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PERTURBATION METHOD, PADÉ APPROXIMANTS AND EXACT SOLUTIONS OF NONLINEAR MECHANICS EQUATIONS

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Abstract. In this article we are suggesting a method for finding exact solutions to integrable and non-integrable nonlinear mechanics equations that is based on the perturbation method. The criterion of equality of sequential diagonal Padé approximants whose minimum order is determined by the pole order of the equation's solution is used for summation of the perturbation series. When the criterion is satisfied, the Padé approximants are the sought exact solutions.

Keywords: perturbation method; Padé approximants; nonlinear evolution equations; exact solutions.

1. Introduction

In the study of dynamic problems of solid mechanics and mechanics of nanostructured materials, nonlinear quasi-hyperbolic equations with complex analytic structure often become the object of analysis. The development of methods for constructing physically realizable classes of their exact solutions is an important task of nonlinear mathematical physics during the last decades [1].

In this work, the exact solutions are constructed using the perturbation method. The technique of Padé summation is applied to the resulting exponential series. It is shown that exact solutions of nonlinear non-integrable equations can be obtained based on solving a sequence of linear problems.

Padé approximants (PA) used in the theory of integrable equations are primarily associated with the concept of padeons, which are the simplest [0/1] approximants resulting from summation of the exponential terms in the perturbation series [2, 3]. The highest approximants [N-1/N] and [N/N] referred to as N-padeons are associated with N-soliton solutions described in [4-6]. This approach is based on the idea suggested by Hirota in [7] of the possibility of applying Padé summation to the perturbation series constructed for the evolution equation from a linearized equation. However, in the original work [7], this idea was deemed unproductive, most likely due to the fact that the computer algebra systems such as Maple or Mathematica [8] had not been available in the 1970s. Instead, Hirota developed a direct bilinear method for constructing multisoliton solutions and Bäcklund transformations to integrable equations.

In [9] PA are used for constructing approximate solutions to the Gross-Pitaevskii equation. In [10] PA are applied in a numerical-analytical method when calculating soliton solutions in the two-field antiferromagnet model. Effectiveness of two-point PA in nonlinear dynamics and solid mechanics is pointed out in [11].

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In this paper we are proposing a method of constructing exact solutions to integrable and non-integrable evolution equations that are based on PA to the corresponding series of the perturbation method. The PA are shown to be the sought exact solutions [12].

Before outlining the method, let us consider the Korteweg-de Vries (KdV) equation:

$$u_t + u_{xxx} + uu_x = 0\,. \tag{1}$$

The KdV equation describes the evolution of long, one-dimensional waves in many physical settings, including shallow-water waves with weakly non-linear restoring forces, ion acoustic waves in plasma and acoustic waves on a crystal lattice.

In accordance with the perturbation method, the solution to (1) may be found in the form of a power series on a small parameter ε:

$$u = \varepsilon u_1(x,t) + \varepsilon^2 u_2(x,t) + \varepsilon^3 u_3(x,t) + \ldots \tag{2}$$

Substituting (2) into (1) and collecting the terms with identical power of ε, we arrive to:

$$\begin{aligned} &u_{1t} + u_{1xxx} = 0,\\ &u_{2t} + u_{2xxx} = -u_1 u_{1x}\,, \\ &u_{3t} + u_{3xxx} = -u_1 u_{2x} - u_2 u_{1x}\,, \ \ldots \end{aligned} \tag{3}$$

Substituting $u_1 = \exp(kx - \omega t)$ from the first equation of the system (3) we get $\omega = k^3$.

Sequentially solving the remaining equations we obtain:

$$u = z - \frac{z^2}{6k^2} + \frac{z^3}{48k^4} - \frac{z^4}{432k^6} + \frac{5z^5}{20736k^8} - \frac{z^6}{41472k^{10}} + \ldots\,, \tag{4}$$

where $z = \varepsilon \exp\left(kx - k^3 t\right)$.

