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DYNAMIC ANALYSIS OF 3D ANISOTROPIC ELASTIC SOLIDS BY A LAPLACE DOMAIN BEM

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Abstract. This work presents a direct boundary element method (BEM) in the Laplace domain for transient dynamic analysis of the three-dimensional (3D) homogeneous linear anisotropic elastic bodies. Proposed boundary element formulation is applied together with the modified Durbin's method for inverse Laplace transform to obtain time domain results. Implementation of the proposed method is tested by comparing the numerical results with those obtained with the finite element method software.

1. Introduction

The boundary element method is a well-established and powerful numerical method and has been widely used for isotropic elastodynamic problems. However, a very few works are dedicated to the transient analysis of anisotropic elastic solids because only integral representations of the dynamic anisotropic Green's functions are available [1]. Most of the formulations employ the dual reciprocity technique first introduced by Nardini and Brebbia [2] because it requires only static fundamental solutions.

In this paper, a Laplace domain direct BEM formulation for the dynamic problems of anisotropic elasticity is presented. Time domain solutions are approximated by a modified Durbin's numerical Laplace transform inversion routine. An example problem is solved using the proposed formulation and the results are compared with solutions obtained using the finite element method.

2. Problem statement and BEM formulation

Consider a three-dimensional, homogeneous and linear elastic anisotropic body and let $\Omega \in R^3$ denote the region of space occupied by this body. The governing equations for linear elasticity without body force are represented in terms of the displacement vector u_i as follows

$$C_{ijkl}u_{k,lj} = \rho\ddot{u}_i \text{ in } \Omega, \ i, j, k, l = \overline{1,3}, \tag{1}$$

where C_{ijkl} is the stiffness tensor and ρ is the mass density.

Boundary conditions are given as follows:

$$u_i = u_i^* \text{ on } \Gamma^u, \tag{2}$$

$$t_i = t_i^* \text{ on } \Gamma^t, \tag{3}$$

where $t_i = \sigma_{ij}n_j$ is the traction vector with n_j being the outward unit normal vector to the boundary $\Gamma = \partial\Omega = \Gamma^u \cup \Gamma^t$, Γ^u and Γ^t denote the Dirichlet and Neumann boundaries, respectively.

Throughout the paper, zero initial conditions are assumed:

$$u_i(\mathbf{x},t) = \dot{u}_i(\mathbf{x},t) = 0 \text{ for } t = 0. \tag{4}$$

In order to suppress time variable we employ the Laplace transform

$$L\{f(\mathbf{x},t)\} = \bar{f}(\mathbf{x},s) = \int_0^\infty f(\mathbf{x},t)e^{-st}dt, \tag{5}$$

where $s = \alpha + i\omega$ is the transform parameter. Applying the Laplace transform to Eqs. (1)-(3) under zero initial conditions we obtain

$$C_{ijkl}\bar{u}_{k,il} = \rho s^2 \bar{u}_k \text{ in } \Omega, \tag{6}$$

$$\bar{u}_i = \bar{u}_i^* \text{ on } \Gamma^u, \tag{7}$$

$$\bar{t}_i = \bar{t}_i^* \text{ on } \Gamma^t, \tag{8}$$

where overline denotes a transformed variable.

BIE and boundary element formulation. The boundary integral representation of Eq. (6) can be obtained by utilizing the reciprocal relation between two elastodynamic states with the same Laplace transform parameter *s*, one of the states being the fundamental solution of Eq. (6), and is formulated as

$$c_{jk}\bar{u}_k(\mathbf{x},s) + (\text{p.v.})\int_\Gamma \bar{h}_{jk}(\mathbf{x},\mathbf{y},s)\bar{u}_k(\mathbf{x},s)\mathrm{d}\Gamma(\mathbf{y}) = \int_\Gamma \bar{g}_{jk}(\mathbf{x},\mathbf{y},s)\bar{t}_k(\mathbf{x},s)\mathrm{d}\Gamma(\mathbf{y}), \quad \mathbf{x} \in \Gamma, \tag{9}$$

where (p.v.) denotes a Cauchy principal value of the singular integral, c_{jk} is the free-term matrix, **x** and **y** represent the source and the field points, $\bar{u}_k$ and $\bar{t}_k$ – transformed displacement and traction vectors components, $\bar{g}_{jk}$ and $\bar{h}_{jk}$ are the dynamic anisotropic elastic displacement and traction fundamental solutions in the Laplace domain. Since closed-forms expressions for fundamental solutions are unavailable, we employ corresponding integral representations [1, 3]. In addition, we use interpolation approach proposed by Wilson and Cruse [4] to calculate the static parts of the fundamental solutions.

