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APPLIED THEORY OF MICROPOLAR ELASTIC THIN PLATES WITH CONSTRAINED ROTATION AND THE FINITE ELEMENT METHOD

S.H. Sargsyan[1*], K.A. Zhamakochyan[2]

[1] Shirak State University, Paruyr Sevak 4 street, Armenia

[2] Shirak State University, Paruyr Sevak 4 street, Armenia

[*]e-mail: s_sargsyan@yahoo.com

Abstract. In the present paper boundary value problems of three-dimensional micropolar theory of elasticity with constrained rotation are considered in thin region of the plate. On the basis of the previously developed hypotheses an applied theory of micropolar thin plates with constrained rotation is constructed, where transverse shear strains are taken into account. The energy balance equation is obtained and the corresponding variation functional is constructed. The finite element method is developed for the boundary problems (statics and natural oscillation) of micropolar plates with constrained rotation. On the basis of the analysis of the corresponding numerical results main properties of the micropolarity of the material are established.

Keywords: micropolar elasticity; constrained rotation; thin plate; applied theory; finite element method.

1. Introduction

Along with the model of the three-dimensional moment (asymmetric, micropolar) elastic medium [1-3] and starting from the article [4], two- and one-dimensional generalized models were also developed, i.e. models of beams, plates and shells. The model, proposed in paper [4], has become one of the most general models of beams, plates and shells and the method of construction (direct method) has been further developed in papers [5-11]. This method initially treats the shell as a material surface, a plate as a material plane, a beam as a material line and establishes the laws of their deformation under the action of generalized internal and external forces and moments. Essentially, this method ignores the spatial structure of the shell (plate) along the thickness, the beam-along the section and does not give constructive methods for reconstructing the volume fields of displacements, rotations, strains, stresses and moment stresses in these thin bodies.

A review of the work of the micropolar model of elastic thin shells and plates is given in papers [12-14].

In papers [15-20], on the basis of the asymptotic properties of the solutions of the three-dimensional micropolar theory of elasticity with free rotation in thin regions, rather general hypotheses are formulated and applied theories of micropolar elastic thin beams, plates and shells with free fields of displacements and rotations are constructed. In papers [21,22], a finite element method for solving boundary value problems of the statics and dynamics of micropolar elastic thin beams and plates with free rotation is developed.

In paper [23], with the help of hypotheses method [15-20] applied theory of micropolar elastic thin shells with constrained rotation [24,25] is constructed.

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In this paper, based on the hypotheses method [15-20,23], an applied theory of micropolar elastic thin plates with constrained rotation is constructed, in which transverse shear deformations are taken into account. The energy balance equation is obtained and a general variation functional is constructed. Further, the finite element method is developed for solving boundary value problems of the applied theory of statics and free vibrations of micropolar elastic thin plates with constrained rotation. Concrete bending problems and natural oscillations of micropolar thin plates are considered, which are solved by the finite element method. Concrete numerical results are obtained, on the basis of analysis of them the effective properties of micropolar materials are approved compared with their classical analogues.

2. Problem statement

An isotropic micropolar elastic plate of constant thickness $2h$ is considered, as a thin three-dimensional body. The axes x_1 and x_2 of the Cartesian coordinate system are directed to the middle plane of the plate, the axis x_3 is perpendicular to this plane.

We start from the basic equations of the three-dimensional micropolar (moment, asymmetric) static theory of elasticity with constrained rotation:

Equilibrium (motion) equations:

$$\begin{gathered}
\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{21}}{\partial x_2}+\frac{\partial\sigma_{31}}{\partial x_3}=0\left(\rho\frac{\partial^2 V_1}{\partial t^2}\right), \frac{\partial\mu_{11}}{\partial x_1}+\frac{\partial\mu_{21}}{\partial x_2}+\frac{\partial\mu_{31}}{\partial x_3}+\sigma_{23}-\sigma_{32}=0\left(J\frac{\partial^2\omega_1}{\partial t^2}\right),\\
\frac{\partial\sigma_{12}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{32}}{\partial x_3}=0\left(\rho\frac{\partial^2 V_2}{\partial t^2}\right), \frac{\partial\mu_{12}}{\partial x_1}+\frac{\partial\mu_{22}}{\partial x_2}+\frac{\partial\mu_{32}}{\partial x_3}+\sigma_{31}-\sigma_{13}=0\left(J\frac{\partial^2\omega_2}{\partial t^2}\right),\\
\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}=0\left(\rho\frac{\partial^2 V_3}{\partial t^2}\right), \frac{\partial\mu_{13}}{\partial x_1}+\frac{\partial\mu_{23}}{\partial x_2}+\frac{\partial\mu_{33}}{\partial x_3}+\sigma_{12}-\sigma_{21}=0\left(J\frac{\partial^2\omega_3}{\partial t^2}\right);
\end{gathered} \tag{1}$$

