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APPROXIMATED ANALYTICAL SOLUTION OF CONTACT PROBLEM ON INDENTATION OF ELASTIC HALF-SPACE WITH COATING REINFORCED WITH INHOMOGENEOUS INTERLAYER

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Abstract. Axisymmetric contact problem on indentation of linearly elastic half-space with coating reinforced with inhomogeneous in depth interlayer is considered. Elastic moduli of the interlayer vary with depth according to arbitrary continuously differentiable independent functions. Construction of the compliance functions is reduced to the solution of Cauchy problems for a system of ordinary differential equations with variable coefficients. Contact problem is reduced to the solution of an integral equation which is solved using the bilateral asymptotic method. Approximated analytical expressions for contact stresses and indentation force are provided. Stresses and displacements inside the half-space and coating are obtained in the form of quadratures.

Keywords: contact; indentation; two-layered coating; functionally graded materials; analytical methods.

1. Introduction

Most of the results for dynamic [1,2] and static [3-7] problems for elastic solids with coatings are obtained for a case when whole coating is functionally graded (FG) [3-5] or piecewise homogeneous [1,2,6,7]. Practically, coatings often have layered structure with one or more layers having FG structure. For instance, to reduce wear and increase operating time of rails and wheel sets specially designed coatings are used [8]. Such coatings consists of a load-bearing skeleton (metal coating with complex nonmonotonic variation of elastic moduli) applied to the rail by the method of electric-arc deposition and an antifriction polymer composite (soft homogeneous layer). Layered structures can also be formed as a result of oxidation of the coating.

The paper addresses to the modeling of contact of a rigid body (punch) and an elastic solid with coating reinforced with a FG interlayer. The main difference between this problem and the one considered by the authors before for a single-layer FG coating [9] is in the scheme of construction of the compliance function. This function characterizes elastic reaction of the considered elastic media under the normal loading. The integral equation of the problem does not differ from those considered earlier [9], its solution is constructed in an approximated analytical form using the bilateral asymptotic method [10]. Torsion of a half-space with coating reinforced with FG interlayer is considered earlier [11].

2. Statement of the problem

An elastic half-space with a two-layered coating is considered. The upper layer of thickness

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h_1 is homogeneous, the lower layer (interlayer of the media) has thickness h_2 and made of a FG material (continuously inhomogeneous in depth). A cylindrical system of coordinates r, φ, z is chosen (see Fig. 1). Lamé parameters of the half-space vary according to the following:

$$\{\mathrm{M},\Lambda\}(z) = \begin{cases} \{\mathrm{M}_0,\Lambda_0\} = \mathrm{const}, & -h_1 < z \le 0, \\ \{\mathrm{M}_1,\Lambda_1\}(z), & -H \le z \le -h_1, \\ \{\mathrm{M}_2,\Lambda_2\} = \mathrm{const}, & -\infty < z < -H. \end{cases} \tag{1}$$

Here $H = h_1 + h_2$ is the total thickness of the coating; $\{\mathrm{M}_1,\Lambda_1\}(z)$ are arbitrary positive continuously differentiable functions; superscripts 0, 1 and 2 correspond to the upper layer, to the interlayer and to the substrate, respectively.

Rigid circular punch with flat base is pressed into surface of the half-space under the action of normal centrally applied force P. The punch moves at a distance δ downward the z-axis. Outside of the contact area the surface is traction-free. Therefore, the boundary conditions take the form:

$$z = 0:\ \tau_{zr}^0 = 0,\ \sigma_z^0 = 0\ (r > a), w^0 = -\delta\ (r \le a). \tag{2}$$

$$z = -h_1:\tau_{zr}^0 = \tau_{zr}^1,\ \sigma_z^0 = \sigma_z^1,\ w^0 = w^1,\ \ u^0 = u^1, \tag{3}$$

The continuity conditions between the layers are satisfied:

$$z = -H:\tau_{zr}^1 = \tau_{zr}^2,\ \sigma_z^1 = \sigma_z^2,\ w^1 = w^2,\ \ u^1 = u^2. \tag{4}$$

