NONLINEAR STATIONARY FLEXURAL-TORSIONAL WAVES
IN AN ELASTIC ROD

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Abstract. The processes of interaction of flexural and torsional waves that lead to the formation of periodic and solitary waves are studied here in the framework of geometrically nonlinear model of an elastic rod.

1. Introduction
In earlier papers dedicated to the dynamics of elastic structures based on rods, a linear theory of propagation of flexural-torsional waves in a thin-walled elastic beams of arbitrary cross section, the general features of the dispersion relation, taking into account the flexural-torsional relationship [1, 2], flexural-torsional vibrations of two rigidly coupled elastic rods with clamped ends [3].

In [4] within the geometrically nonlinear model of bending-torsional vibrations of an elastic rod, the problems of propagation of intense flexural wave in a twisted rod and the spread of intensive torsional waves in a rod with initial deflection.

In this paper, we study processes of interaction of flexural and torsional waves leading to the formation of nonlinear periodic and solitary waves.

2. Mathematical models
Coupled flexural-torsional waves are governed by a system of the form [4]:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + c_s^2 r_y^2 \dfrac{\partial^4 w}{\partial x^4} = 2c_m^2 \dfrac{\partial}{\partial x}\left[\left(\theta^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right)\dfrac{\partial w}{\partial x} + \dfrac{1}{2}r_y^2\left(\dfrac{\partial \theta}{\partial x}\right)^2 \dfrac{\partial w}{\partial x}\right], \\[4mm] \dfrac{\partial^2 \theta}{\partial t^2} - c_\tau^2 \dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{2c_m^2 I_y}{I_\rho}\dfrac{\partial \theta}{\partial x}\left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{\beta}{\rho I_\rho}\left(\dfrac{\partial \theta}{\partial x}\right)^3 + c_m^2 \theta^2 \dfrac{\partial \theta}{\partial x}\right] - \\[4mm] -\dfrac{2c_m^2}{r_\rho^2}\theta^3 - \dfrac{2c_m^2}{r_\rho^2}\theta\left(\dfrac{\partial w}{\partial x}\right)^2 - c_m^2 \theta\left(\dfrac{\partial \theta}{\partial x}\right)^2. \end{cases} \quad (1)$$

Here $w(x,t)$ is the lateral deflection of particles of the centerline of the rod; $\theta(x,t)$ is the angle of rotation of the cross-sectional area; $c_\tau = \sqrt{\mu/\rho}$ is the speed of propagation of shear

wave; $c_s = \sqrt{E/\rho}$ is the speed of propagation of longitudinal wave (rod speed); $c_m = \sqrt{(\lambda+\mu)/\rho}$; E is the Young's modulus; λ, μ are the Lame constants; ρ is the density of the rod material; r_y and r_ρ are the axial and angular radii of inertia defined by the following relations: $r_{y,\rho} = \sqrt{I_{y,\rho}/F}$, where $I_y = \iint\limits_F z^2 dF$ is the axial moment of inertia; $I_\rho = \iint\limits_F (y^2 + z^2)dF$ is the angular moment of inertia; F is the cross-sectional area; $\beta = (\lambda/2+\mu)\iint\limits_F (y^2+z^2)dF$. For a further simplification of equations, let us estimate values of coefficients comprising them. If the rod is made out of the aluminum alloy D16, then $\lambda = 6{,}99 \cdot 10^{10}$ N/m^2, $\mu = 2{,}75 \cdot 10^{10}$ N/m^2 [5], therefore $c_m^2/c_\tau^2 \approx 3$, and $c_m^2/c_s^2 \approx 1{,}125$.

3. Nonlinear waves.

We search for the solution of system (1) in the form of the travelling stationary waves:

$$w = w(\xi), \theta = \theta(\xi), \tag{2}$$

where $\xi = x - Vt$, V is the speed of a stationary wave. Relations (2) reduce (1) to the system of ordinary differential equations:

$$\frac{dW}{d\eta} = X, \quad \frac{dX}{d\eta} = -W + W^3 + W\psi^2 + WY^2, \quad \frac{d\psi}{d\eta} = Y, \quad \frac{dY}{d\eta} = \pm\frac{3}{2}(4WXY + \psi Y^2 - 2\psi^3 - 2\psi W^2), \tag{3}$$

where $\tilde{u} = (2\varepsilon)^{-1/2}W$, $\tilde{\theta} = (2\varepsilon)^{-1/2}\psi$, $\tilde{\theta} = \theta/\theta_0$, $\tilde{U} = U/\theta_0$, $U = \dfrac{dw}{d\xi}$, $\eta = \xi/\Lambda$, $\varepsilon \gg \theta_0^2$, $\Lambda \sim r_{\rho,y}$.

