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A NUMERICAL STUDY OF WAVE PROPAGATION ON POROELASTIC HALF-SPACE WITH CAVITIES BY USE THE BEM AND RUNGE-KUTTA METHOD

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Abstract. The report presents the development of the time-boundary element methodology and a description of the related software based on a stepped method of numerical inversion of the integral Laplace transform in combination with a family of Runge-Kutta methods for analyzing 3-D mixed initial boundary-value problems of the dynamics of poroelastic bodies. The results of the numerical investigation are presented. The investigation methodology is based on direct-approach boundary integral equations of 3-D isotropic linear theory of poroelasticity in Laplace transforms. Poroelastic media are described using Biot models with four functions. With the help of the boundary-element method, solutions in time are obtained, using the stepped method of numerically inverting Laplace transform on the nodes of Runge-Kutta methods.

1. Introduction

The paper is dedicated to the development of 3-D poroelastodynamic problems numerical modeling technique based on Boundary Element Method (BEM) usage in Laplace domain and time-stepping schemes for Laplace transform numerical inversion. Specially designed software allows analyzing wave processes in homogeneous and piecewise homogeneous solids.

There are two major approaches to dynamic processes modeling by means of BEM: solving BIE system directly in time domain [1], or in Laplace or Fourier domain followed by the respective transform inversion [2]. Traditional stepping schemes with spline approximation in scope of the first approach require fundamental solutions in time what severely restricts their usage. Often it’s only possible to construct fundamental solution matrices in Laplace or Fourier domain. Thus first BEM formulations for Biot’s poroelastodynamics, published by G.D. Manolis & D.E. Beskos, employ Laplace transform.

Formulation in time domain has been developed in [3] basing on analytical inversion of Laplace transform for fundamental solutions. The main shortcoming of the approach is again the demand of fundamental solutions in time. They exist for quasi-static problems of poro- & visco-elasticity, but are quite cumbersome; damping effects cannot be accounted at all. As well the methodology is characterized by significant computing costs and low stability rate. In 1988 C. Lubich introduced brand new Convolution Quadrature Method (CQM) [4,5]. Now it’s widely applied in construction of time-step boundary element schemes on the basis of fundamental solutions in Laplace domain [6, 7]. CQM helps to get rid of fundamental solutions in time requirement and shows better stability but is still costly. Work [8] addresses the issue of Runge-Kutta-based CQM usage. Various combinations of this CQM modification with BEM stepping schemes were developed in [9, 10]. An overview of different boundary element approaches based on CQM is presented in [11]. Works [9, 12] estimate the accuracy and precision of Runge-Kutta-based CQM. In this paper a similar boundary element scheme based on the stepping method for Laplace transform numerical inversion is considered. A modification of the scheme with varied integration step relying on highly oscillatory quadrature principles is employed.

2. Problem formulation

The boundary-value problem for full Biot's model of linear saturated poroelastic continuum in Laplace domain concerning 4 basic functions – skeleton displacements $\bar{u}_i$ and pore pressure $\bar{p}$ – takes the following form [6]:

$$G\bar{u}_{i,jj} + \left(K + \frac{G}{3}\right)\bar{u}_{j,ij} - (\alpha - \beta)\bar{p}_{,i} - s^2(\rho - \beta\rho_f)\bar{u}_i = -\bar{F} \tag{3}$$

$$\frac{\beta}{s\rho_f}\bar{p}_{,ii} - \frac{\phi^2 s}{R}\bar{p} - (\alpha - \beta)s\bar{u}_{i,i} = -\bar{a}, \quad x \in \Omega, \tag{4}$$

$$\bar{u}'(x,s) = \tilde{u}', \quad x \in \Gamma^u, \quad \bar{u}' = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{p}), \tag{5}$$

$$\bar{t}'_n(x,s) = \tilde{t}'_n, \quad x \in \Gamma^\sigma, \quad \bar{t}' = (\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{q}) \tag{6}$$

where Γ^u denotes Dirichlet boundary and Γ^σ denotes Neumann boundary, G, K are elastic moduli ϕ is porosity, $\bar{F}_i, \bar{a}$ are bulk body forces,

$$\beta = \frac{\kappa\rho_f\phi^2 s}{\phi^2 + s\kappa(\rho_e + \phi\rho_f)}, \ \alpha = 1 - \frac{K}{K_s}, \tag{7}$$

$$R = \frac{\phi^2 K_f {K_s}^2}{K_f(K_s - K) + \phi K_s(K_s - K_f)} \tag{8}$$

– constants describe the interaction between the skeleton and filler, κ is permeability, ρ, ρ_e, ρ_f are material density, apparent mass density and filler density respectively, K_s, K_f are elastic bulk moduli of the skeleton and filler respectively.