Physical relations of elasticity:

$$\begin{gathered}
\varepsilon_{11}=\frac{1}{E}\left(\sigma_{11}-\nu(\sigma_{22}+\sigma_{33})\right),\ \varepsilon_{13}=\varepsilon_{31}=\frac{\sigma_{13}+\sigma_{31}}{2\mu},\\
\varepsilon_{22}=\frac{1}{E}\left(\sigma_{22}-\nu(\sigma_{11}+\sigma_{33})\right),\ \varepsilon_{23}=\varepsilon_{32}=\frac{\sigma_{23}+\sigma_{32}}{2\mu},\\
\varepsilon_{33}=\frac{1}{E}\left(\sigma_{33}-\nu(\sigma_{11}+\sigma_{22})\right),\ \varepsilon_{12}=\varepsilon_{21}=\frac{\sigma_{12}+\sigma_{21}}{2\mu},
\end{gathered} \tag{2}$$

$$\begin{gathered}
\chi_{11}=\frac{\mu_{11}}{2\gamma},\quad \chi_{22}=\frac{\mu_{22}}{2\gamma},\quad \chi_{33}=\frac{\mu_{33}}{2\gamma},\\
\chi_{21}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{21}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{12},\quad \chi_{12}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{12}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{21},\\
\chi_{31}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{31}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{13},\quad \chi_{13}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{13}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{31},\\
\chi_{32}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{32}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{23},\quad \chi_{23}=\frac{\gamma+\varepsilon}{4\gamma\varepsilon}\mu_{23}-\frac{\gamma-\varepsilon}{4\gamma\varepsilon}\mu_{32};
\end{gathered} \tag{3}$$

Geometrical relations:

$$\begin{gathered}
\varepsilon_{11}=\frac{\partial V_1}{\partial x_1},\quad \varepsilon_{22}=\frac{\partial V_2}{\partial x_2},\quad \varepsilon_{33}=\frac{\partial V_3}{\partial x_3},\\
\varepsilon_{12}=\frac{\partial V_2}{\partial x_1}+\frac{\partial V_1}{\partial x_2},\quad \varepsilon_{13}=\frac{\partial V_1}{\partial x_3}+\frac{\partial V_3}{\partial x_1},\quad \varepsilon_{23}=\frac{\partial V_3}{\partial x_2}+\frac{\partial V_2}{\partial x_3},
\end{gathered} \tag{4}$$

$$\chi_{11}=\frac{\partial\omega_1}{\partial x_1},\quad \chi_{22}=\frac{\partial\omega_2}{\partial x_2},\ \chi_{33}=\frac{\partial\omega_3}{\partial x_3},\ \chi_{12}=\frac{\partial\omega_2}{\partial x_1},\ \chi_{21}=\frac{\partial\omega_1}{\partial x_2},$$
$$\chi_{13}=\frac{\partial\omega_3}{\partial x_1},\quad \chi_{31}=\frac{\partial\omega_1}{\partial x_3},\quad \chi_{23}=\frac{\partial\omega_3}{\partial x_2},\ \chi_{32}=\frac{\partial\omega_2}{\partial x_3},\ \vec{\omega}=\frac{1}{2}rot\vec{V}, \tag{5}$$

$$\omega_1=\frac{1}{2}\left(\frac{\partial V_3}{\partial x_2}-\frac{\partial V_2}{\partial x_3}\right),\omega_2=\frac{1}{2}\left(\frac{\partial V_1}{\partial x_3}-\frac{\partial V_3}{\partial x_1}\right),\ \omega_3=\frac{1}{2}\left(\frac{\partial V_2}{\partial x_1}-\frac{\partial V_1}{\partial x_2}\right). \tag{6}$$

Here (V_1,V_2,V_3) – components of the displacement vector; $(\omega_1,\omega_2,\omega_3)$ – components of the rotation vector; $\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{12},\varepsilon_{13},\varepsilon_{23}$ – components of the deformation tensor; $\chi_{11},\chi_{22},\chi_{33},\chi_{12},\chi_{21},\chi_{13},\chi_{23},\chi_{31},\chi_{32}$ – components of the bending-torsions tensor; $\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{12},\sigma_{21},\sigma_{13},\sigma_{31},\sigma_{23},\sigma_{32}$ – components of the stresses tensor; $\mu_{11},\mu_{22},\mu_{33},\mu_{12},\mu_{21},\mu_{13},\mu_{31},\mu_{23},\mu_{32}$ – components of the moment stresses tensor; $E,\nu,\mu=\frac{E}{2(1+\nu)},\gamma,\varepsilon$ – elastic coefficients of an isotropic micropolar material; ρ – material density; J – measure of inertia of the material during rotation.

On the face planes $(x_3=\pm h)$ it is assumed, that values of stresses $\sigma_{31},\sigma_{32},\sigma_{33},\mu_{31},\mu_{32}$ are given, and on the lateral surface, in general, can be given: either stresses and moment stresses, or displacements and rotations, or mixed conditions.

In the case of the dynamic problem, it is assumed that the initial conditions for the quantities $V_1,V_2,V_3,\frac{\partial V_1}{\partial t},\frac{\partial V_2}{\partial t},\frac{\partial V_3}{\partial t}$ are given when $t=0$.