According to the definition [13], Padé approximant of the function $f(z)$ is a rational function:

$$[P/Q] = \frac{a_0 + a_1 z + .. + a_P z^P}{b_0 + b_1 z + .. + b_Q z^Q},$$

whose decomposition into a Maclaurin series coincides with the decomposition of the function $f(z)$ as long as possible. If P is equal to Q, the approximant is called a diagonal Padé approximant (PA).

Since diagonal PA are optimal approximations to power series [13], let us write the first diagonal approximants of the series (4):

$$\begin{aligned} &[1/1] = \frac{6k^2 z}{6k^2 + z}\,,\\ &[2/2] = \frac{144k^4 z}{(12k^2 + z)^2}\,,\\ &[3/3] = \frac{144k^4 z}{(12k^2 + z)^2}\,,\ \ldots \end{aligned} \tag{5}$$

A solution to Eq. (1) has a pole of the 2nd order, and all approximants starting from [2/2] are expressed similarly. It is not difficult to prove that this expression is the exact solution to Eq. (1). Representing [2/2] approximant as sum of the geometric progression:

$$\frac{144k^4 z}{\left(12k^2 + z\right)^2} = \frac{z}{1 - \left(-\dfrac{z}{6k^2} - \dfrac{z^2}{144k^4}\right)} = z + z\left(-\frac{z}{6k^2} - \frac{z^2}{144k^4}\right) + z\left(-\frac{z}{6k^2} - \frac{z^2}{144k^4}\right)^2 + \ldots =$$

$$= z - \frac{z^2}{6k^2} + \frac{z^3}{48k^4} - \frac{z^4}{432k^6} + \ldots,$$

we get a series matching (4). In other words, applying diagonal PA rebuilds the series of the perturbation method for the KdV equation into a geometric series in such a way that sum of the series coincides with the corresponding approximant and represents the exact solution to the original equation. This transformation forms the basis of the method proposed below.

2. Description of the method

Consider a partial differential equation:

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0. \qquad (6)$$

Step 1. We represent Eq. (6) as a polynomial in $u(x,t)$ and its derivatives. For this purpose, a nonlinear substitution or replacement of radicals and transcendental terms to the corresponding Maclaurin series may be performed.

Step 2. The solution to (6) will be given in the form of the series:

$$u(x,t) = \sum_{n=1}^{\infty} \varepsilon^n u_n(x,t). \qquad (7)$$

In one of the lower-order terms on ε we find a homogeneous equation for the function u_1. The solution to this equation can be presented as $u_1 = \exp(kx - \omega t)$, where ω and k are connected by the dispersion relation $\omega = \omega(k)$.

Sequentially solving inhomogeneous equations arising in higher orders of ε, we express u_2, u_3,… through the corresponding powers of u_1:

$$u_n = K_n u_1^n,$$

where K_n depends on the wave number k and coefficients of the original equation (6).

If we denote $z = \varepsilon u_1$, the series (7) becomes an ordinary power series:

$$u(z) = z + \sum_{n=2}^{\infty} K_n z^n \ . \qquad (8)$$

Step 3. In order to determine the order P of the pole of the sought solution characterized by decomposition (8), it is necessary to carry out standard analysis of the leading terms of Eq. (6). If P is not a natural number, we perform transformation of the series (8), which we call *normalization*:

if $P = m/n,\ m, n \in N$, the series (8) is raised to the power of n;

if $P = 0$, the series (8) is differentiated term by term.

If the resulting series does not contain all natural powers of z, we perform the substitution $z \to Z$ in order for the transformed series to contain all natural powers of the new variable Z.

A normalized series characterizes the solution with the pole order equal to the natural number Q.

