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CONDITIONS ON THE SURFACE OF DISCONTINUITY FOR THE REDUCED COSSERAT CONTINUUM

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Abstract. In this work, we consider a mathematical model for granular medium. Here we claim that reduced Cosserat continuum is a suitable model to describe granular materials. Reduced Cosserat continuum is an elastic medium, where all translations and rotations are independent. Moreover, a force stress tensor is asymmetric and a couple stress tensor is equal to zero. In paper, we are aiming to establish a continuity conditions for nonlinear reduced Cosserat continuum.

1. Introduction

By granular medium, we understand a set of contact relied solid grains. The total volume of such a medium is composed of solid grains and voids, the later could be filled either with air or with a liquid. As for all granular medium grains at the surface can easily undertake a relative movement with respect to their neighbors. However, free movement of the grains requires space around them, in other words the increase of the body volume.

In most of recent papers on this subject [1] - [4] the granular media was systematically presented as discrete medium. In this close to reality description, each case should have its own model that takes into account the structure and nature of the probable distribution of forces and deformations in the medium. Contrary to the previous method, the use of a continuum model for such description is more general. In this type of medium grain’s size and nearest-neighbor distance are roughly comparable, rotational degrees of freedom should be considered along with the translational. To obtain a more accurate model of a granular medium it is necessary to use the continuum model with microstructure [5-7]. This widely known Cosserat continuum is the subject of intensive scientific activities in recent years [8-10]. A practical application of these models requires an experimental determination of a large number of additional constants in constitutive equations.

In this paper, we propose less known the reduced Cosserat continuum as a model of a granular medium. In this continuum, translations and rotations are independent, stress tensor is not symmetric and couple stresses tensor is equal to zero. It is more real the possibility of applying the reduced Cosserat model in practice because it contains less additional constants in constitutive equations. Originally, the idea of an equal footing for rotational and translational degrees of freedom appeared in [11]. Some complementary studies of this model were also performed in more recent works [12-16].

2. Problem formulation

Earlier in papers [12-16] were considered functions that are continuous and differentiable repeatedly. In this paper, we discuss areas on which functions can be discontinuous. Continuity conditions are then needed to connect field solutions in two regions separated by the discontinuity. In this work we are aiming to establish a continuity conditions for reduced Cosserat continuum.

We consider the deformed or current configuration (CC) of the material. So, at some time t radius vector is $\mathbf{r}(x_i;t)$ and a turn tensor is $\mathbf{P}(x_i;t)$. Further, we list conservation laws for the CC for the reduced Cosserat continuum:

linear momentum balance equation

$$\frac{d}{dt}\int_V \rho\mathbf{v}dV = \int_S \mathbf{n}\cdot\boldsymbol{\tau}dS\,, \tag{1}$$

kinetic momentum balance equation

$$\frac{d}{dt}\int_V (\rho\mathbf{J}\cdot\boldsymbol{\omega} + \mathbf{r}\times\rho\mathbf{v})dV = \int_S \mathbf{r}\times(\mathbf{n}\cdot\boldsymbol{\tau})dS\,, \tag{2}$$

the energy balance equation

$$\frac{d}{dt}\int_V (\frac{1}{2}\rho\mathbf{v}^2 + \frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\boldsymbol{\omega} + \rho\Pi)dV = \int_S \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v}dS\,, \tag{3}$$

the law of conservation of mass

$$\frac{d}{dt}\int_V \rho dV = 0\,. \tag{4}$$

Here ρ is a volume density for the CC, $\mathbf{v}$ is a velocity vector $(\mathbf{v}=\dot{\mathbf{r}})$, V is a volume limited by a surface S, $\mathbf{n}$ is an outward unit normal to the surface S, $\boldsymbol{\tau}$ is a stress tensor, $\mathbf{J}$ is a mass density of an inertia tensor, $\boldsymbol{\omega}$ is an angular velocity vector $(\dot{\mathbf{P}}=\boldsymbol{\omega}\times\mathbf{P})$, Π is a mass density of the strain energy. We take the load volume is equal zero. The differential system of nonlinear equations in the actual configuration for the reduced Cosserat medium was obtained in paper [16] from these equations. The volume V is divided by the surface of discontinuity, so it consists of two sub-volumes V_1 and V_2 (Fig. 1).

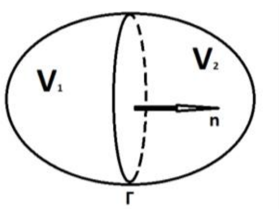

Fig. 1. The surface of discontinuity Γ.

The motion on the both sides of the surface of discontinuity is continuous and irrotational. The surface Γ is a surface of discontinuity. Continuity conditions are then needed to connect field solutions in two regions separated by the discontinuity. These conditions are implied by the differential equations that apply throughout the region. They assure that the fields are consistent with the basic laws, even in passing through the discontinuity.

3. Solution of the problem

The Reynolds transport theorem [19] is a useful way to find conditions on the surface of discontinuity that moves in the volume V.

$$\dot{\Phi} = \dot{\Phi}_1 + \dot{\Phi}_2 + \int_\Gamma [\phi(\mathbf{v}\cdot\mathbf{n}-u)]d\Gamma\,. \tag{5}$$

Here $\Phi = \int_V \phi\, dV$, $\Phi_i = \int_{V_i} \phi\, dV$ $(i=1,2)$, ϕ is any valued property of a material particle at position $\mathbf{r}(x_i;t)$ in the deformed body, $\mathbf{n}$ is an outward unit normal to the surface Γ directed from the volume V_1, u is the velocity of the area Γ – so not necessarily the particle velocity. Brackets mean change in the value enclosed in brackets.