From equations (1)-(5) we can pass to the energy balance equation for the three-dimensional micropolar theory of elasticity with constrained rotation:

$$2W=A, \tag{7}$$

where W is the potential energy of deformation:

$$W=\iint_{(S)}\int_{-h}^{h}W_0dx_1dx_2dx_3, \tag{8}$$

A is work of external surface stresses and moment stresses; W_0 is the density of potential energy of deformation:

$$\begin{aligned}W_0=\frac{1}{2}\Big[&\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11}^2+\varepsilon_{22}^2+\varepsilon_{33}^2\right)+\frac{2E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{11}\varepsilon_{22}+\varepsilon_{11}\varepsilon_{33}+\varepsilon_{22}\varepsilon_{33}\right)+\\&+\mu\left(\varepsilon_{12}^2+\varepsilon_{13}^2+\varepsilon_{23}^2\right)+2\gamma\left(\chi_{11}^2+\chi_{22}^2+\chi_{33}^2\right)+(\gamma+\varepsilon)\chi_{21}^2+(\gamma+\varepsilon)\chi_{12}^2+2(\gamma-\varepsilon)\chi_{12}\chi_{21}+\\&+(\gamma+\varepsilon)\chi_{31}^2+(\gamma+\varepsilon)\chi_{13}^2+2(\gamma-\varepsilon)\chi_{31}\chi_{13}+(\gamma+\varepsilon)\chi_{32}^2+(\gamma+\varepsilon)\chi_{23}^2+2(\gamma-\varepsilon)\chi_{32}\chi_{23}\Big]\end{aligned} \tag{9}$$

Considering that the plate is thin, our aim is to construct an applied theory of bending of micropolar plate and to develop finite element method for solving boundary problems of this applied theory. It should be noted that in the problem of plate bending V_3,ω_1,ω_2 are even functions by x_3, and V_1,V_2,ω_3 are odd functions by x_3.

3. Basic hypotheses. Displacements and rotations, deformations, bending-torsion, stresses and moment stresses

Following rather general hypotheses should be accepted [23]:

1) Kinematic hypothesis of Timoshenko: in the process of deformation, initially straight and normal to the middle plane of the fiber element freely rotates in space at some angle as a

whole rigid body, without changing its length and without remaining perpendicular to the deformed middle plane.

The accepted hypothesis mathematically can be written as follows: tangential displacements are distributed linearly along the thickness of the plate, and the normal displacement does not depend on the transverse coordinate x_3, i.e.

$$V_i = x_3\psi_i(x_1,x_2,t),\ \ i=1,2;\quad V_3 = w(x_1,x_2,t). \tag{10}$$

In this case, for the angles of rotation of the points of the body, we obtain:

$$\omega_1 = \Omega_1(x_1,x_2,t) = \frac{1}{2}\left(\frac{\partial w}{\partial x_2} - \psi_2\right),\qquad \omega_2 = \Omega_2(x_1,x_2,t) = \frac{1}{2}\left(\psi_1 - \frac{\partial w}{\partial x_1}\right),$$
$$\omega_3 = x_3\iota(x_1,x_2,t),\qquad \iota = \frac{1}{2}\left(\frac{\partial \psi_2}{\partial x_1} - \frac{\partial \psi_1}{\partial x_2}\right). \tag{11}$$

2) The assumption that the plate is thin (assumption of the plate thickness $1+\dfrac{2h}{a}\approx 1$, where $2h$ is the plate thickness, $a-$ linear minimum plate size in plan);

3) Stress σ_{33} in the generalized Hooke's law for $\varepsilon_{11},\varepsilon_{22}$ can be neglected in relation to the stresses σ_{11},σ_{22};

4) In the expression for χ_{i3}(from (3)) the moment stresses μ_{3i} can be neglected in relation to $\mu_{i3}(i=1,2)$.

5) For determination the deformations, bending-torsion, stresses and moment stresses, for stresses $\sigma_{3i}\ \ (i=1,2)$ we take:

$$\sigma_{3i} = \overset{0}{\sigma}_{3i}(x_1,x_2,t)\ \ (i=1,2). \tag{12}$$

After determination of the mentioned quantities, values of $\sigma_{3i}\ (i=1,2)$ finally can be obtained as the sum of the values (12) and the result of integration the first and second equilibrium equations (motion) of (1) requiring the condition for each integral that the averaged quantities along the plate thickness are equal to zero.

We begin our study of the problem of bending of micropolar elastic plate with the determination of the components of the strain tensors and torsion-twists, on the basis of the hypotheses adopted above.

Having formulas for the displacements (10) and the rotations (11), using expressions (4) and (5) for the components of the strain tensors and torsion-twisting, it is obtained:

$$\varepsilon_{11}=x_3K_{11},\varepsilon_{22}=x_3K_{22},\varepsilon_{12}=x_3K_{12},\varepsilon_{13}=\Gamma_{13},\ \varepsilon_{23}=\Gamma_{23},\varepsilon_{33}=0,\chi_{11}=k_{11},$$
$$\chi_{22}=k_{22},\chi_{33}=k_{33},\chi_{12}=k_{12},\chi_{21}=k_{21},\chi_{13}=x_3l_{13},\chi_{23}=x_3l_{23},\chi_{31}=0,\chi_{32}=0, \tag{13}$$

where the following notations are obtained:

