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ON CALCULATION OF NATURAL FREQUENCIES OF COMPRESSED RODS WITH VARIABLE CROSS-SECTION

L.M. Kagan-Rosenzweig

Saint-Petersburg State University of Architecture and Civil Engineering, Russia

e-mail: Kagan_R@mail.ru

Abstract. A method of constructing improved basis functions for frequency calculation by means of Bubnov-Galerkin method is proposed. Method efficiency is demonstrated by analysis of cantilever rod with variable cross-section. Natural frequencies of such unloaded rod are calculated, and critical force for a rod compressed by follower force is determined.

1. Introduction

In natural frequencies computation using Bubnov-Galerkin method the excess of number of basis functions over a certain limit leads to a system of equations with ill-conditioned matrix. Comparison of numerical methods [1] has shown that for this reason in problems that require precise frequency values (such as non-conservative problems of elastic stability), Bubnov-Galerkin method is less efficient than other numerical methods.

Below a method of constructing improved basis functions that allows to decrease their number in calculations with high accuracy is presented.

2. Problem definition

A cantilever of variable rigidity $EI = D_0 e_x$ and linear mass $m = m_0 m_x$ (D_0, m_0 are dimensional parameters, e_x, m_x are dimensionless coordinate functions) is compressed by a force P at its top end. Bubnov-Galerkin method is applied to calculate natural frequencies ω and characteristic exponents $\lambda = i\omega$. Improved basis functions are designed to calculate frequencies.

Frequencies are calculated based on the bending equation written in terms of the bending moment amplitude M (hereinafter the moment) [2]

$$M'' + (P / EI)M = -q. \quad (1)$$

In equation (1), load intensity q is proportional to deflection amplitude (hereinafter the deflection) w:

$$q = -m\ddot{w} = -\lambda^2 mw. \quad (2)$$

The form of equations (1) and (2) suggests the idea to construct the system of matched basis functions M_i, w_i, that correspond to the moment M and the deflection w respectively.

The functions M_i are based on equation (1). Correlation between q and w allows to impose M_i not two, but four boundary conditions. After integrating bending equation

$$w'' = -M / EI \quad (3)$$

and considering two the other boundary conditions we have functions w_i corresponding to M_i.

about 0.5 % [3]. $n+1$ functions yield a sufficiently accurate value for the *n*-th frequency.

Cantilever with varying cross-section. The exact solution is known for a sharp wedge ($e_x=(1-x)^3$, $m_x=1-x$). Frequencies $\overline{\omega}$ of such a cantilever are the roots of equation [4]

$$1+\sum_{i=1}^{\infty}\frac{(-1)^i\overline{\omega}^{2i}}{((i+1)!)^2(2i+1)!}=0. \qquad (13)$$

Figures in Table 1 indicate sufficient accuracy for the *n*-th frequency found by using $n+1$ basis function.

Table.

Frequency	Number of basic functions				Exact solution
	1	2	3	4	
Cantilever with constant cross-section					
$\overline{\omega}_1$	3.51636	3.51601	3.51601	3.51601	3.51601
$\overline{\omega}_2$		22.1098	22.0345	22.0345	22.0345
$\overline{\omega}_3$			62.5942	61.7040	61.6972
Wedge cantilever					
$\overline{\omega}_1$	5.53034	5.31563	5.31510	5.31510	5.31510
$\overline{\omega}_2$		16.41208	15.24583	15.20763	15.20717
$\overline{\omega}_3$			32.56778	30.39982	30.01981

4. Stability of cantilever compressed by the follower force

Force *P* is applied to the top end. Influence of dissipation is not considered. Coordinates are the same as in the previous example, $x\equiv x/l$. D_0, m_0 are stiffness and linear mass in bottom section, so that $e_x(0)=1$, $m_x(0)=1$.

Stability analysis consists in computing characteristic exponents λ and finding the load level that results in $\mathrm{Re}(\lambda)>0$.

Stability loss of the cantilever happens dynamically by "swaying" of the system due to interaction of oscillations, that correspond to the two adjacent frequencies. Consequently, at least two frequencies are required to be satisfactorily approximated and we need to use at least three basis functions.