Let us introduce equation (1) for the area V_1:

$$\frac{d}{dt}\int_{V_1}\rho\mathbf{v}dV=\int_{S_1}\mathbf{n}\cdot\boldsymbol{\tau}dS+\int_{\Gamma}\mathbf{n}\cdot\boldsymbol{\tau}_1 d\Gamma,$$

for the area V_2:

$$\frac{d}{dt}\int_{V_2}\rho\mathbf{v}dV=\int_{S_2}\mathbf{n}\cdot\boldsymbol{\tau}dS-\int_{\Gamma}\mathbf{n}\cdot\boldsymbol{\tau}_2 d\Gamma.$$

These results in

$$\frac{d}{dt}\int_{V_1}\rho\mathbf{v}dV+\frac{d}{dt}\int_{V_2}\rho\mathbf{v}dV=\int_{S}\mathbf{n}\cdot\boldsymbol{\tau}dS-\int_{\Gamma}[\mathbf{n}\cdot\boldsymbol{\tau}]d\Gamma. \tag{6}$$

Condition (5) may be substituted into equation (6) to obtain

$$\frac{d}{dt}\int_{V}\rho\mathbf{v}dV=\frac{d}{dt}\int_{V_1}\rho\mathbf{v}dV+\frac{d}{dt}\int_{V_2}\rho\mathbf{v}dV+\int_{\Gamma}[\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}-u)]d\Gamma. \tag{7}$$

We get $\rho\mathbf{v}$ as ϕ. Using equations (1), (6) and (7) we arrive at

$$\int_{\Gamma}[\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{n}\cdot\boldsymbol{\tau}]d\Gamma=0. \tag{8}$$

So that the equation representing the continuity of linear momentum at the surface Γ.
Applying a techniques describing above we can get

$$\int_{\Gamma}[(\rho\mathbf{J}\cdot\boldsymbol{\omega}+\mathbf{r}\times\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{r}\times(\mathbf{n}\cdot\boldsymbol{\tau})]d\Gamma=0. \tag{9}$$

The equation (9) representing the continuity of kinetic momentum at the surface Γ.

$$\int_{\Gamma}\left[(\frac{1}{2}\rho\mathbf{v}^2+\frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\boldsymbol{\omega}+\rho\Pi)(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v}\right]d\Gamma=0. \tag{10}$$

So that the equation representing the continuity of energy at the surface Γ.

Now we consider the law of conservation of mass (4), which is the same for the total volume V and any of its sub-region.

$$\frac{d}{dt}\int_{V_i}\rho dV=0. \tag{11}$$

The transfer theorem (5) in this case looks like

$$\frac{d}{dt}\int_{V}\rho dV=\frac{d}{dt}\int_{V_1}\rho dV+\frac{d}{dt}\int_{V_2}\rho dV+\int_{\Gamma}[\rho(\mathbf{v}\cdot\mathbf{n}-u)]d\Gamma. \tag{12}$$

Let us substitute equations (4), (11) into (12) to get the following

$$\int_{\Gamma}[\rho(\mathbf{v}\cdot\mathbf{n}-u)]d\Gamma=0. \tag{13}$$

The equation (13) representing the continuity of mass conservation law at the surface Γ.

Conditions (8), (9), (10), (13) are hold for any areas $d\Gamma\subset\Gamma$. Since all changes are continuous functions on Γ, the following four relations give the value at the surface of discontinuity.

$$[\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{n}\cdot\boldsymbol{\tau}]=0, \tag{14}$$

$$[(\rho\mathbf{J}\cdot\boldsymbol{\omega}+\mathbf{r}\times\rho\mathbf{v})(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{r}\times(\mathbf{n}\cdot\boldsymbol{\tau})]=0, \tag{15}$$

$$\left[(\frac{1}{2}\rho\mathbf{v}^2+\frac{1}{2}\omega\cdot\rho\mathbf{J}\cdot\boldsymbol{\omega}+\rho\Pi)(\mathbf{v}\cdot\mathbf{n}-u)-\mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{v}\right]=0, \tag{16}$$

$$[\rho(\mathbf{v}\cdot\mathbf{n}-u)]=0. \tag{17}$$

In classical theory of elasticity the fulfilling the continuity conditions for conservation of linear momentum automatically fulfills conditions for conservation of kinetic momentum. In models with microstructure this doesn't work as we can see from equations (14), (15). We shall combine equations (14) and (15). As a result we get conditions for conservation of kinetic momentum.

$$\left[\rho \mathbf{J} \cdot \boldsymbol{\omega}(\mathbf{v} \cdot \mathbf{n} - u)\right] = 0. \qquad (18)$$

4. Summary

In this work, we consider reduced Cosserat continuum as a model for granular medium. In this continuum, translations and rotations are independent, stress tensor is not symmetric and couple stresses tensor equal to zero. We considered areas on which functions can be discontinuous. These surfaces are interesting in that waves propagating in the medium changing their velocity on them. Equations for solving problems in which a moving surface of discontinuity separates regions of continuous functions are presented: the law of conservation of mass (17), the law of conservation of linear momentum (14), the law of conservation of energy (16), the law of conservation of kinetic momentum (18).

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