$$K_{11}=\frac{\partial\psi_1}{\partial x_1},\ K_{22}=\frac{\partial\psi_2}{\partial x_2},\ K_{12}=K_{21}=\frac{\partial\psi_2}{\partial x_1}+\frac{\partial\psi_1}{\partial x_2},\ \Gamma_{13}=\psi_1+\frac{\partial w}{\partial x_1},\ \Gamma_{23}=\psi_2+\frac{\partial w}{\partial x_2},$$
$$k_{11}=\frac{\partial\Omega_1}{\partial x_1},k_{22}=\frac{\partial\Omega_2}{\partial x_2},k_{33}=\iota,k_{12}=\frac{\partial\Omega_2}{\partial x_1},\ k_{21}=\frac{\partial\Omega_1}{\partial x_2},l_{13}=\frac{\partial\iota}{\partial x_1},\ l_{23}=\frac{\partial\iota}{\partial x_2}. \tag{14}$$

Now we turn to the study of the stresses and moment stresses in the plate.

Using hypotheses 3), 4) and 5), on the basis of expressions of the generalized Hooke's law (2), (3) and equilibrium equations, for stresses and moment stresses we obtain:

$$\sigma_{11}=x_3\frac{E}{1-\nu^2}(K_{11}+\nu K_{22}),\ \sigma_{22}=x_3\frac{E}{1-\nu^2}(K_{22}+\nu K_{11}), \tag{15}$$

$$\sigma_{12}+\sigma_{21}=2\mu x_3 K_{12},\ \sigma_{13}+\overset{0}{\sigma}_{31}=2\mu\Gamma_{13},\ \sigma_{23}+\overset{0}{\sigma}_{32}=2\mu\Gamma_{23}, \tag{16}$$

$$\mu_{11}=2\gamma k_{11},\ \ \mu_{22}=2\gamma k_{22},\mu_{33}=2\gamma k_{33},\ \ \mu_{21}=(\gamma+\varepsilon)k_{21}+(\gamma-\varepsilon)k_{12},$$

$$\mu_{12}=(\gamma+\varepsilon)k_{12}+(\gamma-\varepsilon)k_{21},\ \ \mu_{13}=x_3\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{13},\ \mu_{23}=x_3\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{23}, \tag{17}$$

$$\sigma_{33}=-x_3\left[\left(\frac{\partial\sigma_{13}}{\partial x_1}+\frac{\partial\sigma_{23}}{\partial x_2}\right)+\left(-\rho\frac{\partial^2 w}{\partial t^2}\right)\right], \tag{18}$$

$$\sigma_{31}=\overset{0}{\sigma}_{31}+\left(\frac{h^2}{6}-\frac{x_3^2}{2}\right)\left(\frac{\partial\overset{1}{\sigma}_{11}}{\partial x_1}+\frac{\partial\overset{1}{\sigma}_{21}}{\partial x_2}+\left(-\rho\frac{\partial^2\psi_1}{\partial t^2}\right)\right)\quad(1\leftrightarrow 2), \tag{19}$$

$$\begin{gathered}\mu_{31}=-x_3\left(\frac{\partial\mu_{11}}{\partial x_1}+\frac{\partial\mu_{21}}{\partial x_2}\right)+x_3\left(\overset{0}{\sigma}_{32}-\sigma_{23}\right)-\left(-J\frac{\partial^2\omega_1}{\partial t^2}\right),\\ \mu_{32}=-x_3\left(\frac{\partial\mu_{12}}{\partial x_1}+\frac{\partial\mu_{22}}{\partial x_2}\right)+x_3\left(\sigma_{13}-\overset{0}{\sigma}_{31}\right)-\left(-J\frac{\partial^2\omega_2}{\partial t^2}\right).\end{gathered} \tag{20}$$

It should be noted that inertial terms should not be taken into account in the formulas (18) - (20) in the case of static problem (this remark must be taken into account in the future), and they should be taken into account in the case of dynamic problems (according to D'Alembert's principle).

It is easy to see, that using expressions for stresses $\sigma_{31},\sigma_{32},\sigma_{33},\mu_{31},\mu_{32}$, we can satisfy the boundary conditions on the planes $x_3=\pm h$ (in bending):

$$\sigma_{33}\big|_{x_3=\pm h}=\pm\frac{1}{2}q_3,\sigma_{31}\big|_{x_3=\pm h}=\frac{1}{2}q_1\ \ (1\to 2),\ \mu_{31}\big|_{x_3=\pm h}=\pm\frac{1}{2}m_1\ \ (1\to 2). \tag{21}$$

4. Applied theory of bending deformation of micropolar plates with constrained rotation

For the construction of the applied theory of the micropolar plate, we accept averaged integral characteristics over its thickness: forces, moments and hypermoments:

$$\begin{gathered}N_{i3}=\int_{-h}^{h}\sigma_{i3}dx_3,\ \ N_{3i}=\int_{-h}^{h}\sigma_{3i}dx_3,\ \ M_{ii}=\int_{-h}^{h}x_3\sigma_{ii}dx_3,\ \ H_{ij}=\int_{-h}^{h}x_3\sigma_{ij}dx_3,\\ L_{ii}=\int_{-h}^{h}\mu_{ii}dx_3,\quad L_{ij}=\int_{-h}^{h}\mu_{ij}dx_3,\quad \Lambda_{i3}=\int_{-h}^{h}x_3\mu_{i3}dx_3,\ \ i=1,2.\end{gathered} \tag{22}$$

