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BOUNDARY ELEMENT METHOD IN SOLVING DYNAMIC PROBLEM OF POROVISCOELASTIC PRISMATIC SOLID

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Abstract. Boundary-value problem of three-dimensional poroviscoelasticity is considered. The basic equations for fluid-saturated porous media proposed by Biot are modified by applying elastic-viscoelastic principle to classical linear elastic model of the solid skeleton. To describe viscoelastic properties of the solid skeleton model with weakly singular kernel is used. Boundary Integral Equations (BIE) method and Boundary-Element Method (BEM) with mixed discretization are applied to obtain numerical results. Solutions are obtained in Laplace domain. Modified Durbin's algorithm of numerical inversion of Laplace transform is used to perform solutions in time domain. An influence of viscoelastic parameter coefficient on dynamic responses is studied.

1. Introduction

Various types of interactions in advanced dispersed media, such as, porous of viscous media, are of a great interest of many disciplines. Wave propagation in porous/viscous media is an important issue of geophysics, geomechanics, geotechnical engineering etc. Study of wave propagation processes in saturated porous continua began from the works of Y. I. Frenkel and M. Biot [1, 2]. The implementation of the solid viscoelastic effects in the theory of poroelasticity was first introduced by Biot [3]. Although many significant achievements have been made on wave motion in porous media [4], because the complexity of the inertial viscosity and mechanical coupling in porous media most transient response problems can only be solved via numerical methods.

There are two major approaches to dynamic processes modeled by means of BEM: solving BIE system directly in time domain [5] or in Laplace or Fourier domain followed by the respective transform inversion [6]. Shanz and Cheng [7] presented an analytical solution in the Laplace domain and analytical time-domain solution without considering viscous coupling effect, and then developed the governing equation for saturated poroviscoelastic media by introducing the Kelvin–Voigt model and obtained an analytical solution in the Laplace domain for the 1D problem [8].

Special material constants (case where Poison's ratio equals 0) and boundary conditions for 3D formulation to make solution similar to 1D solution are used. That is done to verify numerical approach. Also a classical three-dimensional formulation is employed for study of viscoelastic parameters influence to displacement responses in poroviscoelastic prismatic solid.

2. Problem formulation and solution method

Homogeneous body Ω in three-dimensional Euclidean space R^3 is considered, with the boundary $\Gamma = \partial\Omega$. It's assumed, that body Ω is isotropic poroviscoelastic.

The set of differential equations of poroelasticity for displacements $\hat{u}_i$ and pore pressure $\hat{p}$ in Laplace domain (s – transform parameter) take the following form [4]:

$$G\hat{u}_{i,jj} + \left(K + \frac{1}{3}G\right)\hat{u}_{j,ij} - (\alpha - \beta)\hat{p}_{,i} - s^2(\rho - \beta\rho_f)\hat{u}_i = -\hat{F}_i \qquad (1)$$

$$\frac{\beta}{s\rho_f}\hat{p}_{,ii} - \frac{\phi^2 s}{R}\hat{p} - (\alpha - \beta)s\hat{u}_{i,i} = -\hat{a}\,, \qquad (2)$$

$$\beta = \frac{k\rho_f\phi^2 s^2}{\phi^2 s + s^2 k(\rho_a + \phi\rho_f)}\,, R = \frac{\phi^2 K_f K_s^{\ 2}}{K_f(K_s - K) + \phi K_s(K_s - K_f)}\,,$$

where G,K – material constant from elasticity, ϕ – porosity, k – permeability, α – Biot's effective stress coefficient, ρ, ρ_a, ρ_f – bulk, apparent mass and fluid densities respectively, $\hat{F}_i, \hat{a}$ – bulk body forces per unit volume.

