

## ON THE INFLUENCE OF THE MICROSTRUCTURE ON THE STRESS-STRAIN STATE OF MATERIAL

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**Abstract.** The present paper is devoted to the problem of describing materials capable of structural transformations. Basing on two-component model of material with nonlinear internal force, we investigate the existence of non-stable constitutive curve containing a decreasing segment. For this purpose a kinematic loading of two-component rod is considered. The main goal is to determine the influence of the relative displacement on the stress-strain dependence and to establish the expression, connecting the position of the critical point on the diagram with the parameters of the microstructure.

**Keywords:** non-monotone constitutive curve; two-component model; structural transformations.

### 1. Introduction.

In numerous papers devoted to materials experiencing phase transitions, the assumption of a non-monotone dependence between stress and strain is introduced [1,2,3,4]. In the present paper we consider the possibility of obtaining such constitutive relation, basing on a two-component model of material. If Hooke's law is valid for each component, then its dynamic equations in 1D case are given by [5]:

$$\begin{aligned} E_1 \frac{\partial^2 u_1}{\partial x^2} - \rho_{10} \frac{\partial^2 u_1}{\partial t^2} - R &= 0 \\ E_2 \frac{\partial^2 u_2}{\partial x^2} - \rho_{20} \frac{\partial^2 u_2}{\partial t^2} + R &= 0 \end{aligned} \quad (1)$$

Here  $u_k$  ( $k=1,2$ ) denotes the displacement of each component,  $E_k$  is Young's modulus and  $\rho_{0k}$  is the density. The interaction force is denoted by  $R$ . Taking into account the periodic structure of the lattice, the simplest expression for interaction force can be chosen as:

$$R = K \sin \lambda z, \quad (2)$$

where  $K$  defines its maximum value and the parameter  $\lambda = \frac{2\pi}{d}$  is inversely proportional to the period of the lattice  $d$ . Function  $z = u_1 - u_2$  signifies the relevant displacement, which performs the role of an additional degree of freedom corresponding to the microstructure of material. Then it is convenient to rewrite equations (1) with respect to the relative displacement

$z = u_1 - u_2$  and the center of mass displacement  $U = \frac{\rho_{10}u_1 + \rho_{20}u_2}{\rho_{10} + \rho_{20}}$  [ 6]. In statics they can be

presented as:

$$\frac{\partial^2 U}{\partial x^2} = \alpha \frac{\partial^2 z}{\partial x^2},$$

$$\frac{\partial^2 z}{\partial x^2} = \beta K \sin \lambda z, \quad (3)$$

where the following notation is used:  $\alpha = \frac{E_2 \rho_1 - E_1 \rho_2}{(\rho_1 + \rho_2)(E_1 + E_2)}$ ,  $\beta = \frac{E_1 + E_2}{E_1 E_2}$ .

## 2. Kinematic loading

Consider a kinematic loading of the two-component rod of length  $l$ . One of the butts of the rod is fixed and another one is extended accordingly to the prescribed function of time  $U_0(t)$ . These conditions can be written down as:

$$u_1 = u_2 \Big|_{x=0} = 0, \quad U = \frac{u_1 \rho_1 + u_2 \rho_2}{\rho_1 + \rho_2} \Big|_{x=l} = U_0(t). \quad (4)$$

It is clearly seen that three boundary conditions are not enough to determine the stress-strain dependence. So, we need an additional assumption. Let us suppose that the stress distribution among the components is proportional to their densities

$$E_k \frac{\partial u_k}{\partial x} = \frac{\rho_k \sigma}{\rho_1 + \rho_2}, \quad (5)$$

where  $\sigma = E_1 \frac{\partial u_1}{\partial x} + E_2 \frac{\partial u_2}{\partial x}$  is the total stress in the two-component medium. At least, this assumption does not contradict to the rule of summation for stresses. Obviously, if the velocities  $c_k = \sqrt{\frac{E_k}{\rho_{k0}}}$  of the longitudinal waves are equal, there will be no relative displacement.