Step 4. We calculate sequential diagonal Padé approximants [Q/Q], [$Q+1/Q+1$], [$Q+2/Q+2$] etc. If the condition:

$$\forall Q \geq Q^*,\ [Q/Q] - [Q+1/Q+1] \equiv 0, \qquad (9)$$

is satisfied at arbitrary or specific values of the coefficients of Eq. (6), we verify that, after returning to the variables (x, t), the expression [Q^*/Q^*] gives the exact solution to this equation. In the majority of cases, we found that the coincidence of three sequential approximants indicated that the exact solution had been found.

If the condition (9) is not satisfied, then application of the perturbation method with PA does not give the exact solution.

The above scheme is intended for finding solutions in the neighborhood of zero, since ε in (7) is a small parameter.

Substituting:

$$u = u_0(x,t) + \sum_{n=1}^{\infty} \varepsilon^n u_n(x,t),$$

instead of (7) enables us to find other solutions in the neighborhood of a known solution $u_0(x,t)$ [14], which for many equations may be given as constant or linear function of the variables (x, t).

Several examples of this will be presented below.

3. The 5th-order evolution equation

Consider the equation:

$$u_t + au_{xxxxx} + buu_{xxx} + cu_x u_{xx} + du^2 u_x = 0\,. \tag{10}$$

The dispersion relation for (10) has the form $\omega = ak^5$ and the perturbation series may be presented as:

$$\begin{aligned} u = {} & z - \frac{(b+c)z^2}{30ak^2} - \frac{(10ad - 3b^2 - 5bc - 2c^2)z^3}{2400a^2k^4} + \\ & + \frac{(450abd + 330acd - 79b^3 - 180b^2c - 135bc^2 - 34c^3)z^4}{1836000a^3k^6} + \\ & + \frac{z^5}{2291328000a^4k^8}(15300a^2d^2 - 23940ab^2d - 34620abcd - \\ & - 12840ac^2d + 3125b^4 + 9086b^3c + 9897b^2c^2 + 4820bc^3 + 884c^4) - \ldots \end{aligned} \tag{11}$$

The pole order of the solution (10) is equal to 2. Factorization of the left-hand side of the equalities in the chain:

$[2/2] - [3/3] = 0$,

$[3/3] - [4/4] = 0$, …

reveals a common factor, which can be equated to zero:

$$10ad - b^2 - bc = 0\,. \tag{12}$$

Having found the factor c from (12), we determine that all the approximants $[Q/Q]$, $Q \geq 2$ are equal to:

$$\frac{36k^4b^2z}{\left(6bk^2 + dz\right)^2}\,. \tag{13}$$

The expression (13) is sum of the series (11) and gives the exact solitary wave solution to Eq. (10) after substituting $z = \exp\left(kx - ak^5t + \ln\varepsilon\right)$.

The family of equations (10) describes the evolution of steeper waves of shorter wavelength than those described by the KdV equation does and includes many well-known integrable equations and, in particular, the KdV symmetry equation [15]:

$$u_t = u_{xxxxx} + 10u\,u_{xxx} + 20u_x\,u_{xx} + 30u^2\,u_x, \tag{14}$$

the Sawada-Kotera equation:

$$u_t = u_{xxxxx} + 5u\,u_{xxx} + 5u_x\,u_{xx} + 5u^2\,u_x, \tag{15}$$

and the Kaup-Kupershmidt equation:

$$u_t = u_{xxxxx} + 5u\,u_{xxx} + \frac{25}{2}u_x\,u_{xx} + 5u^2\,u_x. \tag{16}$$

It is easy to check that (14) and (15) satisfy the condition (12), and that (16) does not.

4. The Vakhnenko equation

The Vakhnenko equation [16]:

$$u\,u_{xxt} - u_x u_{xt} + u^2 u_t = 0 \tag{17}$$

is the integrable reduction of the famous Ostrovsky equation, which describes gravity waves propagating down a channel under influence of Coriolis force. Perturbation method lets us find a solution to the Vakhnenko equation in the neighborhood of zero, as well as in the vicinity of a constant. In the first case, the series of the method is given in the form:

$$u = z - \frac{z^2}{3k^2} + \frac{z^3}{12k^4} - \frac{z^4}{54k^6} + \frac{5z^5}{1296k^8} - \frac{z^6}{1296k^{10}} + \dots \tag{18}$$

when an arbitrary dispersion relation is assumed, since any function $\exp(kx - \omega t)$ is a solution to the equation for $u_1(x,t)$.