On the basis of formulas for stresses $(\sigma_{31},\sigma_{32},\sigma_{33},\mu_{31},\mu_{32})$, satisfying the boundary conditions (21), we obtain the equilibrium (motion) equations of the applied theory of micropolar elasticity with constrained rotation for the thin plate:

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}=\left(2\rho h\frac{\partial^2 w}{\partial t^2}\right)-q_3,$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial H_{21}}{\partial x_2}-N_{31}=\left(\frac{2}{3}\rho h^3\frac{\partial^2\psi_1}{\partial t^2}\right)-q_1 h,$$

$$\frac{\partial H_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{32}=\left(\frac{2}{3}\rho h^3\frac{\partial^2\psi_2}{\partial t^2}\right)-q_2h,$$
$$\frac{\partial L_{11}}{\partial x_1}+\frac{\partial L_{21}}{\partial x_2}+N_{23}-N_{32}=\left(2Jh\frac{\partial^2\Omega_1}{\partial t^2}\right)-m_1, \tag{23}$$
$$\frac{\partial L_{12}}{\partial x_1}+\frac{\partial L_{22}}{\partial x_2}+N_{31}-N_{13}=\left(2Jh\frac{\partial^2\Omega_2}{\partial t^2}\right)-m_2,$$
$$\left(\frac{\partial \Lambda_{13}}{\partial x_1}+\frac{\partial \Lambda_{23}}{\partial x_2}\right)-H_{21}+H_{12}=0\ \left(2Jh\frac{\partial^2\Omega_3}{\partial t^2}\right).$$

The elasticity relations of the applied moment theory of the bending of thin plates with constrained rotation will be obtained on the basis of the expressions for the Hooke law in the form (15)-(17):

$$M_{11}=\frac{2Eh^3}{3(1-\nu^2)}(K_{11}+\nu K_{22}),\ M_{22}=\frac{2Eh^3}{3(1-\nu^2)}(K_{22}+\nu K_{11}),$$
$$H_{12}+H_{21}=\frac{4\mu h^3}{3}K_{12},$$
$$N_{13}+N_{31}=4h\mu\Gamma_{13}, \tag{24}$$
$$N_{23}+N_{32}=4h\mu\Gamma_{23},$$
$$L_{11}=4h\gamma k_{11},\ L_{22}=4h\gamma k_{22},$$
$$L_{12}=2h[(\gamma+\varepsilon)k_{12}+(\gamma-\varepsilon)k_{21}],\ L_{21}=2h[(\gamma+\varepsilon)k_{21}+(\gamma-\varepsilon)k_{12}],$$
$$\Lambda_{13}=\frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{13}\right],\ \Lambda_{23}=\frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{23}\right].$$

Geometric relations (14) should be added to the equations (23) and (24).

The equilibrium (motion) equations (23), the elasticity relations (24) and the geometric relations (14) determine the model of micropolar elastic thin plate with constrained rotation. Boundary conditions (and initial conditions in the case of dynamics) should be added to this basic system of equations. What boundary conditions can be set on the contour of the middle plane of the plate? This will be seen below, when we obtain a formula for calculating the work of forces and moments on the specified contour.

The energy balance equation for the applied theory of micropolar elastic thin plates with constrained rotation can be obtained if in the corresponding equation of the three-dimensional theory (7) we take into account the formulas for displacements (10), rotations (11), deformations and bending torsions (13):

$$2\tilde{W}=\tilde{A}, \tag{25}$$

where $\tilde{W}$ is the potential energy of deformation, $\tilde{A}$ is the work of external forces and moments.

Potential energy of deformation is determined by the relation:

$$\tilde{W}=\iint_{(S)}\tilde{W}_0 ds, \tag{26}$$

where (S) is area occupied by the median plane of the plate, $\tilde{W}_0$ is the density of the potential deformation energy of a micropolar plate under bending, which is expressed by the following formula:

$$\tilde{W}_0 = \frac{1}{2}\left[\frac{2Eh^3}{3\left(1-\nu^2\right)}\left(K_{11}^2 + K_{22}^2 + 2\nu K_{11}K_{22}\right) + \frac{2\mu h^3}{3}K_{12}^2 + 2\mu h(\Gamma_{13}^2 + \Gamma_{23}^2) + \right.$$
$$4\gamma h(k_{11}^2 + k_{22}^2 + k_{33}^2) + 2h(\gamma+\varepsilon)k_{21}^2 + 2h(\gamma+\varepsilon)k_{12}^2 + \tag{27}$$
$$\left. 4h(\gamma-\varepsilon)k_{12}k_{21} + \frac{2h^3}{3}\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{13}^2 + \frac{2h^3}{3}\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{23}^2\right],$$

and the work of external distributed forces and moments is determined by the formula:

$$\tilde{A} = \iint_{(S)}\left[\left(hp_1\psi_1 + hp_2\psi_2 + p_3 w\right) + \left(m_1\Omega_1 + m_2\Omega_2 + \iota h m_3\right)\right]ds +$$
$$+ \int_{l_1}\left[\left(M_{11}\psi_1 + M_{12}\psi_2 + N_{13}w\right) + \left(L_{11}\Omega_1 + L_{12}\Omega_2 + \Lambda_{13}\iota\right)\right]dl + \tag{28}$$
$$+ \int_{l_2}\left[\left(M_{21}\psi_1 + M_{22}\psi_2 + N_{23}w\right) + \left(L_{21}\Omega_1 + L_{22}\Omega_2 + \Lambda_{23}\iota\right)\right]dl.$$

