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DYNAMIC ANALYSIS OF 3D COMPOSITE PIEZOELECTRIC SOLIDS USING BEM

L.A. Igumnov*, I.P. Markov, A.V. Amenitsky, I.V. Vorobtsov

Research Institute for Mechanics, Lobachevsky State University of Nizhni Novgorod,

23 Prospekt Gagarina (Gagarin Avenue) BLDG 6, Nizhny Novgorod, 603950, Russia

*e-mail: igumnov@mech.unn.ru

Abstract. A Laplace domain direct boundary element approach for the three-dimensional dynamic analysis of the composite piezoelectric solids is presented. Integral representations of the fundamental solutions are used. Time domain solutions are obtained by the modified Durbin's method. Proposed boundary element formulation is verified through numerical examples.

1. Introduction

Due to the mechanical-electric coupling effects, piezoelectric materials have a broad range of engineering applications in smart devices and structural components such as transducers, sensors, actuators, multilayered composite shells and plates.

The analytical solutions are only available for very simple cases of problems geometries, boundary conditions and loading configurations, such as simply supported piezoelectric laminates subjected to static or harmonic loadings. For arbitrary structure configurations, general boundary conditions or complex dynamic loadings numerical methods have to be applied for structural analysis of the transient problems. Boundary Element Method (BEM) is a robust, efficient and accurate method for the solution of dynamic problems in composite piezoelectric solids and presents well-known significant advantages over other numerical techniques. Most reported piezoelectric boundary element formulations rely on Dual Reciprocity Method that was originally developed by Nardini and Brebbia [1] for the problems of anisotropic elasticity and was later extended to the transient piezoelectric problems, e.g. by Gaul et al. [2].

In this paper, based on the integral expressions of dynamic piezoelectric fundamental solutions, a Laplace domain direct BEM formulation for dynamic analysis of three-dimensional (3D) composite piezoelectric solids is presented. Numerical examples of the harmonic vibration and transient response are provided in order to verify the present boundary element approach and assess its accuracy.

2. Governing equations in Laplace domain

First, we consider a single three-dimensional homogeneous piezoelectric solid of volume $\Omega \subset R^3$ and boundary $\Gamma = \partial\Omega$. Assuming zero initial conditions and in the absence of body forces and free electric charges, Laplace transformed linear coupled equations of motion can be written in the following form

$$C_{ijkl}\overline{U}_{k,il} = \rho s^2 \delta^*_{jk} \overline{U}_k, \text{ in } \Omega \subset R^3, \quad i,l = \overline{1,3},\ j,k = \overline{1,4}, \tag{1}$$

$$\delta^*_{jk} = \begin{cases} \delta_{jk}, & j = k = \overline{1,3}, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where s is the parameter of the Laplace transform, ρ is mass density. For convenience, we employed a unified description of the coupled fields through the following generalized quantities

$$\bar{U}_k = \begin{cases} \bar{u}_k, & k = \overline{1,3}, \\ \bar{\phi}, & k = 4, \end{cases} \tag{3}$$

$$C_{ijkl} = \begin{cases} C^E_{ijkl}, & i, j, k, l = \overline{1,3}, \\ e_{lij}, & i, l, j = \overline{1,3},\ k = 4, \\ e_{ikl}, & i, l, k = \overline{1,3},\ j = 4, \\ -\kappa_{il}, & i, l = \overline{1,3},\ k = j = 4, \end{cases} \tag{4}$$

where $\bar{u}_k$ are the elastic displacements, $\bar{\phi}$ is the electric potential, C^E_{ijkl}, e_{ijk} and κ_{il} denote the tensors of elastic, piezoelectric and dielectric constants, respectively.

The generalized boundary conditions are given as follows

$$\bar{U}_k = \tilde{U}_k \text{ on } \Gamma^U,\ \bar{T}_j = \tilde{T}_j \text{ on } \Gamma^T, \tag{5}$$

with

$$\bar{T}_j = \begin{cases} \bar{t}_j = \bar{\sigma}_{jk} n_k, & j = \overline{1,3}, \\ \bar{q} = \bar{D}_k n_k, & j = 4, \end{cases} \tag{6}$$

where $\bar{t}_j$ are the elastic tractions, $\bar{q}$ is the normal electric charge flux, $\bar{D}_j$ are the electric displacements, n_k is the surface outward normal of the boundary, $\Gamma^U + \Gamma^T = \Gamma$.