The solution to (17) has a 2nd order pole. Starting from the second order, all the PA to (18) coincide and give the exact solitary wave solution to Eq. (17):

$$u = \frac{36k^4 z}{\left(6k^2 + z\right)^2} = \frac{36k^4 \exp\left(kx - \omega t + \ln\varepsilon\right)}{\left(6k^2 + \exp\left(kx - \omega t + \ln\varepsilon\right)\right)^2} \ .$$

In the second case, we seek the solution to (17) in the form:

$$u = C + \sum_{n=1}^{\infty} \varepsilon^n u_n(x,t) \ .$$

The equation for $u_1(x,t)$ becomes an identity at $C = -k^2$ and the series (18) does not change. The exact solution in this case looks like solitary wave on a pedestal and can be written as:

$$u = -k^2 + \frac{36k^4 \exp\left(kx - \omega t + \ln\varepsilon\right)}{\left(6k^2 + \exp\left(kx - \omega t + \ln\varepsilon\right)\right)^2} \ .$$

5. The Konno-Kameyama-Sanuki equation (KKS)

Equations containing a term with the sine of the dependent variable, such as sine-Gordon equation [17] or the KKS equation [18], are widely used in the modeling of atomic chains properties in crystals. The KKS equation:

$$u_{xt} + \alpha u_x^2 u_{xx} + \beta u_{xxxx} - \delta \sin u = 0, \tag{19}$$

is integrable [18], if

$$\alpha = \frac{3}{2}\beta \ . \tag{20}$$

Analysis of the leading terms in Eq. (19) gives a zero-order pole solution.

Replacing sin(u) in (19) with a partial sum of the Maclaurin series and applying the standard procedure of the perturbation method, we get:

$$\omega = \frac{\beta k^4 - \delta}{k} \ ,$$

$$\begin{aligned} u = {} & z - \frac{1}{48}\frac{\left(6\alpha k^4 + \delta\right) z^3}{9\beta k^4 + \delta} + \frac{1}{1280}\frac{\left(100\alpha^2 k^8 + 20\alpha\delta k^4 + 4\beta\delta k^4 + \delta^2\right) z^5}{225\beta^2 k^8 + 34\beta\delta k^4 + \delta^2} - \\ & - \frac{1}{86016}\Big(\big(88200\alpha^3\beta k^{16} + 6664\alpha^3\delta k^{12} + 24500\alpha^2\beta\delta k^{12} + 3864\alpha\beta^2\delta k^{12} + \\ & + 720\beta^3\delta k^{12} + 1652\alpha^2\delta^2 k^8 + 2422\alpha\beta\delta^2 k^8 + 564\beta^2\delta^2 k^8 + 126\alpha\delta^3 k^4 + 87\beta\delta^3 k^4 + \\ & + 3\delta^4\big) z^7\Big) \Big/ \left(99225\beta^4 k^{16} + 28044\beta^3\delta k^{12} + 2638\beta^2\delta^2 k^8 + 92\beta\delta^3 k^4 + \delta^4\right) + \dots \end{aligned} \tag{21}$$

In order to carry out normalization, let us differentiate the series (21) with respect to *z*, substitute $z=\sqrt{Z}$ and multiply every term by Z. Approximants [1/1] and [2/2] of the normalized series are equal in two cases: when (20) is satisfied, or when

$$\alpha=\frac{3\beta k^4-\delta}{8k^4}.$$

In the first, integrable case, the approximants are equal to $16Z/(16+Z)$. After making the reverse transformations: dividing by *Z*, replacing $Z=z^2$ and integration with respect to *z*, we get back to the original variables (*x*, *t*) to obtain a known kink-type solution to the KKS equation:

$$u=4\arctan\left(\exp\left(kx-\frac{\beta k^4-\delta}{k}t+\ln\frac{\varepsilon}{4}\right)\right).$$