Following types of boundary conditions for Ω are considered:

$$u_l(x,s) = f_l(x,s),\ u_4(x,s) = p(x,s) = f_4(x,s), x \in \Gamma^u,\ l = \overline{1,3}\,;$$

$$t_l(x,s) = g_l(x,s),\ t_4(x,s) = q(x,s) = g_4(x,s),\ x \in \Gamma^\sigma,\ l = \overline{1,3}\,.$$

Γ^u and Γ^σ – parts of boundary Γ, where corresponding generalized displacements and generalized tractions are prescribed.

Direct approach of boundary integral equations method is given [4, 9-10]:

Displacement vector at internal points is connected with boundary displacements and tractions as follows:

$$u_l(x,s) = \int_{\Gamma_k} U_{lj}^s(x,y,s)t_j(y,s)d_yS - \int_{\Gamma_k} T_{lj}^s(x,y,s)u_j(y,s)d_yS\,, l = 1,2,3,\quad x \in \Omega\,,$$

where U_{lj} and T_{lj} – components of fundamental and singular solution tensors.

Analytical solutions of one-dimensional problem of axial force t_2 = 1 N/m^2 for displacements and pore pressure can be obtained as follows [4]:

$$\tilde{u}_y = \frac{S_0}{E(d_1\lambda_2 - d_2\lambda_1)}\left[\frac{d_2(e^{-\lambda_1 s(l-y)} - e^{-\lambda_1 s(l+y)})}{s(1+e^{-2\lambda_1 sl})} - \frac{d_1(e^{-\lambda_2 s(l-y)} - e^{-\lambda_2 s(l+y)})}{s(1+e^{-2\lambda_2 sl})}\right],$$

$$\tilde{p} = \frac{S_0 d_1 d_2}{E(d_1\lambda_2 - d_2\lambda_1)}\left[\frac{e^{-\lambda_1 s(l-y)} - e^{-\lambda_1 s(l+y)}}{1+e^{-2\lambda_1 sl}} - \frac{e^{-\lambda_2 s(l-y)} - e^{-\lambda_2 s(l+y)}}{1+e^{-2\lambda_2 sl}}\right].$$

Poroviscoelastic solution is obtained from poroelastic solution by means of the elastic-viscoelastic correspondence principle, applied to skeleton's elastic constants K and G in Laplace domain. Forms of $\hat{K}(s)$ and $\hat{G}(s)$ in case of model with weakly singular kernel of Abel type are following: $\hat{G} = \dfrac{G}{1+hs^{\alpha-1}}\,, \hat{K} = \dfrac{K}{1+hs^{\alpha-1}}$, h and α are model parameters [10, 11].

Thus, poroviscoelastic solution in time domain we get with the help of the Laplace transform inversion. Durbin's algorithm [12] with variable integrating step (5) is used for numerical inversion of Laplace transform.

$$f(0) = \sum_{k=1}^{\infty}\frac{\left(f(\alpha + i\omega_k) + f(\alpha + i\omega_{k+1})\right)}{2\pi}\Delta_k\,,$$
$$f(t) \approx \frac{e^{\alpha t}}{\pi}\sum_{k=1}^{\infty}\frac{(f(\alpha + i\omega_k)e^{it\omega_k} + f(\alpha + i\omega_{k+1})e^{it\omega_{k+1}})}{2}\Delta_k\,. \qquad (5)$$

Integral representation and BIE with integral Laplace transform are used. Fundamental and singular solutions are considered in term of singularity isolation. Realization of boundary element approach for solving problems of poroelasticity/poroviscoelasticity needs in a special calculating method [4].

Numerical scheme is based on the Green-Betti-Somigliana formula. To perform numerical results double accuracy calculations are used. Regularized BIE are considered in order to introduce BE discretization [9].

3. Method verification

The problem of poroviscoelastic rod under Heaviside type load $t_2 = 1N/m^2$ is considered. Material constants are: $K = 4{,}8 \cdot 10^9\ N/m^2$, $G = 7{,}2 \cdot 10^9\ N/m^2$, $\rho = 2458\,kg/m^3$, $\phi = 0{,}19$, $K_s = 3{,}6 \cdot 10^{10}\ N/m^2$, $\rho_f = 1000\,kg/m^3$, $K_f = 3{,}3 \cdot 10^9\ N/m^2$ $k = 1{,}9 \cdot 10^{-10}\ m^4/(N \cdot s)$, $\nu = 0$ (it is important, because we simulate 1D solution through 3D solution).