Note, that to obtain equation (6) we have neglected nonlinear terms in front of $\dfrac{d^2\tilde{\theta}}{d\eta^2}$ as higher-order terms compared to the saved terms.

Points $O(W_0, X_0, \psi_0, Y_0)$: $O_1(0,0,0,0)$, $O_2(-1,0,0,0)$, $O_3(1,0,0,0)$ represent states of equilibrium of equations (3).

Directly from equations (3) we establish the existence of the following integral manifolds of the system in the phase space $G\{W, X, \psi, Y\}$.

$M_1 = \{\psi=0, Y=0\}$ (coordinate plane (W, X)) is an integral manifold of system (3).

We conclude that in the absence of torsion, an existence of two types of stationary flexural waves is possible.

Solitons (kink and antikink) [6], corresponding to the separatrix going from saddle $O_2(-1,0,0,0)$ to saddle $O_3(1,0,0,0)$.

Periodic stationary waves correspond to periodic trajectories inside the separatrix contour. Trajectories near a separatrix loop correspond to cnoidal waves and trajectories located close to the equilibrium state $O_1(0,0,0,0)$ (center) correspond to quasi-harmonic waves.

$M_2 = \{W=0, X=0\}$ (coordinate plane (ψ, Y)) is an integral manifold of system (3). Thus, in the absence of flexural disturbances all possible stationary torsional waves are periodic and according to the terminology from the [6] they are essentially nonlinear.

Consider the dynamics of the system (3) beyond integral manifolds.

As shown by the numerical experiment, stationary motion of the oscillators in system (3) that occur outside the manifolds discussed above are either periodic or quasi-periodic. Consequently, these motions determine either periodic or quasi-periodic flexural-torsional stationary waves, see Figs. 1-4. Characteristic features of these waves are as follows.

When the waveform is shaping, the interaction of flexural and torsional vibrations is weak in the sense that small oscillations of one component at any point remain so at all points

of the wave profile. This situation is shown in Figs. 1 and 2. In the first case, the flexural component is small, and in the second one the torsional wave component is small. The weak interaction of flexural and twisting component of the stationary wave can be explained by the large spacing of their nonlinear frequencies determined by the displacement of both components from the equilibrium state.
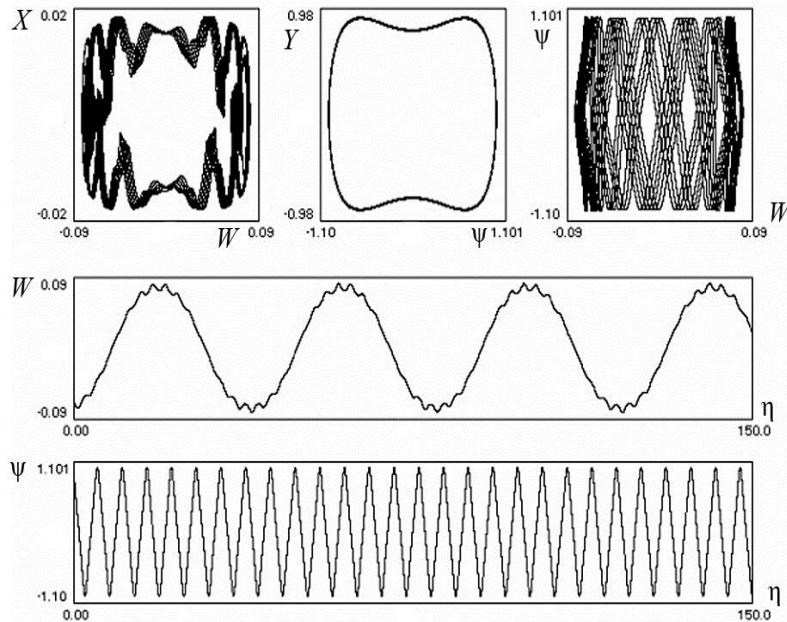
Fig. 1.
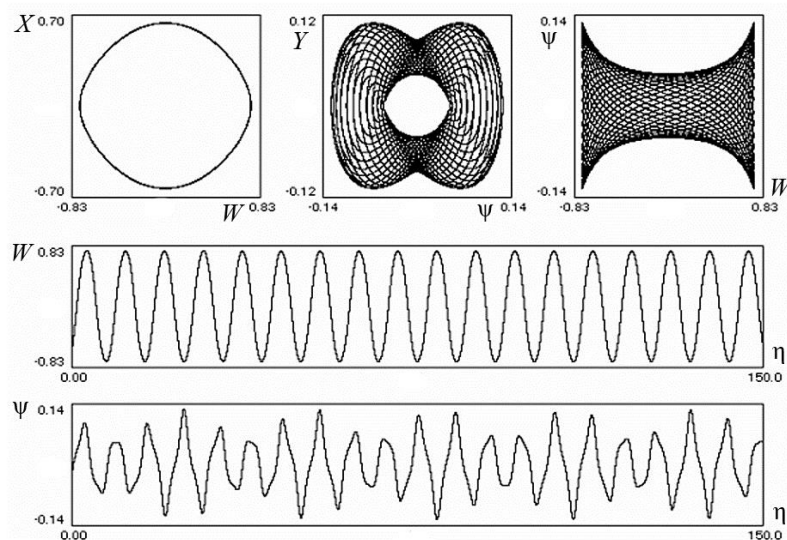
Fig. 2.