In the second, non-integrable case, the approximants are equal to $64Z/(64+Z)$ and lead to the new exact kink-type solution, which has not been previously encountered in research:

$$u=8\arctan\left(\exp\left(kx-\frac{\beta k^4-\delta}{k}t+\ln\frac{\varepsilon}{8}\right)\right).$$

6. The Bretherton equation

The Bretherton equation [19]:

$$u_{tt}-a\,u_{xx}-b\,u_{xxxx}-c\,u+d\,u^3=0$$

is a model equation for studying weakly-nonlinear wave dispersion. For this equation, that has a 2nd-order pole solution, the perturbation method gives:

$$\omega=\pm\sqrt{b\,k^4+ak^2+c}\,,$$

$$u=z+\frac{1}{8}\frac{dz^3}{9bk^4-c}+\frac{1}{64}\frac{d^2z^5}{(9bk^4-c)(25bk^4-c)}+$$
$$+\frac{1}{512}\frac{d^3(17bk^4-c)z^7}{(9bk^4-c)^2(25bk^4-c)(49bk^4-c)}+ \qquad (22)$$
$$+\frac{1}{4096}\frac{d^4(433b^2k^8-50bck^4+c^2)z^9}{(9bk^4-c)^3(25bk^4-c)(49bk^4-c)(81bk^4-c)}+\dots$$

Having normalized the series (22), we require that:

$$[2/2]-[3/3]=0\,,$$
$$[3/3]-[4/4]=0,\ \dots \qquad (23)$$

Factorization of the left-hand side of the equalities (23) shows that the first two contain a common factor $11bk^4+c$. Substituting $c=-11bk^4$ confirms that approximant [2/2] gives the exact solitary wave solution to the Bretherton equation:

$$u=\frac{480z(480bk^4+d\,z^2)bk^4}{(480bk^4-d\,z^2)^2},\quad z=\exp\left(kx\mp\sqrt{-10bk^4+ak^2}\,t+\ln\varepsilon\right).$$

7. The Kuramoto-Sivashinsky equation (KS)

The KS equation was derived to model the diffusive instabilities in the laminar flame front and in its standard form [20]:

$$u_t+uu_x+u_{xx}+a\,u_{xxx}+b\,u_{xxxx}=0 \qquad (24)$$

has a 3rd-order pole solution. In order to simplify solving the equation and reduce the order of the pole to 2, let us consider the KS equation in the potential form:

$$u_{xt} + u_x u_{xx} + u_{xxx} + a u_{xxxx} + b u_{xxxxx} = 0 \;. \qquad (25)$$

The perturbation method for Eq. (25) leads to:

$$\omega = bk^4 + ak^3 + k^2,$$

$$u = z - \frac{z^2}{4(7bk^2+3ak+1)} + \frac{z^3}{12(7bk^2+3ak+1)(13bk^2+4ak+1)} - \\ - \frac{(27bk^2+10ak+3)z^4}{96(7bk^2+3ak+1)^2(13bk^2+4ak+1)(21bk^2+5ak+1)} + \\ + \frac{(90bk^2+25ak+6)z^5}{480(7bk^2+3ak+1)^2(13bk^2+4ak+1)(21bk^2+5ak+1)(31bk^2+6ak+1)} - \ldots \qquad (26)$$