Dimensionless load parameter $k^2=Pl^2/D_0$ is introduced. Bending equation is taken in the form (1) (both sides of (1) are multiplied by e_x):

$$e_xM''+k^2M=\lambda^2e_xm_xw. \qquad (14)$$

At the top end both moment and shear force are missing, that is, conditions (5.1) are valid. The boundary conditions $w(0)=w'(0)=0$ result in

$$M''(0)+k^2M(0)=0,\ M'''+k^2[M'(0)-e_x'(0)M(0)]=0. \qquad (15)$$

Subjecting functions M_i of the form (8)

$$M_i(x)=(1-x)^{i+2}+\beta_i(k)(1-x)^{i+3}+\alpha_i(k)(1-x)^{i+4} \qquad (16)$$

with load-dependent coefficients to boundary conditions (5.1), (15), we have

$$\alpha_i(k)=\{(A_{i+2}^{i+1}+k^2)[A_{i+3}^{i+1}+k^2(3+i-e_0')]-(A_{i+3}^{i+1}+k^2)[A_{i+2}^{i}+k^2(2+i-e_0')]\}B^{-1},$$

$$\beta_i(k)=\{(A_{i+4}^{i+3}+k^2)[A_{i+2}^{i}+k^2(2+i-e_0')]-(A_{i+2}^{i+1}+k^2)[A_{i+4}^{i+2}+k^2(4+i-e_0')]\}B^{-1},$$

$$B=[A_{i+4}^{i+2}+k^2(4+i-e_0')](A_{i+3}^{i+2}+k^2)-(A_{i+4}^{i+3}+k^2)[A_{i+3}^{i+1}+k^2(3+i-e_0')].$$

Here $A_n^m = n(n-1)\dots m$, $e'_0 = e'_x(0)$.

Functions w_i are calculated according to (9).

The oscillation equation (14) takes the form

$$e_x \sum_{i=0}^{n} a_i M_i''(x) + k^2 \sum_{i=0}^{n} a_i M_i(x) = \lambda^2 m_x e_x \sum_{s=0}^{n} a_i w_i(x). \tag{17}$$

Bubnov-Galerkin procedure leads to equation (11) with matrixes f_{ij}, g_{ij} of the following form

$$f_{ij} = \int_0^1 e_x w_i M_j'' dx + k^2 \int_0^1 w_i M_j dx, \; g_{ij} = \int_0^1 e_x m_x w_i(x) w_j(x) dx. \tag{18}$$

Cantilever with constant cross-section and constant linear mass ($e_x = m_x = 1$). The calculation results are shown in Table 2, n is the number of basis functions. As a test solution the result found by sagittary function method (10,000 steps of Runge-Kutt method) [1] is used. It can be seen that five basis functions lead to the coincidence with the test result.

Table 2.

Cantilever with constant cross-section					Test solution
n	2	3	4	5	
k^{*2}	20.43981	20.05211	20.04800	20.05098	20.05095
Wedge cantilever					
n	3	4	5	6	8
k^{*2}	13.65081	13.07312	12.33549	12.38637	12.38636*

*Accuracy increased using the multiplier: $\bar{\lambda}^2 g = (0.1\,\bar{\lambda})^2 100\, g$.

Best result obtained in [1] by the Bubnov-Galerkin method is $k^{*2} = 20.05252$. 12 basis functions of two different types were used. It was impossible to increase the number of functions due to ill-conditioned matrix phenomenon.

Cantilever with varying cross-section. In this case, Bubnov-Galerkin method with basis functions of a conventional type leads to unreliable results [1].

Calculation results for the slightly blunted wedge of constant width ($e_x = (1-0.99x)^3$, $m_x = 1-0.99x$) are shown in Table 2. They demonstrate convergence of the calculation process and appropriate accuracy of the value $k^{*2} = 12.386$, obtained with 6 basis functions.

In this paper all the calculations were made by means of MathCad and special technique to increase the accuracy of numerical results was not used (except of a number in Table 2 marked by *).

5. Conclusion

The described method of construction of basis functions gives us more precise functions since in addition to static and kinematic boundary conditions it partially takes into account restrictions owing to bending equation. Significant reduction in number of basis functions becomes possible when computing frequencies by means of the Bubnov-Galerkin method.

References

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