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PLANE WAVE AND FUNDAMENTAL SOLUTION IN THERMOPOROELASTIC MEDIUM

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Abstract. The present article deals with the study of propagation of plane wave and fundamental solution in the thermoporoelastic medium. It is found that for two dimensional model, their exist three longitudinal waves, namely P_1-wave, P_2-wave and T-wave in addition to transverse wave. Characteristics of waves like phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The representation of the fundamental solution of the system of equations in the thermoporoelastic medium in case of steady oscillations is considered in term of elementary functions. Some basic properties of the fundamental solution are established. Some special cases are also deduced.
Keywords: plane wave; fundamental solution; thermoporoelastic medium; steady oscillations.

1. Introduction

Poroelastisity is the mechanics of poroelastic solids with pores filled with fluid. Mathematical theory of poroelastisity deals with the mechanical behaviour of fluid saturated porous medium. Pore fluid generally includes gas, water and oil. Due to different motions of solid and fluid phases and complicated geometry of pore structures, it is very difficult to study the mechanical behaviour of a fluid saturated porous medium. The discovery of fundamental mechanical effects in saturated porous solids and the formulation of the first porous media theories are mainly due to Fillunger [1],Terzaghi [2,3 ,4] and their successors.

Based on the work of Von Terzaghi [2,3], Biot [5] proposed a general theory of three dimensional consolidation. Taking the compressibility of the soil into consideration, the water contained in the pores was taken to be incompressible. Biot [6,7] developed the theory for the propagation of stress waves in porous elastic solids containing a compressible viscous fluid and demonstrated the existence of two types of compressional waves (a fast and a slow wave) along with one share wave. Biot’s model was broadly accepted and some of his results have been taken as standard references and the basis for subsequent analysis in acoustic, geophysics and other such fields.

For the thermoporoelastisity problems, coupled thermal and poro-mechanical processes play an important role in a number of problems of interest in the geomechanics such as stability of boreholes and permeability enhancement in geothermal reservoirs. A thermoporoelastic approach combines the theory of heat conduction with poroelastic constitutive equations and coupling the temperature fields with the stresses and pore pressure.

Rice and Cleary [8] presented some basic stress-diffusion solutions for fluid saturated elastic porous media with compressible constituents. There exists a substantial literature treating the extension of the well known isothermal theory to account for the effects of thermal

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expansion of both the pore fluid and the elastic matrix [eg. Schiffman [9], Bowen[10], Noorishad[11]].

McTigue [12] developed a linear theory of fluid saturated porous thermoelastic material and this theory allows compressibility and thermal expansion of both the fluid and solid constituents. He presented a general solution scheme in which a diffusion equation with temperature dependent source term governs a combination of the mean total stress and the fluid pore pressure.

Kurashige [13] extends the Rice and Cleary [8] theory to incorporate the heat transportation by a pore fluid flow in addition to the effect of difference in expansibility between the pore fluid and the skeletal solid and presented a thermoelastic theory of fluid-filled porous materials. This theory shows that the displacement field is completely coupled with the pore pressure and temperature field in general, however, for irrotational displacement, the first field is decoupled from the last two, which are still coupled to each other. This pore pressure-temperature coupling involves nonlinearity.

Abousleiman and Ekbote [14] obtained the solutions for the inclined borehole in a porothermoelastic transversely isotropic medium. Bai [15] studied the fluctuation responses of porous media subjected to cyclic thermal loading. Bai and Li [16] obtained the solution for cylindrical cavity in a saturated thermoporoelastic medium.

Jabbari and Dehbani [17] considered the classical coupled thermoporoelastic model of hollow and solid cylinders under radial symmetric loading conditions and presented a unique solution. Ganbin et al. [18] obtained the solution in saturated porous thermoviscoelastic medium, with cylindrical cavity that is subjected to time dependent thermal load by using Laplace transform technique. Gatmiri et al. [19] presented the two-dimensional fundamental solutions for non-isothermal unsaturated deformable porous medium subjected to quasi- static loading in time and frequency domain. Li et al. [20] presented the study state solutions for transversely isotropic thermoporoelastic media in three dimensions.

Jabbari and Dehbani [21] considered the quasi- static porothermoelasticity model of hollow and solid sphere and obtained the displacement, temperature distribution and pressure distribution due mechanical, thermal and pressure source. Liu et al. [22] studied the relaxation effect of a saturated porous media using the two dimensional generalized thermoelastic theory. Belotserkovets and Prevost [23] obtained an analytical solution of thermoporoelastic response of fluid-saturated porous sphere.

Bai [24] derived an analytical method for the thermal consolidation of layered saturated porous material subjected to exponential decaying thermal loading. Mixed variation principal for dynamic response of thermoelastic and poroelastic continua was discussed by Apostolakis and Dargus [25]. Hou et al. [26] discussed the three dimensional Green’s function for transversely isotropic thermoporoelastic biomaterial. Jabbari et.al. [27] presented the thermal buckling analysis of functionally graded thin circular plate made of saturated porous material and obtained the closed form solutions for circular plates subjected to temperature load.

Liu and Chain [28] discussed a micromechanical analysis of the fracture properties of saturated porous media. He et al. [29] studied the dynamic simulation of landslide based on thermoporoelastic approach. Nguyen et al. [30] discussed the analytical study of freezing behaviour of a cavity in thermoporoelastic medium. Wu et al. [31] presented a refined theory of axisymmetric thermoporoelastic circular cylinder.

Svanadze [32, 33, 34] constructed the fundamental solutions in thermoelasticity with microtemperature and micromorphic elastic solid with microtemperature. Svanadze and his co-workers [35, 36, 37, 38] also constructed the fundamental solution and basic properties in thermomicrostretch, micropolar thermoelsticity without energy dissipation and full coupled theory of elasticity for solids with double porosity.

Kumar and Kansal [39, 40, 41] obtained fundamental solution in thermomicrostrech, micropolar thermoelastic diffusion, micropolar thermoelastic diffusion with voids. Kumar and Kansal [42] also investigated the plane wave and fundamental solution in thermoelastic diffusion. Kumar et.al [43] studied the plane wave and fundamental solution in the theory of an electro- microstretch generalized thermoelastic solid.

Kumar and Gupta [44] studied the plane wave propagation in an anisotropic thermoelastic medium with fractional order derivative and void. Sharma and Kumar [45] studied the propagation of Plane waves and fundamental solution in thermoviscoelastic medium with voids. Plane wave propagation in microstretch thermoelastic medium with microtemperature was studied by Kumar and Kaur [46]. Fundamental and plane wave solution in swelling porous medium was studied by Kumar et.al [47].