Therefore, we assume that there is a difference between Young's modulus  $\delta = \frac{E_2 - E_1}{E_2}$ , whereas

their densities are equal. After introducing dimensionless variables  $\xi = \frac{x}{l}$ ,  $w = \lambda z$ ,  $P = \frac{\sigma}{E_1 + E_2}$ ,  $\varepsilon_0 = \frac{U_0}{l}$  the problem is reduced to one equation with mixed

boundary conditions:

$$w_{\xi\xi} - \theta^2 \sin w = 0$$

$$w_{\xi=0} = 0, \quad w \Big|_{\xi=1} + \frac{\partial w}{\partial \xi} \Big|_{\xi=1} \frac{4(1-\delta)}{\delta^2} = \frac{2(2-\delta)\lambda l \varepsilon_0}{\delta}, \quad (6)$$

where  $\theta^2 = \beta K \lambda l^2$ . Equation (6) can be integrated in terms of elliptic functions [7]. However, the exact solution seems to be quite complicated for further analysis and we use the Galerkin procedure instead, taking for simplicity only one form  $w = \Theta \xi$ . After multiplying equation (6) by  $f(\xi) = \xi$  and integrating between the limits  $\xi = 0$  and  $\xi = 1$  we obtain the following equation:

$$\frac{\theta^2}{\Theta} \left( \cos \Theta - \frac{\sin \Theta}{\Theta} \right) - \Theta \left( 1 + \frac{\delta^2}{4\lambda l(2-\delta)(1-\delta)} \right) + \frac{\lambda l \varepsilon_0 \delta}{2(1-\delta)} = 0. \quad (7)$$

The existence of critical points on the constitutive curve leads to an additional equation with respect to  $\Theta$  :

$$\frac{\theta^2}{\Theta} \left( \frac{2}{\Theta^2} - 1 \right) \sin \Theta - \frac{2\theta^2}{\Theta^2} \cos \Theta - 1 = 0 \quad (8)$$

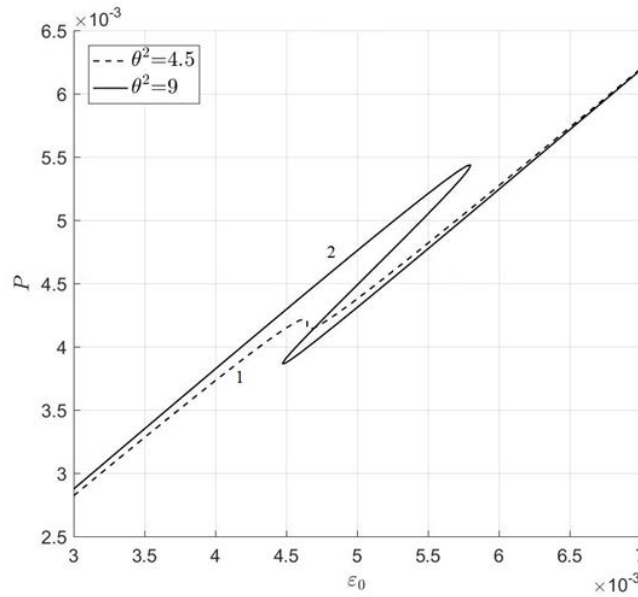
with roots satisfying the inequality

$$\frac{\theta^2}{1 + \frac{\delta^2}{4(1-\delta)}} < \Theta < \theta^2. \quad (9)$$

The numerical integrating of problem (6) is performed by applying the program AUTO 07P, which is widely used for nonlinear problems [8]. Expressions (8) and (9) allow to find the values of the parameters ensuring the existence of a loop-like stress-strain dependence, which is determined by the relative displacement at the end of the rod

$$P(\varepsilon_0) = \varepsilon_0 - \frac{w(1)\delta}{2(2-\delta)\lambda l}. \quad (10)$$

Relation (10) establishes the correlation between the microstructure, presented by the relative displacements of the components, and the macroparameter of material. The results of the numerical calculation are depicted in Fig. 1, where two constitutive curves are shown. Young's moduli, the density and grain size were chosen in accordance with the typical values for metals, and the value of parameter  $K$  characterizing the interaction between the components performs the role of controlled parameter providing the existence of an unstable segment on the stress-strain dependence.



**Fig. 1.** Constitutive curves.

Curve (1) is obtained with the following parameters:  $E_1 = 1 \cdot 10^{11} N \cdot m^{-2}$ ,  $E_2 = 2 \cdot 10^{11} N \cdot m^{-2}$ ,  $\rho_1 = \rho_2 = 4 \cdot 10^3 kg \cdot m^{-3}$ ,  $K = 4 \cdot 10^{10} N \cdot m^{-3}$ ,  $l = 5 \cdot 10^{-3} m$ , when the requirement of unambiguous determination of the stress state by a given strain is observed. The opposite situation is represented by curve 2, which corresponds to the same values of physical characteristics except for the parameter  $K$  whose value was doubled. In this case, we have the infinite values for tangent moduli, which shows the impossibility of the material transition to a new state at a given type of loading.

