# DYNAMIC ANALYSIS OF 3D ANISOTROPIC ELASTIC SOLIDS BY A LAPLACE DOMAIN BEM

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**Abstract.** This work presents a direct boundary element method (BEM) in the Laplace domain for transient dynamic analysis of the three-dimensional (3D) homogeneous linear anisotropic elastic bodies. Proposed boundary element formulation is applied together with the modified Durbin's method for inverse Laplace transform to obtain time domain results. Implementation of the proposed method is tested by comparing the numerical results with those obtained with the finite element method software.

# **1. Introduction**

The boundary element method is a well-established and powerful numerical method and has been widely used for isotropic elastodynamic problems. However, a very few works are dedicated to the transient analysis of anisotropic elastic solids because only integral representations of the dynamic anisotropic Green's functions are available [1]. Most of the formulations employ the dual reciprocity technique first introduced by Nardini and Brebbia [2] because it requires only static fundamental solutions.

In this paper, a Laplace domain direct BEM formulation for the dynamic problems of anisotropic elasticity is presented. Time domain solutions are approximated by a modified Durbin's numerical Laplace transform inversion routine. An example problem is solved using the proposed formulation and the results are compared with solutions obtained using the finite element method.

# 2. Problem statement and BEM formulation

Consider a three-dimensional, homogeneous and linear elastic anisotropic body and let  $\Omega \in \mathbb{R}^3$  denote the region of space occupied by this body. The governing equations for linear elasticity without body force are represented in terms of the displacement vector  $u_i$  as follows

$$C_{iikl}u_{k,li} = \rho \ddot{u}_i \text{ in } \Omega, \ i, j, k, l = \overline{1,3}, \tag{1}$$

where  $C_{iikl}$  is the stiffness tensor and  $\rho$  is the mass density.

Boundary conditions are given as follows:

$$u_i = u_i^* \text{ on } \Gamma^u , \tag{2}$$

$$t_i = t_i^* \text{ on } \Gamma^t, \tag{3}$$

where  $t_i = \sigma_{ij}n_j$  is the traction vector with  $n_j$  being the outward unit normal vector to the boundary  $\Gamma = \partial \Omega = \Gamma^u \cup \Gamma^t$ ,  $\Gamma^u$  and  $\Gamma^t$  denote the Dirichlet and Neumann boundaries, respectively.

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$$f(t) = \frac{e^{\alpha t}}{\pi} \int_{0}^{\infty} \left\{ \operatorname{Re}\left[\bar{f}(\alpha + i\omega)\right] \cos \omega t - \operatorname{Im}\left[\bar{f}(\alpha + i\omega)\right] \sin \omega t \right\} d\omega, \ t > 0.$$
(12)

Durbin [6] developed an approach, namely fast Laplace inverse transform (FLIT) for numerical evaluation of the integrals in Eqs. (11) and (12). In this section, we briefly review a modifications recently proposed by Zhao [7] in order to overcome a drawback of constant integration step in FLIT.

Let R be large real number so we can rewrite Eqs. (11) and (12) as follows

$$f(0) = \frac{1}{\pi} \lim_{R \to \infty} \int_{0}^{R} \operatorname{Re}\left[\bar{f}(\alpha + i\omega)\right] d\omega, \tag{13}$$

$$f(t) = \frac{e^{\alpha t}}{\pi} \lim_{R \to \infty} \int_{0}^{R} \left\{ \operatorname{Re}\left[\bar{f}(\alpha + i\omega)\right] \cos \omega t - \operatorname{Im}\left[\bar{f}(\alpha + i\omega)\right] \sin \omega t \right\} d\omega, \ t > 0.$$
(14)

Defining the nodes as  $0 = \omega_1 < \omega_2 < ... < \omega_n < \omega_{n+1} = R$  we approximate Eqs. (13) and (14) as

$$f(0) = \frac{1}{\pi} \sum_{k=1}^{n} \int_{\omega_{k}}^{\omega_{k+1}} \operatorname{Re}\left[\bar{f}(\alpha + i\omega)\right] d\omega,$$
(15)

$$f(t) = \frac{e^{\alpha t}}{\pi} \sum_{k=1}^{n} \int_{\omega_k}^{\omega_{k+1}} \left[ \operatorname{Re}\left[\bar{f}(\alpha + i\omega)\right] \cos(\omega t) - \operatorname{Im}\left[\bar{f}(\alpha + i\omega)\right] \sin(\omega t) \right] d\omega, \ t > 0.$$
(16)