Scarpetta et.al [48] constructed the fundamental solution in the theory of thermoelasticity for solids with double porosity. Kumar and Gupta [49] studied the Plane wave propagation and domain of influence in fractional order thermoelastic material with three phase lag heat transfer.

The present study deals with the study of propagation of plane waves and fundamental solution in the thermoporoelastic medium. Characteristics of waves like phase velocity and attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The representation of the fundamental solution of the system of equations in the thermoporoelastic medium in the case of steady oscillations is considered in term of elementary functions.

2. Basic Equations

Following Jabbari and Dehbani [50], the field equations are given by

$$(\lambda+\mu)\nabla\nabla.\boldsymbol{u}+\mu\nabla^2\boldsymbol{u}-\alpha\nabla p-\beta\nabla T=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}, \tag{1}$$

$$\frac{k}{\gamma_w}\nabla^2 p-\alpha_p\dot{p}-Y\dot{T}-\alpha div\dot{\boldsymbol{u}}=0, \tag{2}$$

$$K\nabla^2 T-ZT_0\dot{T}+YT_0\dot{p}-\beta T_0 div\dot{\boldsymbol{u}}=0, \tag{3}$$

where $\boldsymbol{u}$ is the displacement component, p is the pore pressure, ρ is the bulk mass density, $\alpha=1-\frac{C_s}{C}$ is the Biot's coefficient, $C_s=3(1-2v_s)/E_s$ is the coefficient of volumetric compression of solid grain, with E_s and v_s being the elastic modulus and Poisson's ratio of solid grain, $C=3(1-2v)/E$ is the coefficient of volumetric compression of solid skeleton, with E and v being the elastic modulus and Poisson's ratio of solid skeleton, T_0 is initial reference temperature, $\beta=\frac{3\alpha_s}{C}$ is the thermal expansion factor, α_s is the coefficient of linear thermal expansion of solid grain, $Y=3(n\alpha_w+(\alpha-n)\alpha_s)$ and $\alpha_p=n(C_w-C_s)+\alpha C_s$ are coupling parameters, α_w and C_w are the coefficients of linear thermal expansion and volumetric compression of pore water, n is the porosity, k is the hydraulic conductivity, γ_w is the unit of pore water and $Z=\frac{(1-n)\rho_s c_s+n\rho_w c_w}{T_0}$ is coupling parameter, ρ_w and ρ_s are densities of pore water and solid grain and c_w and c_s are heat capacities of pore water and solid grain and K is the coefficient of heat conductivity.

3. Formulation of the problem

We consider a homogeneous thermoporoelastic medium. For two dimensional problems, we take displacement vector $\boldsymbol{u}$ as:

$$\boldsymbol{u}=(u_1,0,u_3) \tag{4}$$

We define the dimensionless quantities as:

$$\boldsymbol{x}'=\frac{\omega^*}{c_1}\boldsymbol{x},\ \boldsymbol{u}'=\frac{\omega^*\rho c_1}{\beta T_0}\boldsymbol{u},\ p'=\frac{p}{\beta T_0},\ c_1^2=\frac{\lambda+2\mu}{\rho},$$

$$t'=\omega^* t,\ T'=\frac{T}{T_0},\ \omega^*=\frac{ZT_0c_1^2}{K}, \tag{5}$$

$i = 1,2,3$.

The displacement components u_1 and u_3 are related to the potential functions $\Phi(x_1, x_3, t)$, $\Psi(x_1, x_3, t)$ as:

$$u_1 = \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Psi}{\partial x_3},\ u_3 = \frac{\partial \Phi}{\partial x_3} + \frac{\partial \Psi}{\partial x_1} \tag{6}$$

Using equation (4) on (1)-(3) and applying the dimensionless quantities defined by (5), with the aid of (6), after suppressing the prime, yield

$$\nabla^2\Phi - \alpha p - T - \frac{\partial^2 \Phi}{\partial t^2} = 0, \tag{7}$$

$$\delta^2\nabla^2\Psi - \frac{\partial^2 \Psi}{\partial t^2} = 0, \tag{8}$$

$$b_1\nabla^2 p - b_2\frac{\partial p}{\partial t} - b_3\frac{\partial T}{\partial t} - \frac{\partial}{\partial t}[\nabla^2\Phi] = 0, \tag{9}$$

$$b_4\nabla^2 T - b_5\frac{\partial T}{\partial t} + b_6\frac{\partial p}{\partial t} - \frac{\partial}{\partial t}[\nabla^2\Phi] = 0, \tag{10}$$

where

$$\delta^2 = \frac{\mu}{\lambda+2\mu},\quad b_1 = \frac{k\omega^*\rho}{\gamma_w\alpha},\quad b_2 = \frac{\alpha_p\rho c_1^2}{\alpha},\quad b_3 = \frac{Y\rho c_1^2}{\alpha},\quad b_4 = \frac{K\omega^*\rho}{\beta^2 T_0},\quad b_5 = \frac{Z\rho c_1^2}{\beta^2},\quad b_6 = \frac{Y\rho c_1^2}{\beta^2}$$

and $e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}$, $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

4. Solution of Plane waves

For Plane harmonic waves, we assume the solution of the form:

$$(\Phi, \Psi, p, T) = (\overline{\Phi}, \overline{\Psi}, \overline{p}, \overline{T})e^{i[\xi(x_1 l_1 + x_3 l_3) - \omega t]}, \tag{11}$$

where $\omega(= \xi\, c)$ is the frequency and ξ is the wave number and c is the phase velocity. $\overline{\Phi}, \overline{\Psi}, \overline{p}, \overline{T}$ are undetermined constants that are independent of time t and coordinates x_1, x_3. l_1 and $l_3$are the direction cosines of the wave normal to the x_1x_3 –plane with the property $l_1^2 + l_3^2 = 1$.