In the case of close intensities (when the have the same order) of the flexural and twisting components (respectively, the oscillation frequencies of the corresponding oscillators), their resonant interaction occurs, which results in a periodic stationary wave shown in Fig. 3, or in a quasi-periodic wave with considerable depth of modulation of one of its components, see Fig. 4.
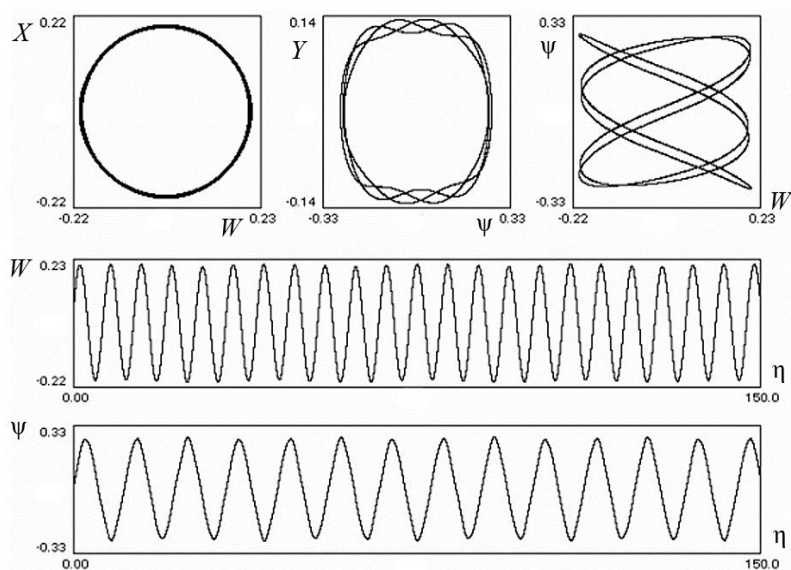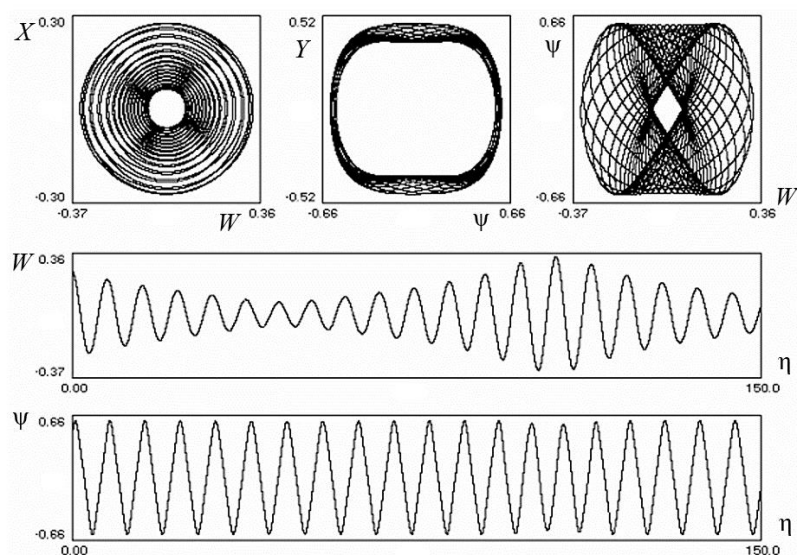
Fig. 3.

Fig. 4.

References
[1] P. Muller // *Journal of Sound and Vibration* **87** (1983) 115.
[2] V.I. Erofeev, O.I. Orekhova // *Volga Scientific Journal* **2** (2011) 7. (In Russian).
[3] N.A. Baranov, A.V. Rodygin // *Information-measuring and Operating Systems* **6** (2008) 47. (In Russian).
[4] V.I. Erofeev, V.V. Kazhaev, O.I. Orekhova // *Journal of Machinery Manufacture and Reliability* **41** (2012) 7.
[5] V.I. Erofeev, *Wave processes in solids with microstructure* (World Scientific, New Jersey, London, Singapore, Hong Kong, 2003).
[6] N.M. Ryskin, D.I. Trubetskov, *Nonlinear waves* (Nauka, Fizmatlit, Moscow, 2000). (In Russian).