Requiring equality of the approximants [2/2] and [3/3] for the expression (26) gives the 1st equation for α and β:

$$(2050\beta^2 - 820\beta + 10)\alpha^2 + (-2940\beta^3 - 3284\beta^2 - 340\beta + 4)\alpha + \\ + 25875\beta^4 - 19758\beta^3 + 364\beta^2 - 82\beta + 1 = 0, \qquad (27)$$

where $\alpha = ak$, $\beta = bk^2$. However, under the single condition (27) the approximants [3/3] and [4/4] are not equal. The requirement of their equality leads to the second equation for α and β, which, due to bulkiness, we present in an abridged form:

$$20160(113375\beta^3 - 91175\beta^2 + 6245\beta - 21)\alpha^8 + \\ + 16(719919875\beta^4 - 1097123350\beta^3 - 6366660\beta^2 + 12202118\beta - 41167)\alpha^7 + \ldots = 0. \qquad (28)$$

Among the 10 pairs of the values $\{\alpha, \beta\}$ giving solution to the system (27), (28), there is only one pair $\{\alpha = 4, \beta = 1\}$ leading to the exact real solution of Eq. (25):

$$u = \frac{60z(z+240)}{(z+120)^2}, \quad z = \exp\left(kx - 6k^2t + \ln\varepsilon\right). \qquad (29)$$

The derivative of (29) with respect to x gives the exact solitary wave solution to the initial equation (24):

$$u = \frac{1728000k\exp\left(kx - 6k^2t + \ln\varepsilon\right)}{\left(\exp\left(kx - 6k^2t + \ln\varepsilon\right) + 120\right)^3}.$$

8. The modified Kuramoto-Sivashinsky equation (mKS)

The mKS equation, [21] often used for modeling dynamics of transition region between two stable states in solidification or reaction-diffusion process:

$$u_t + u^2 u_x + u_{xx} + a u_{xxx} + b u_{xxxx} = 0, \qquad (30)$$

presented in the potential form:

$$u_{tx} + u_x^2 u_{xx} + u_{xxx} + a u_{xxxx} + b u_{xxxxx} = 0, \qquad (31)$$

after application of the perturbation method gives:

$$\omega = bk^4 + ak^3 + k^2,$$

$$u = z - \frac{kz^3}{18(13bk^2+4ak+1)} + \frac{k^2z^5}{120(13bk^2+4ak+1)(31bk^2+6ak+1)} - \\ - \frac{k^3(101bk^2+24ak+5)z^7}{3024(13bk^2+4ak+1)^2(31bk^2+6ak+1)(57bk^2+8ak+1)} + \ldots \qquad (32)$$

The solution to Eq. (31) has the pole of order 1/2, therefore we take a square of the series (32) and substitute $z = \sqrt{Z}$:

$$Z-\frac{kZ^2}{9(13bk^2+4ak+1)}+\frac{(253bk^2+69ak+16)k^2Z^3}{810\,(13bk^2+4ak+1)^2(31bk^2+6ak+1)}-$$
$$-\frac{(113bk^2+22ak+4)k^3Z^4}{945(13bk^2+4ak+1)^2(31bk^2+6ak+1)(57bk^2+8ak+1)}+\dots \quad (33)$$

The numerator of the difference [1/1]-[2/2] contains multipliers:
$(-19bk^2+3ak+2)(13bk^2+4ak+1)(57bk^2+8ak+1)$,
and only the first one is not found among the denominators of (33). Equating this multiplier to zero gives the first condition for the coefficients of Eq. (31):

$$a=\frac{19bk^2-2}{3k}. \quad (34)$$

Similarly, among the multipliers of the numerator of the difference [2/2]-[3/3]
$(2bk^2-1)(23bk^2-1)(209bk^2-7)(463bk^2-17)$
only the first one can be equated to zero. This gives a second condition for the coefficients (31):

$$b=\frac{1}{2k^2}. \quad (35)$$

If the conditions (34) and (35) are satisfied, all the diagonal PA for the (33) coincide and are equal to:

$$\frac{Z}{1+\frac{2k}{315}Z}. \quad (36)$$

When rewritten in the variables (x, t), the expression (36) gives the exact solution to Eq. (31):