Using (11) in (7), (9) and (10), we obtain a system of three homogeneous equations in three unknowns and these equations has non-trivial solution if the determinant of the coefficients of the system vanish, which yields the following characteristic equation in c as:

$$H_1 c^6 + H_2 c^4 + H_3 c^2 + H_4 = 0, \tag{12}$$

where

$$H_1 = \frac{F_4}{\omega^6}, H_2 = \frac{F_3}{\omega^4}, H_3 = \frac{F_2}{\omega^2}, H_4 = F_1,$$

$$F_1 = -b_1 b_4,\ F_2 = b_1 b_4\omega^2 + i\omega(b_1 b_5 + b_2 b_4 + \alpha b_4 + b_1),$$

$$F_3 = -i\omega^3(b_1 b_5 + b_2 b_4) + \omega^2(b_2 b_5 + b_3 b_6 - \alpha b_3 + \alpha b_5 + b_6 + b_2),$$

$$F_4 = -\omega^4(b_2 b_5 + b_3 b_6).$$

The complex coefficient implies that three roots of this equation may be complex. The complex phase velocities of the longitudinal waves, given by c_i, $i = 1, 2, 3$, will be varying with the direction of phase propagation. The complex velocity of a longitudinal wave, i.e. $c_i = c_R + ic_I$, defines the phase propagation velocity $V_i = (c_R^2 + c_I^2)/c_R$, attenuation quality factor $Q_i^{-1} = -2\, c_I/c_R$, Specific loss $R_i = 4\pi\left|\frac{c_I}{c_R}\right|$, Penetration depth $S_i = |(c_R^2 + c_I^2)/\omega c_I|$ for the corresponding wave. Therefore, three waves propagating in such a medium are attenuating. The same directions of wave propagation and attenuation vector of these waves make them homogeneous wave. The waves with phase velocity, attenuation quality factor, specific loss and penetration depth i.e. V_i, Q_i^{-1}, R_i and $S_i (i = 1, 2, 3,)$ may be named as P_1-wave, P_2-wave and T-wave that are propagating with the descending order of their velocities, respectively.

Substituting the value of Ψ from (11) in (8), we obtain:

$$(\omega^2 - \delta^2\xi^2)\overline{\Psi} = 0, \tag{13}$$

which yield

$$\mathrm{c} = \pm\delta \tag{14}$$

The equation (13) corresponds to transverse wave which travel with phase velocity δ.

5. Special Case

In the absence of porosity effect, characteristic equation reduces to:

$$H_5\mathrm{c}^4 + H_6\mathrm{c}^2 + b_4 = 0, \tag{15}$$

where

$H_5 = \frac{F_6}{\omega^4}, H_6 = \frac{F_5}{\omega^2},\ F_5 = -b_4\omega^2 - i\omega(b_{50}+1), F_6 = i\omega^3 b_{50},$

where $b_{50} = \frac{\rho_s^2 c_s c_1^2}{\beta^2 T_0}$

and the equation (13) remains the same because it is not effected by porous effect.

6. Steady Oscillations

For steady oscillations, we assume the displacement vector, pressure and temperature change, of the form:

$$(\boldsymbol{u}(\boldsymbol{x},t), p(\boldsymbol{x},t), T(\boldsymbol{x},t) = (\overline{\boldsymbol{u}}, \bar{p}, \bar{T})e^{-i\omega t} \tag{16}$$

Making use of dimensionless quantities defined by (5), in (1)-(3) and with the aid of (16) yields:

$$(1-\delta^2)\nabla\nabla.\boldsymbol{u} + (\delta^2\nabla^2 + \omega^2)\boldsymbol{u} - \alpha\nabla p - \nabla T = 0, \tag{17}$$

$$a_1\nabla^2 p + i\omega a_2 p + i\omega T + i\omega a_3\, div\boldsymbol{u} = 0, \tag{18}$$

$$a_4\nabla^2 T + i\omega a_5 T + i\omega a_6 p + i\omega a_7 div\boldsymbol{u} = 0, \tag{19}$$

where

$a_1 = \frac{k\omega^*\beta}{\gamma_w Y c_1^2}, a_2 = \frac{\alpha_p\,\beta}{Y}, a_3 = \frac{\alpha\,\beta}{Y\rho c_1^2}, a_4 = \frac{k\omega^*}{c_1^2}, a_5 = ZT_0,\ a_6 = Y\beta T_0,\ a_7 = \frac{\beta^2 T_0}{\rho c_1^2}.$

We introduce the matrix differential operator:

$$\boldsymbol{F}(\boldsymbol{D_x}) = \left\|F_{gh}(\boldsymbol{D_x})\right\|_{5\times5}, \tag{20}$$

where

$F_{mn}(\boldsymbol{D_x}) = (\delta^2\nabla^2 + \omega^2)\delta_{mn} + (1-\delta^2)\frac{\partial^2}{\partial x_{\mathrm{m}}\partial x_{\mathrm{n}}}, F_{m4}(\boldsymbol{D_x}) = -\alpha\frac{\partial}{\partial x_{\mathrm{m}}},$

$F_{m5}(\boldsymbol{D_x}) = -\frac{\partial}{\partial x_{\mathrm{m}}},\ F_{4n}(\boldsymbol{D_x}) = i\omega a_3\frac{\partial}{\partial x_{\mathrm{n}}},\ F_{5n}(\boldsymbol{D_x}) = i\omega a_7\frac{\partial}{\partial x_{\mathrm{n}}},$

$F_{44}(\boldsymbol{D_x}) = i\omega a_2 + a_1\nabla^2,\ F_{45}(\boldsymbol{D_x}) = i\omega,\ F_{54}(\boldsymbol{D_x}) = i\omega a_6,$

$$F_{55}(\boldsymbol{D_x}) = i\omega a_5 + a_4\nabla^2, m,n = 1,2,3, \tag{21}$$

and δ_{mn} is the Kronecker delta.

The system of equations (17)-(19) can be written as:

$$\boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{U}(\boldsymbol{x}) = 0, \tag{22}$$

where $\boldsymbol{U} = (\boldsymbol{u}, p, T)$ is a five component vector function on E^3.

We assume that

$$\delta^2 a_1 a_4 \neq 0 \tag{23}$$

If the condition (23) is satisfied, then $\boldsymbol{F}$ is an elliptic differential operator, Hörmander [55].

Definition: The fundamental solution of the system of equations (17)-(19) (the fundamental matrix of operator $\boldsymbol{F}$) is the matrix $\boldsymbol{G}(\boldsymbol{x}) = \left\|G_{gh}(\boldsymbol{x})\right\|_{5\times5}$, satisfying condition, Hörmander [51]:

$$\boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{U}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{I}(\boldsymbol{x}), \tag{24}$$

where $\delta()$ is the Dirac delta function, $\boldsymbol{I} = \left\|\delta_{gh}\right\|_{5\times5}$ is the unit matrix and $\boldsymbol{x} = E^3$.