It is required to determine the contact normal stresses under the punch:

$$\sigma_z\big|_{z=0} = -q_a(r),\ r \le a. \tag{5}$$

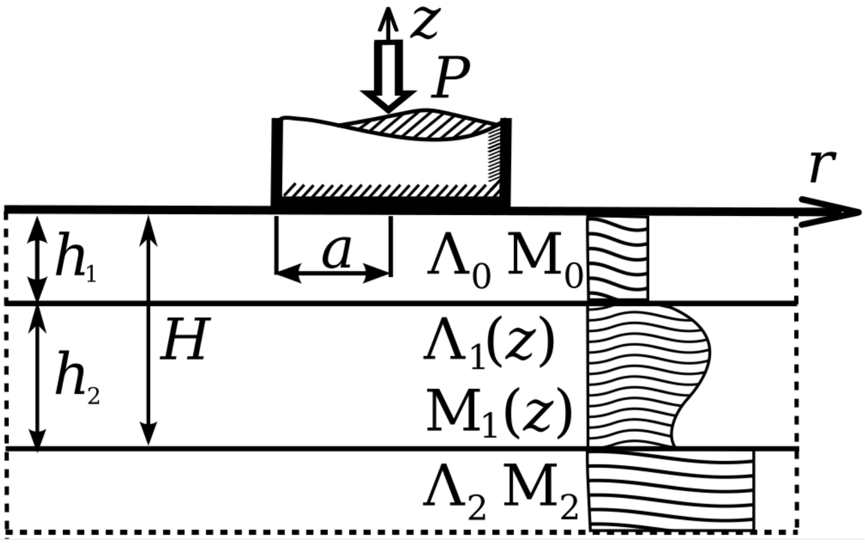

Fig. 1. Statement of the problem.

3. Compliance function

Linear constitutive equations for an inhomogeneous material and equilibrium equations have the form:

$$\sigma_r = 2\mathrm{M}(z)\frac{\partial u}{\partial r} + \Lambda(z)\theta, \sigma_\varphi = 2\mathrm{M}(z)\frac{u}{r} + \Lambda(z)\theta, \sigma_z = 2\mathrm{M}(z)\frac{\partial w}{\partial z} + \Lambda(z)\theta,$$
$$\tau_{rz} = \mathrm{M}(z)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}. \tag{6}$$

$$\frac{\partial}{\partial r}(r\sigma_r) + r\frac{\partial \tau_{rz}}{\partial z} - \sigma_\varphi = 0, \quad \frac{\partial}{\partial r}(r\tau_{rz}) + r\frac{\partial \sigma_z}{\partial z} = 0 \tag{7}$$

Let us use the Hankel transformations:

$$u(r,z) = -\int_0^\infty U(\gamma,z)\,\mathrm{J}_1(\gamma r)\gamma d\gamma, \quad \{w,q_a\}(r,z) = \int_0^\infty \{W,Q_a\}(\gamma,z)\,\mathrm{J}_0(\gamma r)\gamma d\gamma. \tag{8}$$

Rewriting (6) and (7) in terms of Hankel images yield the system of two ordinary differential equations of second order with variable coefficients:

$$\begin{cases} \mathrm{M}_i U'' + \gamma(\mathrm{M}_i + \Lambda_i)W' - \gamma^2(\mathrm{M}_i + \Lambda_i)U + \mathrm{M}_i' U' + \gamma \mathrm{M}_i' W = 0, \\ (2\mathrm{M}_i + \Lambda_i)W'' - \gamma(\mathrm{M}_i + \Lambda_i)U' - \gamma^2 \mathrm{M}_i W + (2\mathrm{M}_i' + \Lambda_i')W' - \gamma\Lambda_i' U = 0. \end{cases} \tag{9}$$

Here f' is the derivative of function f with respect to z.