A numerical implementation of Eq. (9) is carried out by the mixed boundary elements approach [5]. The boundary surface Γ is approximated by quadratic quadrilateral elements, the displacements $\bar{u}_k$ and tractions $\bar{t}_k$ are approximated by quadrilateral linear and constant elements, respectively. After applying the collocation technique for spatial discretization of Eq. (9) and enforcing the boundary conditions we obtain the resulting system of linear equations which is parameterized by the Laplace transform variable *s*. It only remains to invert solution back to the time domain.

3. Numerical inversion of Laplace transform

The inverse Laplace transform is defined as the following contour integral

$$L^{-1}\{\bar{f}(\mathbf{x},s)\} = f(\mathbf{x},t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \bar{f}(\mathbf{x},s)e^{st}ds, \tag{10}$$

where $\alpha > 0$ is the arbitrary real constant greater than the real parts of all singularities in $\bar{f}(\mathbf{x},s)$.

When values of $\bar{f}(\mathbf{x},s)$ are available only at the sample points, analytical evaluation of integral in Eq. (10) is impossible. Supposing $s = \alpha + i\omega$ we have the following expressions [6] (for convenience the spatial variable **x** is omitted hereinafter)

$$f(0) = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\bar{f}(\alpha + i\omega)\right]d\omega, \tag{11}$$

$$f(t)=\frac{e^{\alpha t}}{\pi}\int_0^{\infty}\left\{\mathrm{Re}\left[\bar f(\alpha+i\omega)\right]\cos\omega t-\mathrm{Im}\left[\bar f(\alpha+i\omega)\right]\sin\omega t\right\}d\omega,\ t>0. \tag{12}$$

Durbin [6] developed an approach, namely fast Laplace inverse transform (FLIT) for numerical evaluation of the integrals in Eqs. (11) and (12). In this section, we briefly review a modifications recently proposed by Zhao [7] in order to overcome a drawback of constant integration step in FLIT.

Let R be large real number so we can rewrite Eqs. (11) and (12) as follows

$$f(0)=\frac{1}{\pi}\lim_{R\to\infty}\int_0^{R}\mathrm{Re}\left[\bar f(\alpha+i\omega)\right]d\omega, \tag{13}$$

$$f(t)=\frac{e^{\alpha t}}{\pi}\lim_{R\to\infty}\int_0^{R}\left\{\mathrm{Re}\left[\bar f(\alpha+i\omega)\right]\cos\omega t-\mathrm{Im}\left[\bar f(\alpha+i\omega)\right]\sin\omega t\right\}d\omega,\ t>0. \tag{14}$$

Defining the nodes as $0=\omega_1<\omega_2<...<\omega_n<\omega_{n+1}=R$ we approximate Eqs. (13) and (14) as

$$f(0)=\frac{1}{\pi}\sum_{k=1}^{n}\int_{\omega_k}^{\omega_{k+1}}\mathrm{Re}\left[\bar f(\alpha+i\omega)\right]d\omega, \tag{15}$$

$$f(t)=\frac{e^{\alpha t}}{\pi}\sum_{k=1}^{n}\int_{\omega_k}^{\omega_{k+1}}\left\{\mathrm{Re}\left[\bar f(\alpha+i\omega)\right]\cos(\omega t)-\mathrm{Im}\left[\bar f(\alpha+i\omega)\right]\sin(\omega t)\right\}d\omega,\ t>0. \tag{16}$$

Algorithm 1. In each segment $[\omega_k,\omega_{k+1}]$, $k=\overline{1,n}$, the real and imaginary parts of $\bar f(s)$ are approximated with linear functions as follows

$$\mathrm{Re}[\bar f(\alpha+i\omega)]\approx F_k+\frac{F_{k+1}-F_k}{\omega_{k+1}-\omega_k}(\omega-\omega_k),\ \mathrm{Im}[\bar f(\alpha+i\omega)]\approx G_k+\frac{G_{k+1}-G_k}{\omega_{k+1}-\omega_k}(\omega-\omega_k), \tag{17}$$

where $F_k=\mathrm{Re}[\bar f(\alpha+i\omega_k)]$, $G_k=\mathrm{Im}[\bar f(\alpha+i\omega_k)]$.