Now we construct $\boldsymbol{G}(\boldsymbol{x})$ in terms of elementary functions.

6.1 Fundamental solution of system of equations of steady oscillations. We consider the system of equations:

$\delta^2\nabla^2\boldsymbol{u} + (1-\delta^2)\nabla\nabla.\boldsymbol{u} + i\omega a_3\nabla p + i\omega a_7\nabla T + \omega^2\boldsymbol{u} = \boldsymbol{H};$ (25)
$-\alpha div\boldsymbol{u} + (i\omega a_2 + a_1\nabla^2)p + i\omega a_6 T = L;$ (26)
$-div\boldsymbol{u} + i\omega p + (i\omega a_5 + a_4\nabla^2)T = M,$ (27)
where $\boldsymbol{H}$ is three components of vector function on E^3 and L, M are scalar functions of E^3.The system of equations (25)-(27) may be written in the form:

$\boldsymbol{F}^{tr}(\boldsymbol{D}_x)\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{Q}(\boldsymbol{x}),$ (28)

where $\boldsymbol{F}^{tr}$ is the transpose of $\boldsymbol{F}, \boldsymbol{Q} = (\boldsymbol{H}, L, M)$ and $\boldsymbol{x} = E^3$.

Applying the operator div to (25), we obtain:

$(\nabla^2 + \omega^2)div\boldsymbol{u} + i\omega a_3\nabla^2 p + i\omega a_7\nabla^2 T + \omega^2 div\boldsymbol{u} = div\boldsymbol{H}$ (29)

The equations (26), (27) and (29) can be written in the form:

$\boldsymbol{N}(\Delta)\boldsymbol{S} = \bar{\boldsymbol{Q}},$ (30)

where $\boldsymbol{S} = (div\boldsymbol{u}, p, T), \bar{\boldsymbol{Q}} = (d_1, d_2, d_3) = (div\boldsymbol{H}, L, M)$, and

$$\boldsymbol{N}(\Delta) = \|N_{mn}(\Delta)\|_{3\times3} = \begin{Vmatrix} \nabla^2+\omega^2 & i\omega a_3\nabla^2 & i\omega a_7\nabla^2 \\ -\alpha & i\omega a_2 + a_1\nabla^2 & i\omega a_6 \\ -1 & i\omega & i\omega a_5 + a_4\nabla^2 \end{Vmatrix} \quad (31)$$

The equation (30) can also be written as:

$\Gamma_1(\Delta)\boldsymbol{S} = \boldsymbol{\Psi},$ (32)

where $\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3)$, $\Psi_n = e^*\sum_{m=1}^{3} N_{mn}^*\, d_m$,

$\Gamma_1(\Delta) = e^* \det\ \boldsymbol{N}(\Delta), e^* = \frac{1}{a_1 a_4}, n = 1,2,3,$ (33)

and N_{mn}^* is the cofactor of the elements N_{mn} of the matrix $\boldsymbol{N}$.

From (31) and (33), we see that

$\Gamma_1(\Delta) = \prod_{m=1}^{3}(\Delta + \lambda_m^2),$ (34)

where $\lambda_m^2, m = 1,2,3$ are the root of the equation $\Gamma_1(-\kappa) = 0$ (with respect to κ).

Applying operator $\Gamma_1(\Delta)$ on (25), we have:

$\Gamma_1(\Delta)[\delta^2\nabla^2\boldsymbol{u} + (1-\delta^2)\nabla\nabla.\boldsymbol{u} + i\omega a_3\nabla p + i\omega a_7\nabla T + \omega^2\boldsymbol{u}] = \Gamma_1(\Delta)\boldsymbol{H}$

$\Gamma_1(\Delta)[\delta^2\nabla^2\boldsymbol{u} + \omega^2\boldsymbol{u}] = \Gamma_1(\Delta)\boldsymbol{H} - grad[(1-\delta^2)\Psi_1 + i\omega a_3\Psi_2 + i\omega a_7\Psi_3]$

$\Gamma_1(\Delta)\left[\nabla^2 + \frac{\omega^2}{\delta^2}\right]\boldsymbol{u} = \frac{1}{\delta^2}[\Gamma_1(\Delta)\boldsymbol{H} - grad\{(1-\delta^2)\Psi_1 + i\omega a_3\Psi_2 + i\omega a_7\Psi_3\}]$

$\Gamma_1(\Delta)[\nabla^2 + \lambda_4^2]\boldsymbol{u} = \frac{1}{\delta^2}[\Gamma_1(\Delta)\boldsymbol{H} - grad\{(1-\delta^2)\Psi_1 + i\omega a_3\Psi_2 + i\omega a_7\Psi_3\}]$

$\Gamma_1(\Delta)[\nabla^2 + \lambda_4^2]\boldsymbol{u} = \boldsymbol{\Psi}'',$ (35)

where $\lambda_4^2 = \frac{\omega^2}{\delta^2}$,

$\boldsymbol{\Psi}'' = \frac{1}{\delta^2}[\Gamma_1(\Delta)\boldsymbol{H} - grad\{(1-\delta^2)\Psi_1 + i\omega a_3\Psi_2 + i\omega a_7\Psi_3\}]$ (36)

From equations (32) and (35), we obtain:

$\boldsymbol{\Theta}(\Delta)\boldsymbol{U}(\boldsymbol{x}) = \widetilde{\boldsymbol{\Psi}}(\boldsymbol{x}),$ (37)

where $\widetilde{\boldsymbol{\Psi}}(\boldsymbol{x}) = (\boldsymbol{\Psi}'', \Psi_2, \Psi_3)$,

and $\boldsymbol{\Theta}(\Delta) = \|\Theta_{gh}(\Delta)\|_{5\times5}$; $\Theta_{mm}(\Delta) = \Gamma_1(\Delta)[\nabla^2 + \lambda_4^2], m = 1,2,3;$

$\Theta_{44}(\Delta) = \Theta_{55}(\Delta) = \Gamma_1(\Delta), \Theta_{gh}(\Delta) = 0,\ g, h = 1,2,3,4,5,\ \ g \neq h.$

The equations (33) and (36) can be written in the form:

$\boldsymbol{\Psi}'' = \frac{1}{\delta^2}[\Gamma_1(\Delta)\boldsymbol{H} - grad(1-\delta^2)e^*\sum_{m=1}^{3}N_{m1}^*\, d_m + i\omega a_3\ grad\ e^*\sum_{m=1}^{3}N_{m2}^*\, d_m + i\omega a_7\ grad\ e^*\sum_{m=1}^{3}N_{m3}^*\, d_m],$

$\boldsymbol{\Psi}'' = \frac{1}{\delta^2}\Gamma_1(\Delta)\boldsymbol{H} + \frac{1}{\delta^2}e^* grad\ div\ \boldsymbol{H}\{-(1-\delta^2)N_{11}^* + i\omega a_3 N_{12}^* + i\omega a_7 N_{13}^*)\} + \frac{1}{\delta^2}e^* grad\ L\{-(1-\delta^2)N_{21}^* + i\omega a_3 N_{22}^* + i\omega a_7 N_{23}^*)\} + \frac{1}{\delta^2}e^* grad\ M\{-(1-\delta^2)N_{31}^* + i\omega a_3 N_{32}^* + i\omega a_7 N_{33}^*)\},$

$\boldsymbol{\Psi}'' = \left[\frac{1}{\delta^2}\Gamma_1(\Delta)\boldsymbol{J} + q_{11}(\Delta) grad\ div\right]\boldsymbol{H} + q_{21}(\Delta) grad\ L + q_{31}(\Delta) grad\ M,$

$\Psi_2 = q_{12}(\Delta)\ div\boldsymbol{H} + q_{22}(\Delta)\ L + q_{32}(\Delta)\ M,$ (38)

$\Psi_3 = q_{13}(\Delta)\, div\boldsymbol{H} + q_{23}(\Delta)\, L + q_{33}(\Delta)\, M,$
where $\boldsymbol{J} = \left\|\delta_{gh}\right\|_{3\times 3}$ is the unit matrix and
$q_{11}(\Delta) = \frac{1}{\delta^2} e^*\{-(1-\delta^2)N_{11}^* + i\omega a_3 N_{12}^* + i\omega a_7 N_{13}^*)\},$
$q_{21}(\Delta) = \frac{1}{\delta^2} e^*\{-(1-\delta^2)N_{21}^* + i\omega a_3 N_{22}^* + i\omega a_7 N_{23}^*)\},$
$q_{31}(\Delta) = \frac{1}{\delta^2} e^*\{-(1-\delta^2)N_{31}^* + i\omega a_3 N_{32}^* + i\omega a_7 N_{33}^*)\},$
$q_{m2}(\Delta) = e^* N_{m2}^*$, $q_{m3}(\Delta) = e^* N_{m3}^*$, m=1,2,3.

Now from equations (38), we have:

$$\widetilde{\boldsymbol{\Psi}}(x) = \boldsymbol{R}^{tr}(\boldsymbol{D_x})\boldsymbol{Q}(\boldsymbol{x}), \tag{39}$$

where $\boldsymbol{R}^{tr}$ is the transpose of $\boldsymbol{R}$, and
$\boldsymbol{R} = \|R_{mn}\|_{5\times 5},$
$R_{mn}(\boldsymbol{D_x}) = \frac{1}{\delta^2}\Gamma_1(\Delta)\delta_{mn} + q_{11}\frac{\partial^2}{\partial x_m \partial x_n}$, $R_{m4}(\boldsymbol{D_x}) = q_{21}\frac{\partial}{\partial x_m}$, $R_{m5}(\boldsymbol{D_x}) = q_{31}\frac{\partial}{\partial x_m}$,
$R_{4n}(\boldsymbol{D_x}) = q_{12}\frac{\partial}{\partial x_n}$, $R_{5n}(\boldsymbol{D_x}) = q_{13}\frac{\partial}{\partial x_n}$, $m, n = 1,2,3,$

$$R_{45}(\boldsymbol{D_x}) = q_{22},\ R_{46}(\boldsymbol{D_x}) = q_{32},\ R_{54}(\boldsymbol{D_x}) = q_{23}\ ,\ R_{55}(\boldsymbol{D_x}) = q_{33}. \tag{40}$$

Now from equations (28), (37) and (39), we have:

$$\boldsymbol{\Theta U} = \boldsymbol{R}^{tr}\boldsymbol{F}^{tr}\boldsymbol{U}. \tag{41}$$

It implies that $\boldsymbol{R}^{tr}\boldsymbol{F}^{tr} = \boldsymbol{\Theta}$, and hence

$$\boldsymbol{F}(\boldsymbol{D_x})\boldsymbol{R}(\boldsymbol{D_x}) = \boldsymbol{\Theta}(\Delta). \tag{42}$$

We assume that: $\lambda_m^2 \neq \lambda_n^2 \neq 0$, $m, n = 1,2,3,4,\ m \neq n.$

Let $\boldsymbol{Y}(\boldsymbol{x}) = \|Y_{rs}(\boldsymbol{x})\|_{5\times 5},\ Y_{mm}(\boldsymbol{x}) = \sum_{n=1}^{4} r_{1n}\xi_n(\boldsymbol{x}),\ m = 1,2,3,$
$Y_{44}(\boldsymbol{x}) = Y_{55}(\boldsymbol{x}) = \sum_{n=1}^{3} r_{2n}\xi_n(\boldsymbol{x})$
$Y_{uv}(\boldsymbol{x}) = 0,\quad u, v = 1,2,3,4,5,\quad u \neq v,$
where
$\xi_n(\boldsymbol{x}) = -\frac{1}{4\pi|x|}\exp(i\lambda_n|\boldsymbol{x}|),\ n = 1,2,3,4,$
$r_{2l} = \prod_{m=1, m\neq l}^{4}(\lambda_m^2 - \lambda_l^2)^{-1}$, $\quad l = 1,2,3,4,$

$$r_{2v} = \prod_{m=1, m\neq v}^{4}(\lambda_m^2 - \lambda_v^2)^{-1},\quad v = 1,2,3,4, \tag{43}$$

We will prove the following Lemma:
The matrix $\boldsymbol{Y}$ defined above is the fundamental matrix of operator $\boldsymbol{\Theta}(\Delta)$, that is

$$\boldsymbol{\Theta}(\Delta)\boldsymbol{Y}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{I}(\boldsymbol{x}). \tag{44}$$

Proof: To prove the lemma, it is sufficient to prove that:

$$\Gamma_1(\Delta)(\Delta + \lambda_4^2)Y_{11}(\boldsymbol{x}) = \delta(\boldsymbol{x})\ ,\Gamma_1(\Delta)Y_{44}(\boldsymbol{x}) = \delta(\boldsymbol{x}). \tag{45}$$