The variation principle of Lagrange for the applied theory of micropolar elastic thin plates with constrained rotation can be written as follows:

$$\delta\tilde{\Pi} = 0, \text{ where } \tilde{\Pi} = \tilde{W} - \tilde{A}. \tag{29}$$

5. The stiffness matrix of finite element of a micropolar elastic plate with constrained rotation

The obtaining of the equations of the finite element method in displacements and rotations is based on the Lagrange variation principle (29).

We consider rectangular finite element. The main kinematic parameters in the problem of the bending of a micropolar plate with constrained rotation are the displacement of the point of the median plane w; the angles of rotation of a linear element normal to the median plane in the planes x_1x_3 and x_2x_3 - ψ_1, ψ_2. We will approximate the distribution of the accepted basic kinematic variables along the element of the rectangle of the middle plane of the element plate by polynomials. For the deflection $w(x_1, x_2)$ it is necessary to put:

$$w(x_1,x_2) = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_1^2 + \alpha_5 x_2^2 + \alpha_6 x_1 x_2 + \alpha_7 x_1^2 x_2 + \alpha_8 x_1 x_2^2 + \alpha_9 x_1^3 +$$
$$\alpha_{10}x_2^3 + \alpha_{11}x_1^3 x_2 + \alpha_{12}x_1 x_2^3, \tag{30}$$

for ψ_1, ψ_2 we have completely analogous expressions as for w, but with other constant coefficients the designation of which begins with α_{13} in ψ_1 and ends with α_{36} in the expression for ψ_2.

Vector of nodal kinematic parameters of a finite four-node element $\boldsymbol{\delta_e}$ in general form is represented as $\boldsymbol{\delta_e} = \{\boldsymbol{\delta_i}\}^T$, where $\boldsymbol{\delta_i}$ $(i = 1,2,3,4)$ is the vector of unknown i nodes; “T” is the symbol of the transpose operation and

$$\boldsymbol{\delta_i} = \left\{\boldsymbol{w_i}, \left(\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{x_1}}\right)_i, \left(\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{x_2}}\right)_i, \boldsymbol{\psi_{1i}}, \left(\frac{\partial \boldsymbol{\psi_1}}{\partial \boldsymbol{x_1}}\right)_i, \left(\frac{\partial \boldsymbol{\psi_1}}{\partial \boldsymbol{x_2}}\right)_i, \boldsymbol{\psi_{2i}}, \left(\frac{\partial \boldsymbol{\psi_2}}{\partial \boldsymbol{x_1}}\right)_i, \left(\frac{\partial \boldsymbol{\psi_2}}{\partial \boldsymbol{x_2}}\right)_i\right\}^T.$$

Further transformations can be performed in accordance with the algorithm for deriving the finite element stiffness matrix (FEM).

Realizing the variation algorithm of the FEM theory, we get the system of equilibrium equations for the finite element of the plate:

$$\boldsymbol{K_e}\boldsymbol{\delta_e} = \boldsymbol{F_e}, \tag{31}$$

where $\boldsymbol{K_e}$ is stiffness matrix of finite element of a micropolar plate of dimension 36×36; $\boldsymbol{F_e}$ is vector of equivalent nodal forces and moments; $\boldsymbol{\delta_e}$ - vector of nodal kinematic parameters of the finite element.

Equilibrium equations of the whole plate in the matrix representation can be represented as follows:

$$\boldsymbol{K_\Sigma \Delta} = \boldsymbol{F_\Sigma}, \tag{32}$$

where $\boldsymbol{K_\Sigma}$ is global stiffness matrix of micropolar plate; $\boldsymbol{F_\Sigma}$ - global vector of equivalent nodal forces and moments of the whole plate; $\boldsymbol{\Delta}$ - global vector of nodal unknowns:

$$\boldsymbol{\Delta} = \{\boldsymbol{\delta_1}, \boldsymbol{\delta_2}, ..., \boldsymbol{\delta_i}, ..., \boldsymbol{\delta_N}\}^T,$$

where N is the total number of nodes in the system.

From the solution of the global system of equilibrium equations obtained with allowance for the boundary conditions, we determine the distributions of the node parameters along the middle plane of the plate.