Substituting Eq. (17) into Eqs. (15) and (16) and making direct integration we obtain

$$f(0)\approx\sum_{k=1}^{n}\left[\frac{(F_{k+1}-F_k)\Delta_k}{2\pi}\right], \tag{18}$$

$$f(t)\approx\frac{e^{\alpha t}}{\pi t^2}\sum_{k=1}^{n}\left[\frac{F_{k+1}-F_k}{\Delta_k}\left(\cos(\omega_{k+1}t)-\cos(\omega_k t)\right)+\frac{G_{k+1}-G_k}{\Delta_k}\left(\sin(\omega_{k+1}t)-\sin(\omega_k t)\right)\right],\ t>0, \tag{19}$$

where $\Delta_k=\omega_{k+1}-\omega_k$.

Algorithm 2. The real and imaginary parts of $\bar f(s)$ are now approximated with Subbotin-splines

$$\mathrm{Re}[\bar f(\alpha+i\omega)]\approx F_k+\frac{1}{2}(Z_{k+1}+Z_k)(\omega-\tau_k)+\frac{1}{2\Delta_k}(Z_{k+1}-Z_k)(\omega-\tau_k)^2, \tag{20}$$

$$\mathrm{Im}[\bar f(\alpha+i\omega)]\approx F_k+\frac{1}{2}(Y_{k+1}+Y_k)(\omega-\tau_k)+\frac{1}{2\Delta_k}(Y_{k+1}-Y_k)(\omega-\tau_k)^2, \tag{21}$$

where $\tau_k=(\omega_{k+1}+\omega_k)/2$, $F_k=\mathrm{Re}[\bar f(\alpha+i\tau_k)]$, $G_k=\mathrm{Im}[\bar f(\alpha+i\tau_k)]$.

Z_k, $k=\overline{1,n+1}$ are determined from the following tridiagonal system

$$\begin{aligned}&3\Delta_1 Z_1+\Delta_1 Z_2=8(F_1-F_0),\\&\Delta_{k-1}Z_{k-1}+3(\Delta_k+\Delta_{k-1})Z_k+\Delta_k Z_{k+1}=8(F_k-F_{k-1}),\\&3\Delta_n Z_{n+1}+\Delta_n Z_n=8(F_{n+1}-F_n),\end{aligned} \tag{22}$$

where $F_0=\mathrm{Re}[\bar f(\alpha+i\omega_1)]$, $F_{n+1}=\mathrm{Re}[\bar f(\alpha+i\omega_{n+1})]$.

$Y_k,\ k=\overline{1,n+1}$ are determined from

$$
\begin{aligned}
&3\Delta_1 Y_1 + \Delta_1 Y_2 = 8(G_1 - G_0),\\
&\Delta_{k-1} Y_{k-1} + 3(\Delta_k + \Delta_{k-1}) Y_k + \Delta_k Y_{k+1} = 8(G_k - G_{k-1}),\\
&3\Delta_n Y_{n+1} + \Delta_n Y_n = 8(G_{n+1} - G_n),
\end{aligned}
\tag{23}
$$

where $G_0 = \mathrm{Im}[\bar f(\alpha + i\omega_1)]$, $G_{n+1} = \mathrm{Im}[\bar f(\alpha + i\omega_{n+1})]$.

Substituting Eqs. (20) and (21) into Eqs. (15) and (16) and making direct integration we finally obtain

$$
f(0) \approx \frac{1}{\pi}\sum_{k=1}^{n}\left[F_k + \frac{\Delta_k}{24}(Z_{k+1} - Z_k)\right]\Delta_k, \tag{24}
$$

$$
\begin{aligned}
f(t) \approx \frac{e^{\alpha t}}{\pi}\Bigg\{ &\frac{1}{t} g_1(t) + \frac{1}{t^2} g_2(t) - \sum_{k=1}^{n}\frac{1}{t^3\Delta_k}\big[(Z_{k+1} - Z_k)(\sin\omega_{k+1}t - \sin\omega_k t) + \\
&+ (Y_{k+1} - Y_k)(\cos\omega_{k+1}t - \cos\omega_k t)\big]\Bigg\}, \quad t > 0,
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
g_1(t) = &\left[F_n + \frac{1}{8}(Z_n + 3Z_{n+1})\Delta_n\right]\sin\omega_{n+1}t + \frac{1}{8}(Y_2 + 3Y_1)\Delta_1 - G_1 + \\
&+\left[G_n + \frac{1}{8}(Y_n + 3Y_{n+1})\Delta_n\right]\cos\omega_{n+1}t,
\end{aligned}
\tag{26}
$$

$$
g_2(t) = -Z_1 + Z_{n+1}\cos\omega_{n+1}t - Y_{n+1}\sin\omega_{n+1}t. \tag{27}
$$