We find that:

$$r_{11} + r_{12} + r_{13} + r_{14} = \frac{t_1 + t_2 + t_3 + t_4}{t_5}, \tag{46}$$

$t_1 = (\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2)$, $t_2 = (\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2),$
$t_3 = (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)$, $t_4 = (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2),$

$$t_5 = (\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2). \tag{47}$$

Using (47) in (46), yield

$$r_{11} + r_{12} + r_{13} + r_{14} = 0, \tag{48}$$

$$r_{12}(\lambda_1^2 - \lambda_2^2) + r_{13}(\lambda_1^2 - \lambda_3^2) + r_{14}(\lambda_1^2 - \lambda_4^2) = 0, \tag{49}$$

$$r_{13}(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2) + r_{14}(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2) = 0, \tag{50}$$

$$r_{14}(\lambda_1^2 - \lambda_4^2)(\lambda_2^2 - \lambda_4^2)(\lambda_3^2 - \lambda_4^2) = 1, \tag{51}$$

$$(\Delta + \lambda_4^2)\xi_n(\boldsymbol{x}) = \delta(\boldsymbol{x}) + (\lambda_m^2 - \lambda_n^2)\xi_n(\boldsymbol{x})\ , m, n = 1,2,3,4. \tag{52}$$

Now consider:
$\Gamma_1(\Delta)(\Delta + \lambda_4^2)Y_{11}(\boldsymbol{x}) = (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\sum_{n=1}^{4} r_{1n}\xi_n(\boldsymbol{x}) =$
$= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\sum_{n=1}^{4} r_{1n}[\delta(\boldsymbol{x}) + (\lambda_1^2 - \lambda_n^2)\xi_n(\boldsymbol{x})]=$
$= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)[\delta(\boldsymbol{x})\sum_{n=1}^{4} r_{1n} + \sum_{n=2}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)\xi_n(\boldsymbol{x})] =$

$= (\Delta + \lambda_2^2)(\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\sum_{n=2}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)\xi_n(\boldsymbol{x}) =$
$= (\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\sum_{n=2}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)[\delta(\boldsymbol{x}) + (\lambda_2^2 - \lambda_n^2)\xi_n(\boldsymbol{x})] =$
$= (\Delta + \lambda_3^2)(\Delta + \lambda_4^2)\sum_{n=2}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)(\lambda_2^2 - \lambda_n^2)\xi_n(\boldsymbol{x}) =$
$= (\Delta + \lambda_4^2)\sum_{n=3}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)(\lambda_2^2 - \lambda_n^2)[\delta(\boldsymbol{x}) + (\lambda_3^2 - \lambda_n^2)\xi_n(\boldsymbol{x})] =$
$= (\Delta + \lambda_4^2)\sum_{n=3}^{4} r_{1n}(\lambda_1^2 - \lambda_n^2)(\lambda_2^2 - \lambda_n^2)(\lambda_3^2 - \lambda_n^2)\xi_n(\boldsymbol{x}) =$
$= (\Delta + \lambda_4^2)\xi_n(\boldsymbol{x}) = \delta(\boldsymbol{x}).$ (53)

Similarly, $\Gamma_1(\Delta)Y_{44}(\boldsymbol{x}) = \delta(\boldsymbol{x})$, can be proved.

We introduce the matrix:

$\mathbf{G}(\boldsymbol{x}) = \boldsymbol{R}(\mathbf{D}_x)\boldsymbol{Y}(\boldsymbol{x}).$ (54)

From equations (42), (44) and (54), we have:

$\boldsymbol{F}(\mathbf{D}_x)\mathbf{G}(\boldsymbol{x}) = \boldsymbol{F}(\mathbf{D}_x)\boldsymbol{R}(\mathbf{D}_x)\boldsymbol{Y}(\boldsymbol{x}) = \delta(\boldsymbol{x})\boldsymbol{I}(\boldsymbol{x}),$ (55)

hence $\mathbf{G}(\boldsymbol{x})$ is the solution to the equations (24).

Therefore we have proved the following theorem.

Theorem: The matrix $\mathbf{G}(\boldsymbol{x})$ defined by (54) is the fundamental solution of the system of equations (17)-(19).

6.2 Basic properties of the matrix $\mathbf{G}(\boldsymbol{x})$.

Property 1. Each column of the matrix $\mathbf{G}(\boldsymbol{x})$ is the solution of system of equations (17)-(19) to every point $\boldsymbol{x} \in E^3$ except the origin.

Property 2. The matrix $\mathbf{G}(\boldsymbol{x})$ can be written in the form:

$\boldsymbol{G} = \left\|G_{gh}\right\|_{5\times5}$,

$G_{mn}(x) = R_{mn}(\boldsymbol{D}_x)Y_{11}(\boldsymbol{x}),\ G_{p4}(\boldsymbol{x}) = R_{p4}(\boldsymbol{D}_x)Y_{44}(\boldsymbol{x}),$

$G_{p5}(\boldsymbol{x}) = R_{p5}(\boldsymbol{D}_x)Y_{44}(\boldsymbol{x}),\quad G_{4p}(\boldsymbol{x}) = R_{4n}(\boldsymbol{D}_x)Y_{44}(\boldsymbol{x}),\quad G_{5p}(\boldsymbol{x}) = R_{5n}(\boldsymbol{D}_x)Y_{44}(\boldsymbol{x}),$

$m, n = 1,2,3,\ \ p = 1,2,3,4,5.$

7. Numerical results and discussion

With the view of illustrating the theoretical results and for numerical discussion we take a model for which the values of the various physical parameters are taken from Jabbari and Dehbani [54]:

$E = 6 \times 10^5 Pa$, $\nu = 0.3$, $T_0 = 293\ °K$, $K_s = 2 \times 10^{10} Pa$, $K_w = 5 \times 10^9 Pa$, $K = 0.5 W/m$ °C, $\alpha_s = 1.5 \times 10^{-5} 1/$°C, $\alpha_W = 2 \times 10^{-4} 1/$°C, $c_s = 0.8 J/g$°C, $c_w = 4.2 J/g$°C, $\rho_s = 2.6 \times 10^6\ g/m^3$, $\rho_w = 1 \times 10^6 g/m^3$, $n = 0.4$, $\alpha = 1$

The software Matlab 7.0.4 has been used to determine the values of phase velocity, attenuation coefficient, specific loss and penetration depth of plane waves, i.e. P_1-wave, P_2-wave and T-wave. The variations of phase velocity, attenuation coefficients, specific loss and penetration depth with respect to frequency are shown in Figs. 1-12 respectively. In all the Figures, the solid line corresponds to thermoporoelastic medium (PTE) and dotted line corresponds to thermoelastic medium (TE).