6. Model calculation of micropolar elastic plates with constrained rotation

As an example, we'll consider a micropolar square plate, which is simple supported on all four sides and bends by a normal load $p_3 = const$, (in this case $p_1 = 0, p_2 = 0, p_3 \neq 0, m_1 = 0, m_2 = 0, m_3 = 0$). For the simple supported boundary conditions we have:

$$w = 0, \quad M_{11} = 0, \quad L_{12} = 0, \quad \Omega_1 = 0, \quad \psi_2 = 0, \quad \Lambda_{13} = 0, \text{ on } x_1 = 0; a$$

$$w = 0, \quad M_{22} = 0, \quad L_{21} = 0, \quad \Omega_2 = 0, \quad \psi_1 = 0, \quad \Lambda_{23} = 0, \text{ on } x_2 = 0; a. \tag{33}$$

After construction the stiffness matrix $\boldsymbol{K_\Sigma}$ and the vector of equivalent nodal forces and moments $\boldsymbol{F_\Sigma}$, taking into account the boundary conditions (33) we formulate a system of linear algebraic equations corresponding to the problem under consideration, for different numbers of smashing the plate into finite elements.

It is very important to know what accuracy can be achieved in the problem under consideration when the size of the elements decreases. We note, that we can estimate the error in this model problem by comparing the numerical solution with the known exact solution (which can be obtained in the form of a trigonometric Fourier series [15]).

The calculations were carried out with the following data:

$$p_3 = 0.5 \cdot 10^2 \frac{N}{m^2}, \ a = b = 0.1\,m, h = 0.1 \cdot 10^{-2}\,m,$$

$$E = 3{,}06 \cdot 10^8 \frac{N}{m^2}, \mu = 1.093 \cdot 10^8 \frac{N}{m^2},$$

$$\nu = 0.399, \gamma = 24N, \varepsilon = 24N.$$

Table 1. The maximum deflections of the micropolar and classical plate.

	Micropolar plate			Classical plate			$\frac{w_{max}^{class} - w_{max}^{mic}}{w_{max}^{class}}$
	Exact value	4 element	16 element	Exact value	4 element	16 element	
w_{max} m	$0.6 \cdot 10^{-4}$	$0.51 \cdot 10^{-4}$	$0.58 \cdot 10^{-4}$	$0.8 \cdot 10^{-4}$	$0.72 \cdot 10^{-4}$	$0.8 \cdot 10^{-4}$	0.275

As can be seen from the given values of Table 1, the micropolarity of the material of the plate increases the stiffness of the plate compared with the classical case of the material.

7. Dynamic problem of a micropolar elastic plate with constrained rotation

The general form of the functional of the total mechanical energy (the sum of the potential energy of deformation and kinetic energy) of a micropolar-elastic plate is expressed as follows:

$$\tilde{U} = \iint_{(S)} \left(\tilde{W}_0 + \rho h \frac{\partial^2 w}{\partial t^2} \cdot w + \frac{\rho h^3}{3} \frac{\partial^2 \psi_1}{\partial t^2} \cdot \psi_1 + \frac{\rho h^3}{3} \frac{\partial^2 \psi_2}{\partial t^2} \cdot \psi_2 + \right.$$
$$\left. + Jh \frac{\partial^2 \Omega_1}{\partial t^2} \cdot \Omega_1 + Jh \frac{\partial^2 \Omega_2}{\partial t^2} \cdot \Omega_2 + \frac{Jh^3}{3} \frac{\partial^2 \iota}{\partial t^2} \cdot \iota \right) dS. \tag{34}$$

With free oscillations, we represent the main kinematic functions of the problem in this way:

$$w(x_1, x_2) = \left(\alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_1^2 + \alpha_5 x_2^2 + \alpha_6 x_1 x_2 + \alpha_7 x_1^2 x_2 + \alpha_8 x_1 x_2^2 + \alpha_9 x_1^3 + \right.$$
$$\left. \alpha_{10} x_2^3 + \alpha_{11} x_1^3 x_2 + \alpha_{12} x_1 x_2^3 \right) \sin \omega t, \tag{35}$$

where ω is frequency of natural oscillation. For ψ_1, ψ_2 we have completely analogous expressions as for w, but with other constant coefficients, starting from α_{13} (for ψ_1) and ending with α_{36} (for ψ_2).

Substituting (35) into (34), minimizing the functional (34) we obtain the minimum of the function of thirty-six independent variables:

$$\frac{\partial U}{\partial \delta_{36}} = 0 \quad (k = 1,2,3,...,36).$$

Calculating the corresponding partial derivatives, we obtain the following matrix equation:

$$\left(K - \omega^2 M \right) \cdot \{ \delta \} = 0, \tag{36}$$

where K is stiffness matrix of finite element, M is matrix of masses of a finite element.

We present the results of numerical calculations, the data of the problem as follows: $E = 2 \cdot 10^{11} \frac{N}{m^2}$, $\mu = 7 \cdot 10^{10} \frac{N}{m^2}$, $\nu = 0.3$, $\gamma = 24N$, $\varepsilon = 24N$; density and measure of inertia when the material rotates: $\rho = 7700 \frac{kg}{m^3}$, $J = 5.3 \cdot 10^{-6} \frac{kg}{m}$.

Table 2. The lowest frequency of free oscillation ω.