4. Numerical example

In order to assess the accuracy of the proposed BEM formulation the following problem is solved (see Figure 1): three-dimensional anisotropic elastic prismatic body is clamped at its left end, and subjected to uniaxial and uniform impact loading $t_2 = t_2^* H(t)$, $t_2^* = -1\,\mathrm{N/m^2}$ at the right end, $H(t)$ is a Heaviside step function. The remaining surfaces are traction free. The considered material is fully anisotropic with mass density $\rho = 1600\,\mathrm{kg/m^3}$ and with the following stiffness tensor [8]:

$$
\mathbf{C} = \begin{bmatrix}
60.23 & 18.67 & 18.96 & -7.69 & 15.60 & -25.28\\
 & 21.26 & 9.36 & -3.74 & 4.21 & -8.47\\
 & & 47.04 & -8.82 & 15.28 & -8.31\\
 & & & 10.18 & -9.54 & 5.69\\
 & \mathrm{sym.} & & & 21.19 & -8.54\\
 & & & & & 20.75
\end{bmatrix}\mathrm{GPa}. \tag{28}
$$

In this example, we employ the following relation to produce the integration steps $\Delta_k = \omega_{k+1} - \omega_k$ in both Zhao's algorithms:

$$
\omega_k = e^{(kx)^m} - 1, \tag{29}
$$

where $m = 0.8$, $k = 1,2,\ldots,750$ and $x = \left(\ln(\omega_{\max} + 1)\right)^{1/m}/k$, with $\omega_{\max} = 330$.

To discretize the problem boundary 686 elements with a 688 nodes were used. Figure 2 shows the displacement $u_2(t)$ at the center point of the loaded end and tractions $t_2(t)$ at the center point of the clamped end. As it can be observed, BEM results obtained with both Zhao's algorithms are almost identical and in a very good agreement with the corresponding FEM solutions.

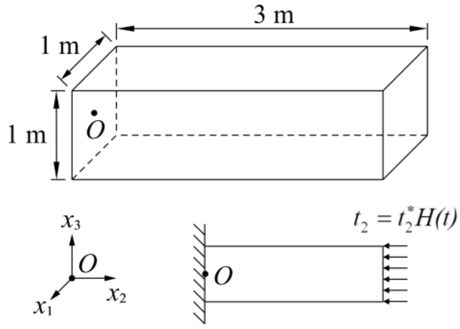

Fig 1. Prismatic body under impact loading.

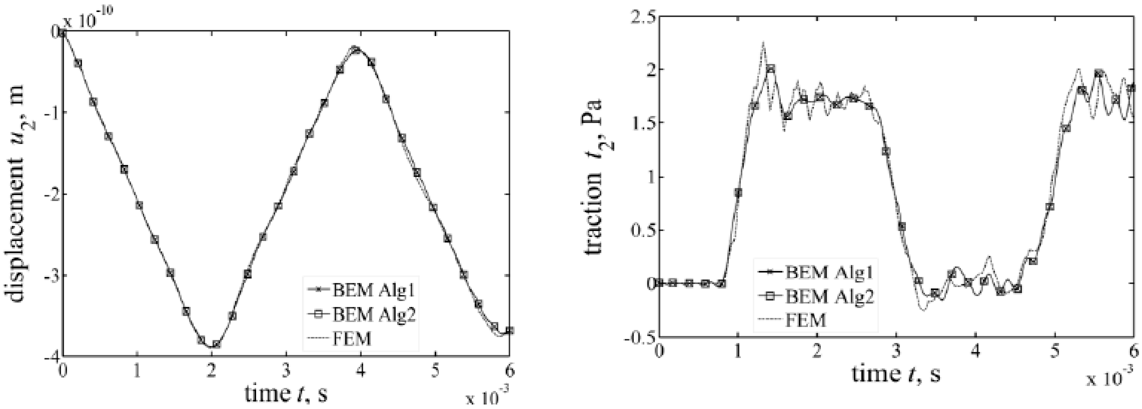

Fig 2. Response of displacements $u_2(0{,}3{,}0;t)$ and tractions $t_2(0{,}0{,}0;t)$ using BEM and FEM.

5. Conclusions

A Laplace domain BEM formulation based on integral representations of the fundamental solutions has been presented for the analysis of three-dimensional anisotropic elastodynamic problems. To verify the proposed formulation a transient analysis for general anisotropic solid is carried out. The accuracy is confirmed by comparing obtained boundary element solutions with the corresponding finite element results. The modified Durbin's method we used to invert solution to the time domain is proved to be an accurate and efficient method particularly well suited for the dynamic problems of the linear anisotropic elasticity.

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