Algorithm 1. In each segment  $[\omega_k, \omega_{k+1}]$ ,  $k = \overline{1, n}$ , the real and imaginary parts of  $\overline{f}(s)$  are approximated with linear functions as follows

$$\operatorname{Re}[\bar{f}(\alpha+i\omega)] \approx F_{k} + \frac{F_{k+1} - F_{k}}{\omega_{k+1} - \omega_{k}}(\omega - \omega_{k}), \quad \operatorname{Im}[\bar{f}(\alpha+i\omega)] \approx G_{k} + \frac{G_{k+1} - G_{k}}{\omega_{k+1} - \omega_{k}}(\omega - \omega_{k}), \quad (17)$$
where  $F_{k} = \operatorname{Re}[\bar{f}(\alpha+i\omega)] = G_{k} - \operatorname{Im}[\bar{f}(\alpha+i\omega)]$ 

where  $F_k = \operatorname{Re}[f(\alpha + i\omega_k)], G_k = \operatorname{Im}[f(\alpha + i\omega_k)].$ 

Substituting Eq. (17) into Eqs. (15) and (16) and making direct integration we obtain

$$f(0) \approx \sum_{k=1}^{n} \left[ \frac{(F_{k+1} - F_k)\Delta_k}{2\pi} \right],$$
(18)

$$f(t) \approx \frac{e^{\alpha t}}{\pi t^2} \sum_{k=1}^{n} \left[ \frac{F_{k+1} - F_k}{\Delta_k} \left( \cos(\omega_{k+1}t) - \cos(\omega_k t) \right) + \frac{G_{k+1} - G_k}{\Delta_k} \left( \sin(\omega_{k+1}t) - \sin(\omega_k t) \right) \right], \quad t > 0, \quad (19)$$
where  $\Delta_k = \omega_{k+1} - \omega_k$ .

Algorithm 2. The real and imaginary parts of  $\bar{f}(s)$  are now approximated with Subbotin-splines

$$\operatorname{Re}[\bar{f}(\alpha + i\omega)] \approx F_{k} + \frac{1}{2}(Z_{k+1} + Z_{k})(\omega - \tau_{k}) + \frac{1}{2\Delta_{k}}(Z_{k+1} - Z_{k})(\omega - \tau_{k})^{2}, \qquad (20)$$

$$\operatorname{Im}[\bar{f}(\alpha + i\omega)] \approx F_{k} + \frac{1}{2}(Y_{k+1} + Y_{k})(\omega - \tau_{k}) + \frac{1}{2\Delta_{k}}(Y_{k+1} - Y_{k})(\omega - \tau_{k})^{2}, \qquad (21)$$

where  $\tau_k = (\omega_{k+1} + \omega_k)/2$ ,  $F_k = \operatorname{Re}[\bar{f}(\alpha + i\tau_k)], G_k = \operatorname{Im}[\bar{f}(\alpha + i\tau_k)].$ 

 $Z_k$ ,  $k = \overline{1, n+1}$  are determined from the following tridiagonal system

$$3\Delta_{1}Z_{1} + \Delta_{1}Z_{2} = 8(F_{1} - F_{0}),$$

$$\Delta_{k-1}Z_{k-1} + 3(\Delta_{k} + \Delta_{k-1})Z_{k} + \Delta_{k}Z_{k+1} = 8(F_{k} - F_{k-1}),$$

$$3\Delta_{n}Z_{n+1} + \Delta_{n}Z_{n} = 8(F_{n+1} - F_{n}),$$
where  $F_{0} = \operatorname{Re}[\bar{f}(\alpha + i\omega_{1})], F_{n+1} = \operatorname{Re}[\bar{f}(\alpha + i\omega_{n+1})].$ 
(22)

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$$Y_{k}, \ k = \overline{1, n+1} \text{ are determined from} \\ 3\Delta_{1}Y_{1} + \Delta_{1}Y_{2} = 8(G_{1} - G_{0}), \\ \Delta_{k-1}Y_{k-1} + 3(\Delta_{k} + \Delta_{k-1})Y_{k} + \Delta_{k}Y_{k+1} = 8(G_{k} - G_{k-1}), \\ 3\Delta_{n}Y_{n+1} + \Delta_{n}Y_{n} = 8(G_{n+1} - G_{n}),$$
(23)

where  $G_0 = \text{Im}[\bar{f}(\alpha + i\omega_1)], \ G_{n+1} = \text{Im}[\bar{f}(\alpha + i\omega_{n+1})].$ 