		Micropolar plate sec^{-1}			Classical plate sec^{-1}		
$a = b$ (m)	h (m)	Exact value	4 element	16 element	Exact value	4 element	16 element
$2 \cdot 10^{-3}$	$5 \cdot 10^{-5}$	$0{,}84 \cdot 10^6$	$0{,}74 \cdot 10^6$	$0{,}79 \cdot 10^6$	$0{,}74 \cdot 10^6$	$0{,}65 \cdot 10^6$	$0{,}69 \cdot 10^6$
10^{-7}	$0{,}5 \cdot 10^{-9}$	$1{,}34 \cdot 10^{11}$	$0{,}94 \cdot 10^{11}$	$0{,}98 \cdot 10^{11}$	$2{,}99 \cdot 10^9$	$2{,}73 \cdot 10^9$	$2{,}99 \cdot 10^9$
10^{-8}	$0{,}5 \cdot 10^{-10}$	$1{,}34 \cdot 10^{12}$	$0{,}94 \cdot 10^{12}$	$0{,}98 \cdot 10^{12}$	$2{,}99 \cdot 10^{10}$	$2{,}73 \cdot 10^{10}$	$2{,}99 \cdot 10^{10}$

As can be seen from the tables above, the micropolarity of the material increases the frequency of oscillations, and in the nanosized region, the frequencies are in the terahertz range.

8. Conclusion

In the present paper applied theory of micropolar elastic thin plates with constrained rotation is constructed, taking into account transverse shear deformations. The energy balance equation is obtained and the functional of the general variation principle is constructed. The finite element method is developed for solving boundary value problems of the applied theory of micropolar elastic thin plates with constrained rotation. The finite element method is used to solve problems of static equilibrium and free oscillations of micropolar elastic rectangular plates. On the basis of numerical analysis, some effective properties of calculation of the micropolarity of the material are established.

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References

[1] E. et F. Cosserat, *Theorie des deformables* (Herman et Fils, Paris, 1909).
[2] E.L. Aero, E.V. Kuvshinski // *Solid State Physics* **2** (1960) 1399.
[3] V.A. Palmov // *Applied Mathematics and Mechanics* **28** (1964) 1117.
[4] J.L. Ericksen, C. Truesdell // *Arch. Ration. Mech. Anal.* **1** (1957) 295.
[5] A.E. Green, P.M. Naghdi, W.L. Wainwright // *Arch. Ration. Mech. Anal.* **20** (1965) 287.
[6] P.A. Zhilin, Dynamics and strength of machines, In: *Proceedings of the Leningrad Polytechnic Institute* **386** (1982) 29.
[7] P.A. Zhilin, Applied mechanics. Basics of the theory of shells, In: *Publishing of Polytechnic University* (2006) 167.
[8] V.V. Eliseev, Mechanics of Solids, In: *Proceedings of the USSR Academy of Sciences* (1976) 163.
[9] L.I. Shkutin, *Mechanics of deformations of flexible bodies* (Nauka, Novosibirsk, 1988).
[10] M.B. Rubin, *Cosserat theories: shells, rods and points* (Kluwer Acad. Publ., Dordrech, 2000).
[11] V.A. Eremeev, L.M. Zubov, *Mechanics of Elastic Shells* (Nauka, Moscow, 2008).
[12] J. Altenbach, H. Altenbach, V.A. Eremeyev // *Arch. Mech.* **80** (2010) 73.
[13] H. Altenbach, V.A. Eremeyev // *Z Angew. Math. Mech. (ZAMM)* **89** (2009) 242.
[14] V. Eremeyev, H. Altenbach, Basics of Mechanics of Micropolar Shells, In: *Shell-like Structures. CISM International Centre for Mechanical Sciences (Courses and Lectures)*, ed. by H. Altenbach, V. Eremeyev (Springer, Cham, 2017) Volume 572.
[15] S.H. Sargsyan // *Journal of Materials Science and Engineering* **2** (2012) 98.
[16] S.H. Sargsyan // *Journal of Applied Mechanics and Technical Physics* **53** (2012) 275.
[17] S.H. Sargsyan // *Journal of Physical Mezomechanics* **15** (2012) 69.
[18] S.H. Sargsyan // *Doklady Physics* **56** (2011) 39.
[19] S.H. Sargsyan // *International Journal of Mechanics* **8** (2014) 93.
[20] S.H. Sargsyan // *Advanced Structured Materials. Shell-like Structures. Non-classical Theories and Applications* (Springer-Verlag Berlin Heidelberg) **15** (2011) 91.
[21] K.A. Zhamakochyan, S.H. Sargsyan // *Computational continuum mechanics* **9** (2016) 375.
[22] K.A. Zhamakochyan, S.H. Sargsyan // *Proceedings of NAS of Armenia. Mechanics* **70** (2017) 22.
[23] S.H. Sargsyan, *Elasticity and inelasticity. Proceedings of the International Scientific Symposium on the Problems of Mechanics of Deformable Bodies dedicated to the 100th anniversary of the birth of A.A. Ilyushin* (Moscow, 2011) p. 231.
[24] V.T. Koiter // *Mechanics. Period. sat. trans. foreign articles* **3** (1965) 89.
[25] R.D. Mindlin // *Mechanics. Period. sat. trans. foreign articles* **4** (1964) 115.