3. BEM formulation and fundamental solutions

Based on the extended Somigliana identity, the Laplace domain displacement boundary integral equations (BIEs) for linear piezoelectricity can be expressed as

$$c_{jk}\bar{U}_k(\mathbf{x}, s) = \int_\Gamma \bar{g}_{jk}(\mathbf{x}, \mathbf{y}, s)\bar{T}_k(\mathbf{y}, s)\mathrm{d}\Gamma(\mathbf{y}) - pv\int_\Gamma \bar{h}_{jk}(\mathbf{x}, \mathbf{y}, s)\bar{U}_k(\mathbf{y}, s)\mathrm{d}\Gamma(\mathbf{y}), \quad \mathbf{x} \in \Gamma, \tag{7}$$

where c_{jk} are the free term constants. *pv* stands for the Cauchy principal value integral, **x** and **y** are referred to as the source and field points of the Laplace transformed dynamic piezoelectric displacement and traction fundamental solutions $\bar{g}_{jk}$ and $\bar{h}_{jk}$, respectively.

For spatial discretization of Eq. (7), a collocation method with mixed boundary element approach is implemented. Quadrilateral 8-noded elements are used for approximation of the boundary surface Γ. Generalized displacements and tractions in each element are approximated by linear and constant shape functions, respectively. In the case where the problem consists of arbitrary number of sub-domains, which are assumed to be perfectly bonded together, the displacement BIEs are applied to each sub-domain. The resulting system of algebraic equations is obtained be enforcing the continuity conditions of generalized displacements and tractions at interfaces of all adjacent sub-domains. In order to obtain time domain solutions we need to perform numerical inversion of the Laplace transform. For this purpose we employ modified Durbin's method [3] proposed by Zhao [4].

Fundamental solutions. The piezoelectric displacement fundamental solution can be expressed as a sum of dynamic regular and static singular parts as

$$\bar{g}_{jp}(\mathbf{x}, \mathbf{y}, s) = \bar{g}_{jp}(\mathbf{r}, s) = g^S_{jp}(\mathbf{r}) + \bar{g}^R_{jp}(\mathbf{r}, s), \tag{8}$$

$$\mathbf{r} = \mathbf{y} - \mathbf{x}, r = |\mathbf{r}|. \tag{9}$$

Expressions of regular and singular parts are given in as a two-dimensional integral over a half of a unit sphere and one-dimensional integral over a unit circle [5], as follows

$$\bar{g}^{R}_{jp}(\mathbf{r},s) = -\frac{1}{8\pi^2}\int\limits_{\substack{|\mathbf{n}|=1\\ \mathbf{n}\cdot\mathbf{r}>0}}\sum_{m=1}^{Q}\frac{k_m\bar{P}^{m}_{jp}}{\rho c_m^2}e^{-k_m|\mathbf{n}\cdot\mathbf{r}|}\mathrm{d}S(\mathbf{n}),\quad j,p=\overline{1,4}, \tag{10}$$

$$g^{S}_{jp}(\mathbf{r}) = \frac{1}{8\pi^2 r}\int\limits_{|\mathbf{d}|=1}\Gamma^{-1}_{jp}(\mathbf{d})\mathrm{d}\Omega(\mathbf{d}), \tag{11}$$

$$\bar{P}^{m}_{jp} = \begin{cases} P^{m}_{jp}, & j,p=\overline{1,3},\\ -\Gamma_{k4}P^{m}_{jk}/\Gamma_{44}, & j=\overline{1,3},\ p=4,\\ \Gamma_{4k}P^{m}_{kl}\Gamma_{l4}/\Gamma^{2}_{44}, & j=p=4, \end{cases}\quad P^{m}_{jp}=\frac{A^{m}_{jp}}{A^{m}_{ii}},\quad A^{m}_{jp}=\mathrm{adj}\left(L_{jp}-\rho c_m^2\delta_{jp}\right), \tag{12}$$