3. BEM application

The boundary-value problem can be reduced to the following boundary integral equation [6, 13, 14]:

$$\alpha_\Omega \bar{u}_k(x,s) + \int_\Gamma \left(\tilde{T}_{ik}(x,y,s)\bar{u}_i(y,s) - \tilde{T}^0_{ik}(x,y,s)\bar{u}_i(x,s) - \tilde{U}_{ik}(x,y,s)\bar{t}_i(y,s)\right)d\Gamma = 0,$$
$$(x \in \Gamma), \quad \bar{t} = (\bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{q})^T, \quad \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{p})^T, \tag{19}$$

where $\tilde{U}(x,s)$, $\tilde{T}(x,s)$ are fundamental and singular solutions for boundary integral equation (BIE), $\tilde{T}^0(x,s)$ contains the singularities isolation.

To approximate the boundary consider its decomposition to a set of quadrangular and triangular 8-node biquadratic elements, where triangular elements are treated as singular quadrangular. Every element is mapped to a reference one (square $\xi = (\xi_1, \xi_2) \in [-1,1]^2$ or triangle $0 \le \xi_1 + \xi_2 \le 1,\ \ \xi_1 \ge 0, \xi_2 \ge 0$). Interpolation nodes for boundary unknowns are a subset of geometrical boundary-element grid nodes. Local approximation follows the Goldshteyn's generalized displacement-stress matched model [14]: generalized boundary displacements are approximated by bilinear elements whereas generalized tractions are approximated by constant. For BIE discretization the collocation method is used; the set of collocation nodes coincides with the set of approximation nodes for the boundary functions. Integrals in discretized BIE's are calculated using Gaussian quadrature in combination with singularity decreasing and eliminating algorithms [14].

4. Laplace transform inversion

CQM and time-step method for numerical Laplace transform inversion are quite close to each other but whereas CQM is based on the convolution theorem and intended for convolution integral calculation, time-step method for numerical Laplace transform inversion is based on the integration theorem and dedicated to the calculation of the original function integral.

In order to employ time-step method for obtaining the original function $f(t)$ such as $f(0)=0$ from known image $\bar{f}(s)$ we need to apply the integration theorem to its derivative:

$$f(t) = \int_0^t f'(\tau) d\tau = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{c-iR}^{c+iR} \bar{f}(s) s \int_0^t e^{s(\tau)} d\tau ds = ... \tag{20}$$

Following the theorem, the integral can be found as (basing on [4–6]):

$$f(0)=0, \quad f(n\Delta t) = \sum_{k=1}^{n} \omega_k(\Delta t), \quad n = 1,...N, \tag{21}$$

$$\omega_n(\Delta t) = \frac{R^{-n}}{L} \sum_{l=0}^{L-1} \bar{f}(s) s e^{-in\varphi}, \quad s = \frac{\gamma(z)}{\Delta t}, z = Re^{i\varphi}, \quad \varphi = 2\pi \tfrac{l}{L} \tag{22}$$

Here R is the radius of the analyticity region for $\bar{f}(\gamma(z)/\Delta t)$ and $\gamma(z)$ is the characteristic function for the linear multistep method applied to the Cauchy problem arising within the integral evaluation. Backward differentiation (BDF) based methods of order ≤ 6 are applicable in scope of this solution scheme. For BDF-2 we have $\gamma(z) = 3/2 - 2z + z^2/2$. For Runge-Kutta method applied instead of linear multistep method we obtain (based on [8-10]):

$$f(0)=0, \quad f(n\Delta t) = b^T A^{-1} \sum_{k=1}^{n} \omega_k(\Delta t), \quad n = 1,...N. \tag{23}$$

Runge-Kutta method is referred to as Butcher tableau

$$\frac{c|A^T}{b^T}, \ A \in R^{m\times m}, \ b, c \in R^m. \tag{24}$$

Formula for coefficients $\omega_n(\Delta t)$ retains its written form with the only difference that now it's a matrix formula. The characteristic function writes as $\gamma(z) = A^{-1} - zA^{-1}[1]b^T A^{-1}$, $[1] = (1,...,1)^T$.

5. Half-space with a cavity

A problem of vertical load to a surface element of poroelastic rock half-space with spherical or cubic cavity is considered (Fig. 1). The poroelastic material is rock with the following properties: $K = 8 \cdot 10^9\ N/m^2$, $G = 6 \cdot 10^9\ N/m^2$, $\rho = 2458\ kg/m^3$, $\phi = 0.19$, $K_s = 3.6 \cdot 10^{10}\ N/m^2$, $\rho_f = 1000\ kg/m^3$, $K_f = 3.3 \cdot 10^9\ N/m^2$, $\kappa = 1.9 \cdot 10^{-10}\ m^4/(N \cdot s)$.The load $P(t) = P_0 f(t)$, $P_0 = 1000\,N/m^2$ acts on the area 1 m^2; the cube side/sphere diameter is 10 m;

the center of cavity is located at a depth of $h = 7.5m$ under the excitation area. The half-space surface is impermeable: q=0. The problem is treated with the help of Radau-based scheme with N=250 and Ł=1000 on $[0, \pi]$. Boundary element mesh takes into account 2 symmetry planes; quarter of the mesh contains 420 elements on the surface and 324 elements on the cavity boundary. Responses of displacements and pore pressure at point 15 *m* from the excitation area are shown in Figs. 2-3 in comparison with the case of poroelastic half-space without cavity.