Let us introduce the notations:

$$\{U^*,W^*\}(\gamma,z) = -\Theta_0\gamma\frac{\{U,W\}(\gamma,z)}{Q_a(\gamma)},\ \Theta_i = \frac{2\mathrm{M}_i(\mathrm{M}_i+\Lambda_i)}{(2\mathrm{M}_i+\Lambda_i)},\ i=0,2. \tag{10}$$

Because of the upper layer and the substrate have constant elastic moduli, the solution of system (9) for $-h_1<z\le 0$ (i=0) and $-\infty<z<-H$ (i=2) can be obtained analytically:

$$\begin{aligned} U_0^*(\gamma,z) &= (d_{01}+\gamma z d_{02})\mathrm{e}^{-\gamma z} + (d_{03}+\gamma z d_{04})\mathrm{e}^{\gamma z}, \\ W_0^*(\gamma,z) &= (-d_{01}-k_0 d_{02}-\gamma z d_{02})\mathrm{e}^{-\gamma z} + (d_{03}-k_0 d_{04}+\gamma z d_{04})\mathrm{e}^{\gamma z}, \end{aligned} \tag{11}$$

$$\begin{aligned} U_2^*(\gamma,z) &= (d_{21}+\gamma z d_{22})\mathrm{e}^{\gamma z}, \\ W_2^*(\gamma,z) &= (d_{21}-k_2 d_{22}+\gamma z d_{22})\mathrm{e}^{\gamma z}. \end{aligned} \tag{12}$$

Vanishing of the displacements for $z\to-\infty$ was accounted for in (12). Constants d_{ij} are to be determined for a fixed γ; $k_i = (\Lambda_i+3\mathrm{M}_i)/(\Lambda_i+\mathrm{M}_i)$, i = 0,2; j = 1,…,4 are some constants depending on elastic moduli of the upper layer and the substrate.

Boundary conditions (2)–(5) using (8) and (10) can be rewritten in the form:

$$\begin{matrix} z=-h_1\,(i=1) \\ z=-H\,(i=2) \end{matrix}: \begin{cases} \mathrm{M}_{i-1}\left(U_{i-1}'^* + \gamma W_{i-1}^*\right) = \mathrm{M}_i\left(U_i'^* + \gamma W_i^*\right), W_{i-1}^* = W_i^*,\ U_{i-1}^* = U_i^*, \\ (2\mathrm{M}_{i-1}+\Lambda_{i-1})W_{i-1}'^* - \gamma\Lambda_{i-1}U_{i-1}^* = (2\mathrm{M}_i+\Lambda_i)W_i'^* - \gamma\Lambda_i U_i^*, \end{cases} \tag{13}$$

$$z=0: \begin{cases} \mathrm{M}_0\left(U_0'^* + \gamma W_0^*\right) = 0, \\ (2\mathrm{M}_0+\Lambda_0)W_0'^* - \gamma\Lambda_0 U_0^* = \gamma\Theta_0. \end{cases} \tag{14}$$

Boundary conditions (13) are satisfied for $z = -h_1$ (i=1) and $z = -H$ (i = 2).

For the FG interlayer (i = 1) let us rewrite (9) in the matrix form:

$$\mathbf{w}' = \mathbf{A}\mathbf{w}, \quad \mathbf{w} = \left(U_1^*, U_1'^*, W_1^*, W_1^*\right)^{\mathrm{T}}. \tag{15}$$

Let us seek the solution of the system (15) in the form:

$$\mathbf{w}(\gamma,z) = d_{21}(\gamma)\mathbf{a}_1(\gamma,z)e^{\gamma z} + d_{22}(\gamma)\mathbf{a}_2(\gamma,z)e^{\gamma z}. \tag{16}$$

By substituting (16) into (13)–(15) and taking into account (12) two Cauchy problems for determining $\mathbf{a}_i(\gamma,z), i=1,2$ for a fixed γ are obtained:

$$\mathbf{a}_i' = \mathbf{A}\mathbf{a}_i - \gamma\mathbf{a}_i, \quad -H \le z \le -h_1, \tag{17}$$

$$\mathbf{a}_1(\gamma,-H) = \left(1, \gamma\frac{2\mathrm{M}_2 - \mathrm{M}_1(-H)}{\mathrm{M}_1(-H)}, 1, \gamma\frac{2\mathrm{M}_2 + \Lambda_1(-H)}{2\mathrm{M}_1(-H)+\Lambda_1(-H)}\right), \tag{18}$$