7.1. Phase velocity. Fig. 1 depicts the variations of phase velocity of V_1 with frequency ω for PTE and TE. In case of PTE, the value of V_1 increases exponentially as ω increases, whereas in case of TE, the value of V_1 oscillates in the range $0 \le \omega \le 2.5$ and then increase as ω increases.

Fig. 2 shows the variations of V_2 with ω for PTE and TE. The value of V_2 for PTE and TE increases gradually but due to the effect of porosity the value of V_2 is less for PTE as compared to TE as ω increases.

Fig.3 shows that the value of V_3 for PTE when $\alpha = 0.5$ increases gradually as ω increases whereas there is sharp increase in the value of V_3 when $\alpha = 0.75$ as ω increases. Due to the effect of porosity the value of V_3, when $\alpha = 0.5$ is more as compared to $\alpha = 0.75$ in the range $0 \le \omega \le 0.4$ and then show the opposite behaviour for higher values of ω.

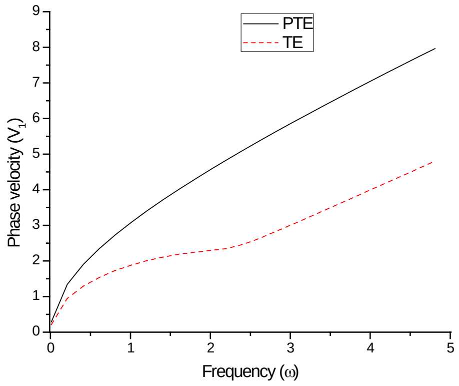

Fig. 1. Variation of phase velocity w.r.t. frequency.

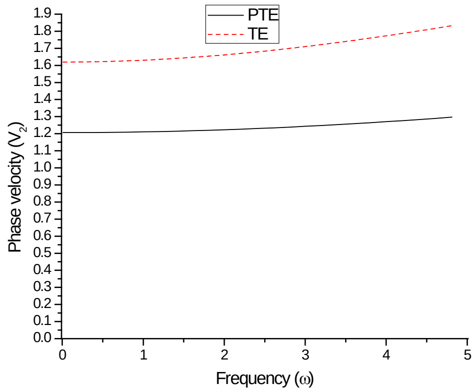

Fig. 2. Variation of phase velocity w.r.t. frequency.

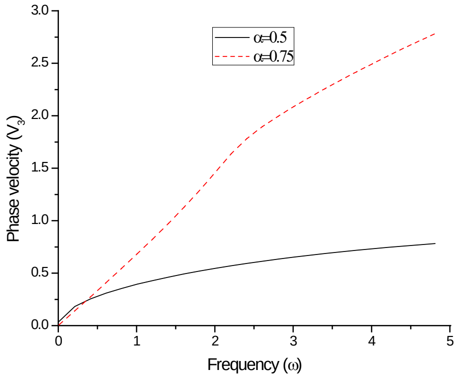

Fig. 3. Variation of phase velocity w.r.t. frequency.

7.2. Attenuation quality factor. Fig. 4 shows the variation of Q_1 with ω for PTE and TE. The value of Q_1 for PTE and TE increases gradually as ω increases but due to the effect of porosity the value of Q_1 for PTE is more as compared to TE as ω increases.

The variation of Q_2 with ω for PTE and TE is shown in Fig.5. The value of Q_2 is consistent for PTE whereas for TE, there is a small increase in the value of Q_2 as ω increases. The value of Q_2 for TE is more as compared to PTE due to the effect of porosity as ω increases.

Fig. 6 depicts the variations of Q_3 with frequency ω for PTE. When $\alpha = 0.5$, there is a small increase in the value of Q_3 as ω increases, whereas for $\alpha = 0.75$, the value of Q_3 decreases exponentially for higher values of ω.

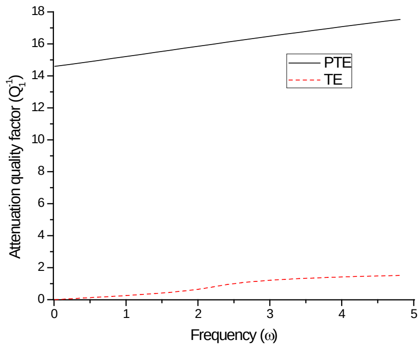

Fig. 4. Variation of attenuation quality factor w.r.t frequency.

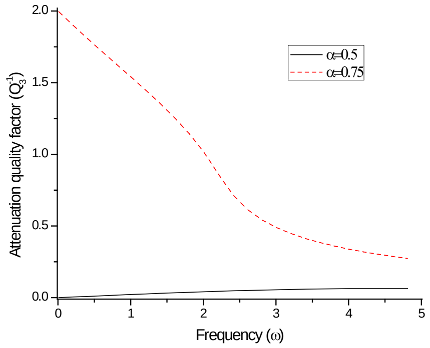

Fig. 5. Variation of attenuation quality factor w.r.t frequency.

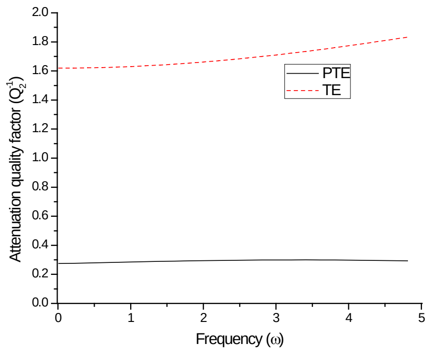

Fig. 6. Variation of attenuation quality factor w.r.t . frequency.

7.3. Specific loss. The variation of specific loss R_1 with frequency ω is shown in Fig. 7. There is a small increase in the value of R_1 for PTE as ω increases whereas for TE, the value of R_1 increases exponentially as ω increases. Due to the effect of porosity the value of R_1 for PTE is more as compared to TE as ω increases.

Fig. 8 shows the variations of R_2 with ω for PTE and TE. The value of R_2 for PTE increases in the range $0 \le \omega \le 3.5$ then starts decreasing as ω increases whereas for TE, the value of R_2 decreases exponentially for higher values of ω. The value of R_2 for PTE is less as compared to TE due to the effect of porosity as ω increases.