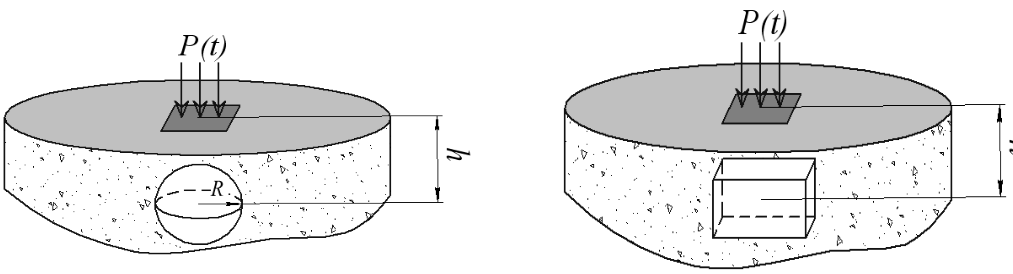

Fig. 1. Half-space with a cavity.

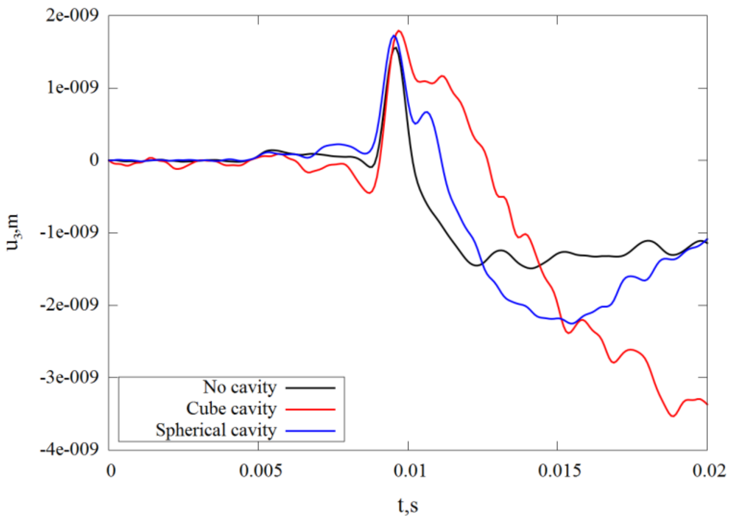

Fig. 2. Vertical displacements.

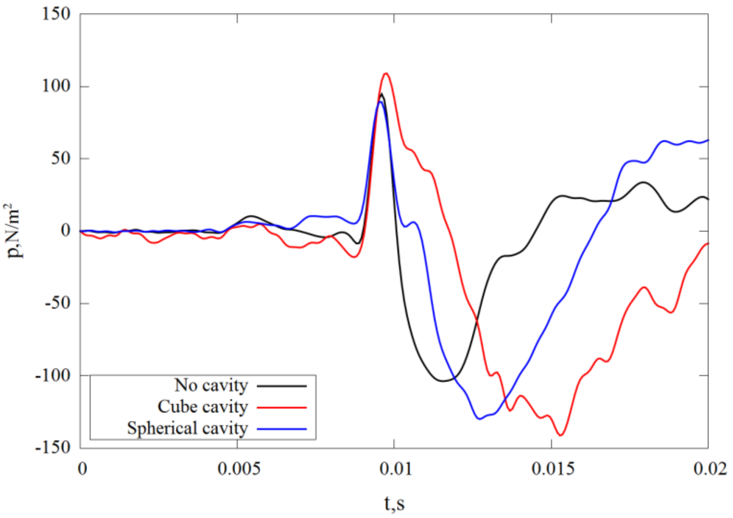

Fig. 3. Pore pressure.

It's seen from the diagrams that the shape of the cavity affects the responses form qualitatively. The displacements and pore pressure for the case without cavity are closer to the spherical cavity case than to the cubic cavity case. However, there is still certain difference. For the cavity case vertical displacements gain one more peak ($t \approx 0.011$ sec) following the arrival of Rayleigh wave in the central part of the diagram ($t \approx 0.0095$ sec); in the horizontal displacements diagram the peak right before the Rayleigh wave arrival becomes much stronger ($t \approx 0.009$ sec).

6. Conclusions

A boundary element approach utilizing Runge-Kutta based time-step method for numerical Laplace transform inversion is developed and applied to 3-D poroelastodynamic boundary value problems. In combination with varied integration step and highly oscillatory quadrature employed here it helps to reduce computational costs for solution. The problem of a vertical load on a half-space with a cavity is considered. An influence of cavity form on displacement and pore pressure responses in the point located on half-space surface in 15m from applied load is studied.

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