$$\begin{aligned} \mathbf{a}_2(\gamma,-H) = \Bigg(&-\gamma H, \frac{k_2\mathrm{M}_1(-H) + \mathrm{M}_2(1-k_2) - \gamma H(2\mathrm{M}_2 - \mathrm{M}_1(-H))}{\mathrm{M}_1(-H)}, \\ &-k_2 - \gamma H, \frac{(1-k_2)(2\mathrm{M}_2+\Lambda_2) - \gamma H(2\mathrm{M}_2 + \Lambda_1(-H))}{2\mathrm{M}_1(-H)+\Lambda_1(-H)}\Bigg). \end{aligned} \tag{19}$$

Function:

$$L(\gamma) = W_0^*(\gamma/H,0) = -d_{01}(\gamma/H) - k_0 d_{02}(\gamma/H) + d_{03}(\gamma/H) - k_0 d_{04}(\gamma/H) \tag{20}$$

hereafter will be called *compliance function*. From the boundary conditions (13) for $i = 1$ and from (14) a system of linear algebraic equations for calculating constants $d_{21}(\gamma)$, $d_{22}(\gamma)$, $d_{0j}(\gamma)$, j=1,...,4 can be obtained.

4. Integral equation of the problem and its solution

Let us use dimensionless variables:

$$\{r',\lambda\}=\{r,H\}\ /a\,,\ q(r')=q_a(r'a). \tag{21}$$

From (2) using (8), (10) and (21), omitting the primes, integral equation is obtained:

$$\int_0^1 q(t)tdt\int_0^\infty L(u)\,\mathrm{J}_0\big(ur\lambda^{-1}\big)\,\mathrm{J}_0\big(ut\lambda^{-1}\big)du=\lambda\Theta_0\delta/a\,,\ \ r\le 1. \tag{22}$$

The solution of (22) is constructed by the bilateral asymptotic method [10]. The main idea of this method is to use approximation of the compliance function $L(u)$ in the form of product of fractional-quadratic functions:

$$L(u)\approx L_N(u)=\prod_{i=1}^{N}\big(u^2+A_i^2\big)/\big(u^2+B_i^2\big). \tag{23}$$

By substituting (23) into (22) the approximated integral equation of the problem is obtained. Its solution is constructed similar to [9]:

$$q(r)=\frac{2\Theta_2\delta}{\pi a}\left(\frac{1}{\sqrt{1-r^2}}+\sum_{i=1}^{N}C_i\left[\frac{\cosh(A_i\lambda^{-1})}{\sqrt{1-r^2}}-\frac{A_i}{\lambda}\int_r^1\frac{\sinh(A_i\lambda^{-1}t)dt}{\sqrt{t^2-r^2}}\right]\right). \tag{24}$$

Constants C_i (i=1,…,N) are the solution of the system of linear algebraic equations below:

$$\sum_{i=1}^{N}C_i\ \frac{A_i\cosh(A_i\lambda^{-1})+B_k\,\sinh(A_i\lambda^{-1})}{A_i^2-B_k^2}=\frac{1}{B_k},k=1,...,N.$$

Expression (24) is asymptotically exact for $\lambda\to0$ and $\lambda\to\infty$ [10].

Stresses and displacements of any point inside the three-layered half-space is reduced to calculation of some quadratures. Equations(6)–(8) and (10) yields:

$$\begin{aligned}&u(r,z)=I_1(r,z),\quad w(r,z)=-I_3(r,z),\\&\sigma_z(r,z)=-(\Lambda(z)+2\mathrm{M}(z))I_4+\Lambda(z)I_6,\quad \sigma_\varphi(r,z)=2\mathrm{M}(z)a^{-1}r^{-1}I_1+\Lambda(z)(I_6-I_4),\\&\sigma_r(r,z)=(\Lambda(z)+2\mathrm{M}(z))I_6-\Lambda(z)I_4-2\mathrm{M}(z)a^{-1}r^{-1}I_1,\quad \tau_{rz}(r,z)=\mathrm{M}(z)(I_2+I_5).\end{aligned} \tag{25}$$