$$u=\frac{3\sqrt{35}\exp\left(kx-4k^2t+\ln\varepsilon\right)}{\sqrt{315+2k\exp\left(2\left(kx-4k^2t+\ln\varepsilon\right)\right)}}, \quad (37)$$

and the derivative of (37) with respect to x is the exact solitary wave solution to the initial equation (30):

$$u=\frac{945\sqrt{35}k\exp\left(kx-4k^2t+\ln\varepsilon\right)}{\left(315+2k\exp\left(2\left(kx-4k^2t+\ln\varepsilon\right)\right)\right)^{3/2}}.$$

9. Conclusion

We demonstrated that the proposed method may be effectively used for solving nonlinear evolutionary equations having no variable coefficients. The method is not applicable to equations that have a non-constant separant and that are reducible to equations with a constant separant by means of differential substitutions and the transformations of equivalence [22].

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References

[1] G.A. Maugin, *Non-Classical Continuum Mechanics: A Dictionary* (Springer Nature Singapore Pte Ltd, 2017).
[2] F. Lambert, M. Musette // *J. Comp. Appl. Math.* **15** (1986) 235.
[3] F. Lambert, M. Musette // *Lect. Notes Math.* **1071** (1984) 197.
[4] C. Liverani, G. Turchetti // *J. Math. Phys*. **24** (1983) 53.

[5] F. Lambert, M. Musette, Diagonal Padé approximants to KdV multisolitons, In: *Proceedings of the international seminar on high energy physics and quantum field theory* (Protvino, 1980) pp. 336-338.
[6] G. Turchetti // *Lett. al Nuovo Cimento.* **27** (1980) 107.
[7] R. Hirota // *Phys. Rev. Lett.* **27** (1971) 1192.
[8] D. Baldwin, U. Goktas, W. Hereman, L. Hong, R.S. Martino, J.C. Miller // *J. Symb. Comp.* **37** (2004) 669.
[9] N.G. Berloff // *J. Phys. A: Math. Gen.* **37(48)** (2004) 1617.
[10] A.I. Bogolyubsky, S.L. Skorokhodov // *Program. Comp. Softw.* **30** (2004) 95.
[11] I.V. Andrianov, J. Awrejcewicz // *Appl. Mech. Rev.* **54(1)** (2001) 69.
[12] A.V. Bochkarev, A.I. Zemlyanukhin // *Comput. Math. and Math. Phys.* **57(7)** (2017) 1111.
[13] G.A. Baker Jr., P. Graves-Morris, *Padé Approximants* (Cambridge U.P., Cambridge, 1996).
[14] V.A. Baikov, K.R. Khusnutdinova // *J. Nonlin. Math. Phys.* **3** (1996) 139.
[15] A.V. Mikhailov, A.B. Shabat, V.V. Sokolov, The symmetry approach to classification of integrable equations, In: *What is integrability?*, ed. by V.E. Zakharov (Springer-Verlag, Berlin/New York, 1991).
[16] V.O. Vakhnenko, E.J. Parkes // *Nonlinearity* **11** (1998) 1457.
[17] E.L. Aero, A.N. Bulygin, Y.V. Pavlov // *Theor. Math. Phys.* **158(3)** (2009) 313.
[18] K. Konno, W. Kameyama, H. Sanuki // *J. Phys. Soc. Japan* **37** (1974) 171.
[19] N.A. Kudryashov, D.I. Sinelshchikov, M.V. Demina // *Phys. Lett. A* **375** (2011) 1074.
[20] N.A. Kudryashov, E.D. Zargaryan // *J. Phys. A: Math. Gen.* **29** (1996) 8067.
[21] N.G. Berloff, L.N. Howard // *Stud. Appl. Math.* **99** (1997) 1.
[22] N.H. Ibragimov, *Elementary Lie group analysis and ordinary differential equations* (John Wiley & Sons, Chichester, 1999).