$$L_{jp}=\Gamma_{jp}-\frac{\Gamma_{j4}\Gamma_{4p}}{\Gamma_{44}},\quad c_m=\sqrt{\lambda_m/\rho},\quad k_m=s/c_m,\quad \Gamma_{jk}(\mathbf{a})=C_{ijkl}a_i a_l, \tag{13}$$

where $\mathbf{n}$ is the wave propagation vector, λ_m are the eigenvalues of L_{jp}, Q is number of distinct eigenvalues, c_m denotes the phase velocities of elastic waves.

The traction fundamental solution can obtained by

$$\bar{h}_{jp}(\mathbf{x},\mathbf{y},s) = C_{ijkl}\bar{g}_{kp,l}n_i(\mathbf{y}), \tag{14}$$

where $n_i(\mathbf{y})$ are the components of the unit outward normal at the field point.

4. Numerical example

A three-layer cross-ply square piezoelectric plate with lamination [0/90/0] is considered (see Figure 1). The total thickness is $h=0.05\,\mathrm{m}$ and side length is $a=0.2\,\mathrm{m}$. The plate is made of PVDF piezoelectric material with mass density $\rho=1780\,\mathrm{kg/m^3}$ and the following mechanical and electrical properties:

$$\mathbf{C}^{E}=\begin{bmatrix} 238 & 3.98 & 2.19 & 0 & 0 & 0\\ 3.98 & 23.6 & 1.92 & 0 & 0 & 0\\ 2.19 & 1.92 & 10.6 & 0 & 0 & 0\\ 0 & 0 & 0 & 2.15 & 0 & 0\\ 0 & 0 & 0 & 0 & 4.4 & 0\\ 0 & 0 & 0 & 0 & 0 & 6.43 \end{bmatrix}\mathrm{GPa},\ \boldsymbol{\kappa}=\begin{bmatrix} 1.1068 & 0 & 0\\ 0 & 1.0607 & 0\\ 0 & 0 & 1.0607 \end{bmatrix}\cdot 10^{-10}\,\mathrm{C/Vm},$$

$$\mathbf{e}=\begin{bmatrix} 0 & 0 & 0 & 0 & -0.01 & 0\\ 0 & 0 & 0 & -0.01 & 0 & 0\\ -0.13 & -0.14 & -0.28 & 0 & 0 & 0 \end{bmatrix}\mathrm{C/m^2}.$$

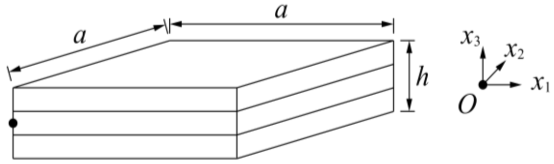

Fig. 1. Geometry of the problem.

The following boundary conditions are prescribed:

at $x_1=0,a,\ \bar{T}_1=\bar{U}_2=\bar{U}_3=\bar{U}_4=0,$ (15)

at $x_2=0,a,\ \bar{T}_2=\bar{U}_1=\bar{U}_3=\bar{U}_4=0,$ (16)

at $x_3=-h/2,\ \bar{T}_1=\bar{T}_2=\bar{T}_3=\bar{U}_4=0.$ (17)

Two cases of load conditions applied on the top surface $x_3 = h/2$ are considered: an applied sinusoidal potential (Case 1) $\phi = \sin(\pi t)\sin(x_1\pi/a)\sin(x_2\pi/a)\cdot \mathrm{V}$ and an applied uniform transverse load (Case 2) $t_3 = H(t)\cdot 10^5\,\mathrm{Pa}$ with top surface grounded. For both cases, the rest of the top surface is traction free. Figure 2 shows the comparison of the exact [6] and obtained numerical results for Case 1. Figure 3 shows the transient results for Case 2.