Following notations were used above:

$$\begin{aligned}&I_1(r,z)=\frac{2\Theta_2\delta a}{\pi\Theta_0}\int_0^\infty Q(\gamma)U^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_1\big(r\gamma\big)d\gamma,\ I_2(r,z)=\frac{2\Theta_2\delta a}{\pi\Theta_0}\int_0^\infty Q(\gamma)U'^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_1\big(r\gamma\big)d\gamma,\\&I_3(r,z)=\frac{2\Theta_2\delta a}{\pi\Theta_0}\int_0^\infty Q(\gamma)W^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_0\big(r\gamma\big)d\gamma,\ I_5(r,z)=\frac{2\Theta_2\delta}{\pi\Theta_0}\int_0^\infty Q(\gamma)W^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_1\big(r\gamma\big)\gamma d\gamma,\\&I_4(r,z)=\frac{2\Theta_2\delta a}{\pi\Theta_0}\int_0^\infty Q(\gamma)W'^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_0\big(r\gamma\big)d\gamma,\ I_6(r,z)=\frac{2\Theta_2\delta}{\pi\Theta_0}\int_0^\infty Q(\gamma)U^*\!\left(\frac{\gamma}{a},z\right)\mathrm{J}_0\big(r\gamma\big)\gamma d\gamma,\end{aligned} \tag{26}$$

$$Q(\gamma)=\sum_{i=1}^{N}C_i\ \frac{\cosh(A_i\lambda^{-1})\gamma\sin\gamma+A_i\lambda^{-1}\sinh(A_i\lambda^{-1})\cos\gamma}{\gamma^2+A_i^{\,2}\lambda^{-2}}+\frac{\sin\gamma}{\gamma}. \tag{27}$$

Values of functions $U^*(\gamma,z)$,$W^*(\gamma,z)$ and their derivatives with respect to z can be calculated numerically from (17)–(19). Function $Q(\gamma)$ is the Hankel image of contact pressure $q(r)$.

Calculating of integrals (26) involves some difficulties, because integrands are strongly oscillating functions. Decay rate of integrands with respect to γ depends on the value of z

coordinate. For a small z they decay very slowly. These factors must be taken into account in the numerical calculation of $I_i(r,z)$.

5. Numerical results

As an example, let us consider the silicon substrate (Si, Young's modulus E = 146 GPa, Poisson's ratio ν=0.22). Let the coating properties vary linearly from the pure nickel (Ni, Young's modulus E=203 GPa, Poisson's ratio ν = 0.31) on the surface to pure silicon in depth. In addition, let us also consider the case when the coating was oxidized near its surface. As a result of oxidation a thin layer of NiO is occurred (Young's modulus E = 90 GPa, Poisson's ratio ν = 0.21).

Figure 2 contains the graphs of the compliance function constructed for different thickness of the oxide layer. As it is seen, presence of the oxide upper layer sufficiently changes the compliance function and, as the result, the stress-strain state of the whole half-space also changes sufficiently. If thickness of the oxide layer is small (h_1<0.2H) then the compliance function has nonmonotonic variation.