The problem has analytical solution in Laplace domain [4], which just transforms back to time domain with the help of the Durbin's method for Laplace transform.

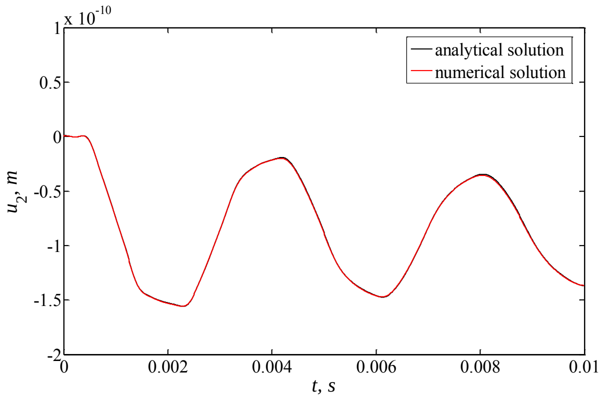

Fig. 1. Numerical and analytical poroviscoelastic solutions comparison in case of weakly singular kernel model, h =1 and $\alpha = 0.7$.

On Fig. 1 the comparison of numerical and analytical solutions is shown. Maximum relative error is less than 2 % at considered time interval. Our numerical approach demonstrates a satisfactive accuracy, so we can apply it to 3D problems.

4. Numerical example

Problem of a prismatic poroviscoelastic column clamped at one end and subjected to a Heaviside type load at another and is considered (Fig. 2).

Fig. 2. Problem formulation (left) and boundary element mesh visualization (right).

Poroelastic material is Berea sandstone. Poroelastic material constants are: $K = 8 \cdot 10^9\ N/m^2$, $G = 6 \cdot 10^9\ N/m^2$, $\rho = 2458\ kg/m^3$, $\phi = 0.66$, $K_s = 3.6 \cdot 10^{10}\ N/m^2$, $\rho_f = 1000\,kg/m^3$, $K_f = 3.3 \cdot 10^9\ N/m^2$, $k = 1.9 \cdot 10^{-10}\,m^4/(N \cdot s)$ (Fig. 3).

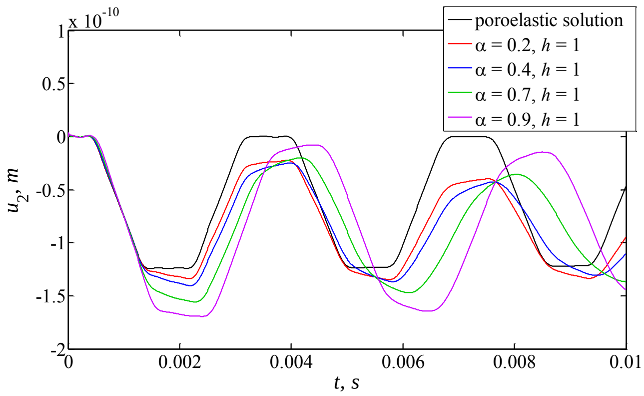

Fig. 3. Displacements u_2 at points B in case of considered model for different values of viscoelastic parameters.

5. Conclusions

An influence of viscoelastic parameters on displacement responses is demonstrated on the example of a problem of the prismatic poroviscoelasticsolid under Heaviside-type load. Boundary integral equations method and boundary element method are applied in order to solve three dimensional boundary-value problems. We verified numerical approach based on BEM by comparison with analitics. The poroviscoelastic media modelling is based on Biot's theory of porous material in combination with the elastic-viscoelastic corresponding principle. Viscous properties are described by model with weakly singular kernel.

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