - \vartheta)\left[\frac{d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P}{\sqrt{\operatorname{tr}(d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P)}} - \frac{1}{3}\mathbf{I}\right]\right\}$$

and

$$\mathbf{s} = \frac{2k}{\sqrt{3}\cos 3\psi}\left\{\cos(2\psi + \frac{\pi}{6})\frac{\sqrt{2}\, d\tilde{\boldsymbol{\varepsilon}}^P}{\sqrt{\operatorname{tr}(d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P)}} + 2\sqrt{3}\sin(\psi - \frac{\pi}{6})\left[\frac{d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P}{\sqrt{\operatorname{tr}(d\tilde{\boldsymbol{\varepsilon}}^P \cdot d\tilde{\boldsymbol{\varepsilon}}^P)}} - \frac{1}{3}\mathbf{I}\right]\right\} \tag{15}$$

The aforementioned constitutive equation (15) in combination with the incompressibility equation

$$\operatorname{tr} d\boldsymbol{\varepsilon}^P = 0$$

are equivalent to the associated flow rule. Consequently, the stress tensor deviator for stress states corresponding to the edge of Coulomb-Tresca prism is related to the incremental tensor of plastic deformations by an essentially nonlinear tensor equation, which fully determines the plastic flow in its fully free modes.

Acknowledgements

The paper is partially supported by Ministry of Education and Science of the Russian Federation in the framework of the part of governmental grant for Samara State Technical University (project № 16.2518.2014/K).

References

[1] D.D. Ivlev, *Theory of Perfect Plasticity* (Nauka, Moscow, 1966). (In Russian).

[2] L.M Kachanov, *Foundations of Theory of Plasticity* (Nauka, Moscow, 1969). (In Russian).

[3] Y.N. Radayev, *Three-Dimensional Problem of the Mathematical Theory of Plasticity* (Samara State University Publisher, Samara, 2006).

[4] W.T. Koiter // *Quarterly of Applied Mathematics* **11** (1953) 350.

[5] A.Y. Ishlinskii // *Uchenye Zapiski MGU. Mechanika. (Scientific notes of the Moscow State University. Mechanics)* **117** (1946) 90.

[6] V.V. Novozhilov // *Applied Mathematics and Mechanics* **15** (1951) 183.