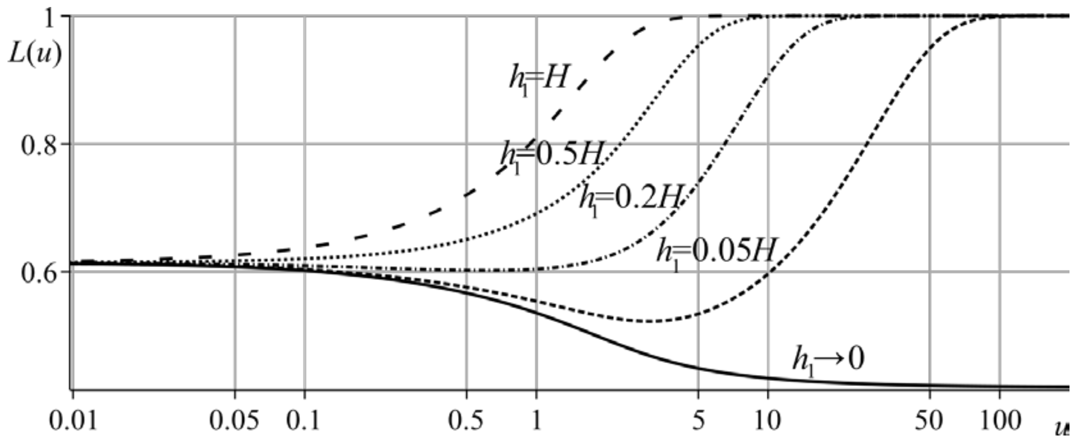

Fig. 2. Compliance functions constructed for different thickness of the oxide layer.

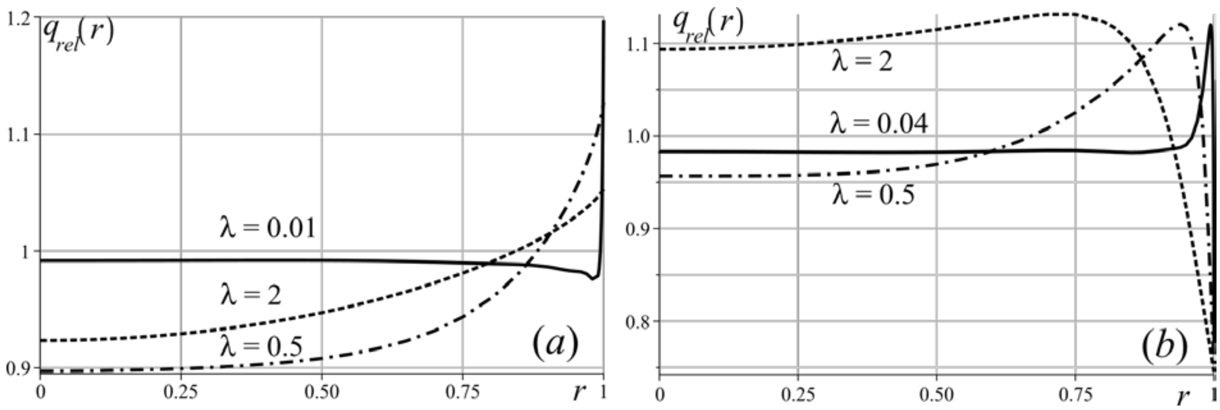

Fig. 3. Relative contact stresses for a coating with oxide layer (b) and without it (a).

To illustrate the distribution of the contact stresses it is convenient to consider relative function: $q_{\mathrm{rel}}(r) = q(r)/q_{\mathrm{hom}}(r)$ where $q_{\mathrm{hom}}(r)$ are the contact stresses appearing on the surface of a homogeneous half-space with elastic moduli of the substrate. Figure 3 contains graphs of relative contact stresses in assumption that the Ni layer is oxidized to 5% of its thickness, i.e. $h_1 = 0.05H$. Presence of oxide layer is sufficiently increases the contact stresses in the central area of contact and reduces them in the neighborhood of contact boundaries. Presence of oxide layer also leads to nonmonotonic variation of relative contact stresses. For thin coatings (small values of λ), a narrow peak can be observed on q_{rel} plot near to the boundaries of contact.

6. Conclusion

Shape of the punch and the origin of the normal loading (5) do not change the compliance function (20). The scheme described in the paper can be easily used in process of solution of a wide range of contact problems, which integral equations were solved previously. For example, contact of two elastic solids with coatings [9], indentation by a conical punch [12], bending of a plate lying on an elastic half-space with coating [13], etc. Solution of the contact problems can be obtained with the high accuracy by the method used in the paper even in the case of complex nonmonotonic variation of elastic moduli in the coating [14] and in the case when elastic moduli of the substrate greater/smaller than the one in coating by two order of magnitude and even more [13]. The method was also used to solve contact problems of electroelasticity [15], thermoelasticity, elastohydrodynamically lubricated contact [16], etc.

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