Substituting Eqs. (20) and (21) into Eqs. (15) and (16) and making direct integration we finally obtain

$$f(0) \approx \frac{1}{\pi} \sum_{k=1}^{n} \left[ F_k + \frac{\Delta_k}{24} (Z_{k+1} - Z_k) \right] \Delta_k,$$
(24)

$$f(t) \approx \frac{e^{\alpha t}}{\pi} \left\{ \frac{1}{t} g_1(t) + \frac{1}{t^2} g_2(t) - \sum_{k=1}^n \frac{1}{t^3 \Delta_k} \left[ (Z_{k+1} - Z_k) (\sin \omega_{k+1} t - \sin \omega_k t) + (Y_{k+1} - Y_k) (\cos \omega_{k+1} t - \cos \omega_k t) \right] \right\}, \quad t > 0,$$
(25)

where

$$g_{1}(t) = \left[F_{n} + \frac{1}{8}(Z_{n} + 3Z_{n+1})\Delta_{n}\right]\sin\omega_{n+1}t + \frac{1}{8}(Y_{2} + 3Y_{1})\Delta_{1} - G_{1} + \left[G_{n} + \frac{1}{8}(Y_{n} + 3Y_{n+1})\Delta_{n}\right]\cos\omega_{n+1}t,$$
(26)

$$\int_{2}^{2} (t) = -Z_1 + Z_{n+1} \cos \omega_{n+1} t - Y_{n+1} \sin \omega_{n+1} t.$$
(27)

#### 4. Numerical example

In order to assess the accuracy of the proposed BEM formulation the following problem is solved (see Figure 1): three-dimensional anisotropic elastic prismatic body is clamped at its left end, and subjected to uniaxial and uniform impact loading  $t_2 = t_2^* H(t)$ ,  $t_2^* = -1 \text{N/m}^2$  at the right end, H(t) is a Heaviside step function. The remaining surfaces are traction free. The considered material is fully anisotropic with mass density  $\rho = 1600 \text{ kg/m}^3$  and with the following stiffness tensor [8]:

$$\mathbf{C} = \begin{bmatrix} 60.23 & 18.67 & 18.96 & -7.69 & 15.60 & -25.28 \\ 21.26 & 9.36 & -3.74 & 4.21 & -8.47 \\ & 47.04 & -8.82 & 15.28 & -8.31 \\ & 10.18 & -9.54 & 5.69 \\ sym. & 21.19 & -8.54 \\ & & 20.75 \end{bmatrix} \mathbf{GPa.}$$
(28)

In this example, we employ the following relation to produce the integration steps  $\Delta_k = \omega_{k+1} - \omega_k$  in both Zhao's algorithms:

$$\omega_k = e^{(kx)^m} - 1, \tag{29}$$

where m = 0.8, k = 1, 2, ..., 750 and  $x = (\ln(\omega_{\text{max}} + 1))^{1/m}/k$ , with  $\omega_{\text{max}} = 330$ .

To discretize the problem boundary 686 elements with a 688 nodes were used. Figure 2 shows the displacement  $u_2(t)$  at the center point of the loaded end and tractions  $t_2(t)$  at the center point of the clamped end. As it can be observed, BEM results obtained with both Zhao's algorithms are almost identical and in a very good agreement with the corresponding FEM solutions.







**Fig 2.** Response of displacements  $u_2(0,3,0;t)$  and tractions  $t_2(0,0,0;t)$  using BEM and FEM.

## **5.** Conclusions

A Laplace domain BEM formulation based on integral representations of the fundamental solutions has been presented for the analysis of three-dimensional anisotropic elastodynamic problems. To verify the proposed formulation a transient analysis for general anisotropic solid is carried out. The accuracy is confirmed by comparing obtained boundary element solutions with the corresponding finite element results. The modified Durbin's method we used to invert solution to the time domain is proved to be an accurate and efficient method particularly well suited for the dynamic problems of the linear anisotropic elasticity.

# **Acknowledgements**

This work is supported by the Russian Science Foundation under grant No. 16-19-10237.

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