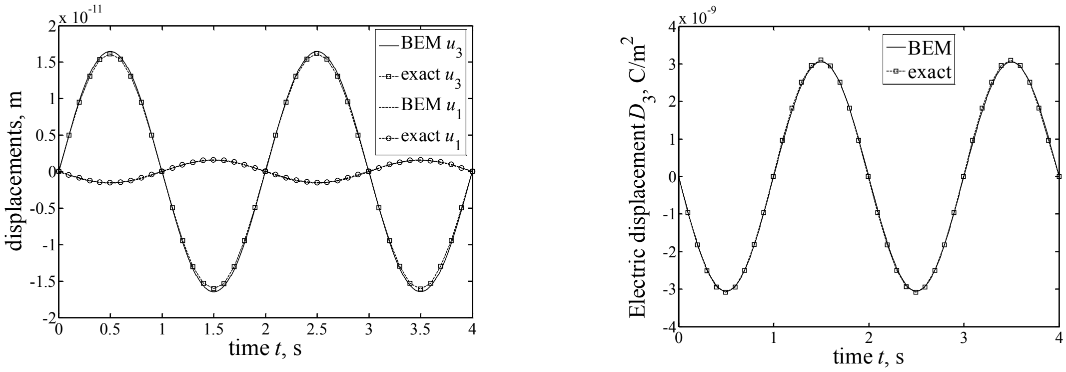

Fig. 2. Case 1: top surface displacements $u_1(0, a/2, h/2; t)$ and $u_3(a/2, a/2, h/2; t)$; top surface electric displacements $D_3(a/2, a/2, h/2; t)$.

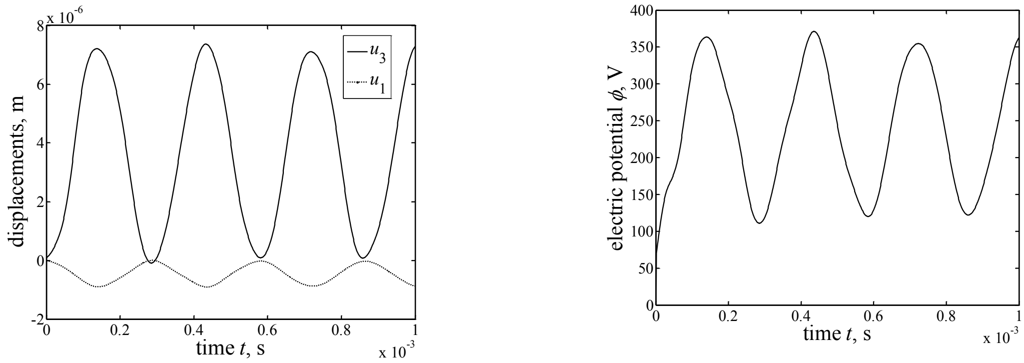

Fig. 3. Case 2: top surface displacements $u_1(0, a/2, h/2; t)$ and $u_3(a/2, a/2, h/2; t)$; electric potential $\phi(a/2, a/2, h/6; t)$.

5. Conclusions

A Laplace domain direct BEM approach, based on the idea of mixed boundary elements, for harmonic vibration and transient analyses of three-dimensional composite piezoelectric solids is presented. Comparison between obtained results and the exact solutions of numerical examples clearly shows that proposed boundary element formulation gives reliable results.

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References

[1] D. Nardini, C.A. Brebbia // *Applied Mathematical Modelling* **7** (1983) 157.
[2] L. Gaul, M. Kogl, M. Wagner, *Boundary Element Methods for Engineers and Scientists* (Springer-Verlag Berlin Heidelberg, 2003).
[3] F. Durbin // *The Computer Journal* **17** (1974) 371.
[4] X. Zhao // *International Journal of Solids and Structures* **41** (2004) 3653.
[5] C.Y. Wang, Ch. Zhang // *Engineering Analysis with Boundary Elements* **29** (2005) 454.
[6] G. Qing, J. Xu, P. Li, J. Qiu // *Composite Structures* **78** (2007) 457.