Fig. 9 shows the variation of R_3 with frequency ω for PTE. There is small increase in the value of R_3 for PTE when $\alpha = 0.5$ as ω increases whereas for $= 0.75$, the value of R_3 decreases sharply for higher values of ω.

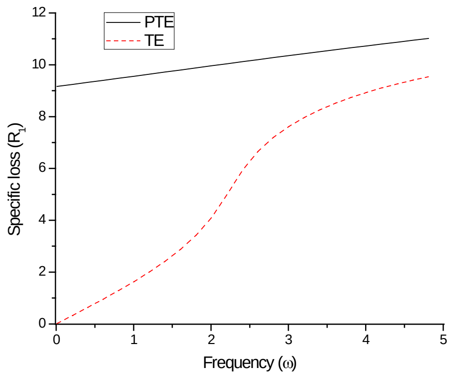

Fig. 7. Variation of specific loss w.r.t frequency.

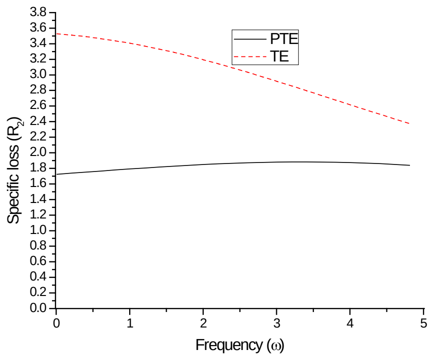

Fig. 8. Variation of specific loss w.r.t frequency.

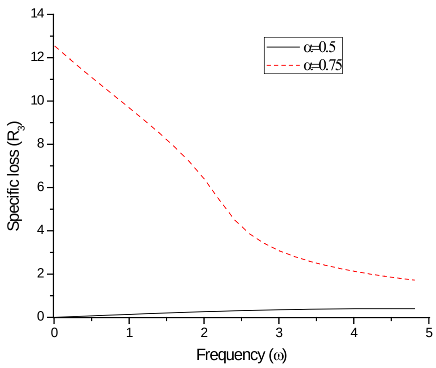

Fig. 9. Variation of specific loss w.r.t frequency.

7.4. Penetration depth. The variation of S_1 with ω for PTE and TE is shown in Fig.10. The value of S_1, for PTE, decreases exponentially whereas for TE, with small initial decrease, the value of S_1 becomes consistent for higher values of ω. Due to the effect of porosity the value of S_1 for PTE is more as compared to TE as ω increases.

Fig. 11 depicts the variation of S_2 with frequency ω.The value of S_2 decreases gradually for all values of ω for PTE and TE but the value of PTE is less as compared to TE in the range $0 \leq \omega \leq 3$ and then show the opposite behaviour for higher values of ω due to the effect of porosity.

Fig. 12 shows the variations of S_3 with ω for PTE. The value of S_3 for PTE decreases gradually for both values of α, but due to the effect of porosity, the value of S_3 is less for $\alpha = 0.5$ as compared to the value of $\alpha = 0.75$.

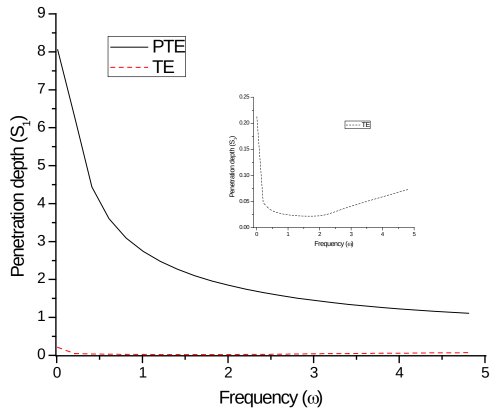

Fig. 10. Variation of penetration depth w.r.t frequency.

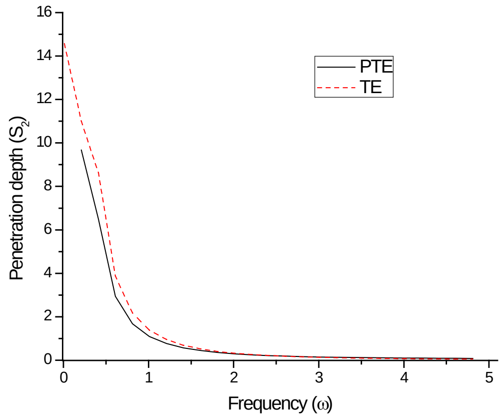

Fig. 11. Variation of penetration depth w.r.t frequency.

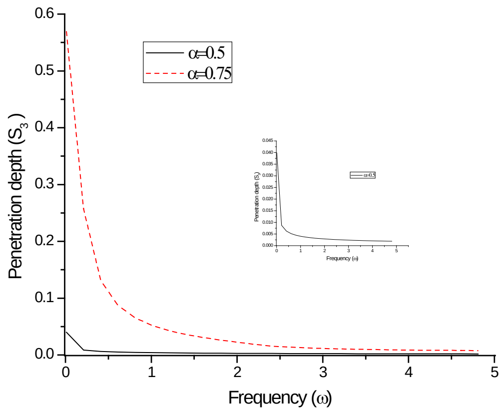

Fig. 12. Variation of penetration depth w.r.t frequency.

8. Conclusion

The present study deals with the propagation of plane wave and fundamental solution in the thermoporoelastic medium. It is found that for two dimensional model, their exist three longitudinal waves, namely P_1-wave, P_2-wave and T-wave in addition to transverse wave. The phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. The fundamental solution of the system of equations in the thermoporoelastic medium in the case of steady oscillations is considered in term of elementary functions.

Due to the presence of porous effect, the phase velocities of P_1-wave for PTE remains more in comparison to TE, whereas the phase velocity of T-wave is less for PTE in comparison to TE for all values of frequency.

Attenuation quality factor, penetration depth and specific loss of P_1-wave for PTE is more in comparison to TE and reverse behaviour is shown in case of T-wave. The value of attenuation quality factor, penetration depth and specific loss of P_2-wave for $\alpha = 0.5$ is less as compared to $\alpha = 0.75$ and opposite behaviour is shown for penetration depth whereas the value of phase velocity of P_2-wave is more for $\alpha = 0.5$ as compared to $\alpha = 0.75$ for initial values of ω and shows the opposite behaviour for higher values of ω.

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