### 3. Critical points

The obtained results lead to the problem of defining the critical value for the deformation corresponding to the local extremum on the constitutive curve, at which a structural transformation begins. To set up its dependence on the parameters of the two-component model, we shall seek the solution of equation (6) as the sum:

$$w(\xi) = w_{lin}(\xi) + w_1(\xi), \quad (11)$$

where  $w_{lin}(\xi)$  is determined by the solution of linearized problem for near-zero values of relative displacements and the second term  $w_1(\xi)$  represents a small perturbation for the solution of linear operator. After substitution (11) into (6) we arrive at the following equation with variable coefficients and zero boundary conditions:

$$\frac{\partial^2 w_1}{\partial \xi^2} - \theta^2 w_1 \cos(w_{lin}) = \theta^2 \sin(w_{lin}) - \frac{\partial w_{lin}}{\partial \xi^2}. \quad (12)$$

Structural transformation of material corresponds to the rapid changes in the form of the solution, when it reaches the turning point. For equation (12) this condition is given by:

$$\cos(w_{lin}) = 0. \quad (13)$$

Before critical point on the constitutive curve (1) there are no turning points inside the gap  $0 \leq \xi \leq 1$ . Fig. 2. demonstrates, how they gradually appears there with increasing the deformation  $\varepsilon_0(t)$

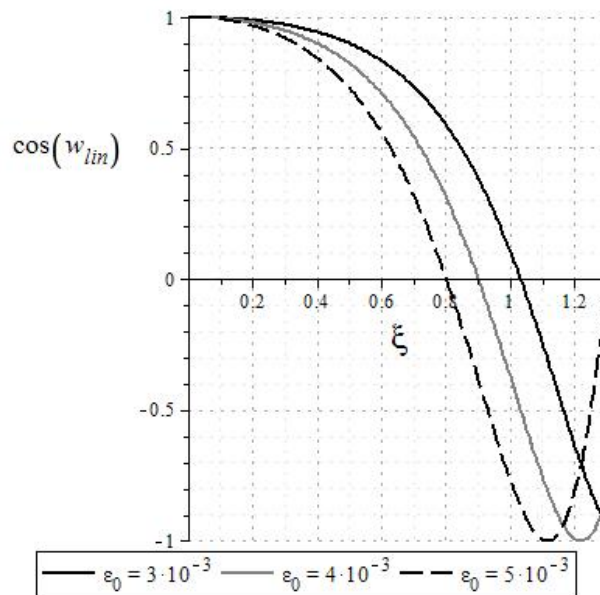


Fig. 2. Returning points.

Equation (13) allows to determine the critical deformation. Substituting into it the solution of the linearized problem at  $\xi = 1$ , one can obtain:

$$\varepsilon_{0cr} = \frac{\pi(1-\delta)\theta}{2\delta\lambda \tanh \theta}. \quad (14)$$

This problem is analogous to the problem of determining the critical load of Euler, which is found from the condition of existence of nontrivial solution for boundary value problem. Here this condition is formulated as the requirement that the return point belongs to the region  $0 \leq \xi \leq 1$ . Note that the value  $\varepsilon_{0cr} = 4 \cdot 10^{-3}$  found using relation (14) correlates well to the critical point in Fig. 1.

#### 4. Conclusion

Thus, in the present paper the two-component model with nonlinear interaction force depending on their relative displacement is proposed. The introduction of this additional degree of freedom responsible for structural transformations of material allows to demonstrate the existence of a nonmonotonic stress-strain dependence. For this purpose we consider a kinematic loading of the rod with complex internal structure. As a result, the relation between its microparameters and the critical deformation on the constitutive curve is obtained.

#### References

- [1] J. Sikora, J.P. Cusumano, W.A. Jester // *Physica D: Nonlinear Phenomena* **121(3)** (1998) 275.
- [2] C. Făciu, A. Molinari // *International journal of solids and structures* **43(3)** (2006) 497.
- [3] S.N. Gavrilov, E.V. Shishkina // *Continuum Mechanics and Thermodynamics* **22(4)** (2010) 299.
- [4] A. Berezovski, G.A. Maugin // *European Journal of Mechanics-A/Solids* **24(1)** (2005) 1.
- [5] D.A. Indeitsev, V.N. Naumov, B.N. Semenov // *Mechanics of Solids* **42(5)** (2007) 672.
- [6] E.L. Aero, A.N. Bulygin // *Mechanics of Solids* **42(5)** (2007) 807.
- [7] P.F. Byrd, M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physics* (Springer-Verlag, Berlin, 1954).
- [8] E.J. Doedel, B.E. Oldeman, *AUTO-07P: Continuation and bifurcation software for ordinary differential equations* (Concordia University, Montreal, Canada, 2008).