Materials Physics and Mechanics 36 (2018) 137-141 Received: November 2, 2017

APPROXIMATE ANALYTICAL APPROACH IN ANALYZING AN ORTHOTROPIC RECTANGULAR PLATE WITH A CRACK

Dmitriy P. Goloskokov[1], Alexander V. Matrosov[2*]

[1]The Bonch-Bruevich Saint-Petersburg State University of Telecommunications, 61 Moika, St. Petersburg, 191186, Russia

[2]Saint Petersburg State University, 7-9, Universitetskaya nab., St. Petersburg, 199034, Russia

[*]e-mail: avmatrosov@mail.ru

Abstract. The stretching of a rectangular plate with a crack parallel to one of its edges under the action of a uniformly distributed load is considered. An approximate analytical solution in the form of trigonometric polynomials is constructed by the superposition method using two solutions obtained by the method of initial functions (MIF). The stress-strain state in the neighborhood of the crack sides and the crack tip for orthotropic plates is investigated.

Keywords: method of initial functions; superposition method; cracks; method of analytical decomposition; plane problem of elasticity theory.

1. Introduction

Nowadays numerical techniques are widely used to analyze elastic structures but analytical approaches to these problems continue to develop. The superposition method is used to analyze 2D and 3D problems with approximate analytical solutions for rectangles and finite cylinders [1-4]. The exact solutions using Lagrange expansion in terms of Fadle-Papkovich functions for a half-strip and elastic rectangle are derived in [5-7]. Of course these approaches are developing for bodies of canonical form when their boundaries are parallel to the coordinate axes (a rectangle, a cylinder, a strip and so on). To construct an analytical solution for the structure with a complex geometry doesn't seem to be possible. However we can decompose a complex structure on areas for which common solutions in a form of series with indeterminate coefficients are known. These coefficients are evaluated meeting the boundary conditions of the whole structure and the conjugation conditions of the contiguous areas. In this paper the general solutions for the orthotropic and isotropic rectangles in accordance with the superposition method is composed of two solutions obtained by the method of initial functions (MIF) in the form of trigonometric series with undetermined coefficients.

If an elastic structure may be presented by a number of contiguous rectangles with finite dimensions then we can use general solutions constructed for each of the rectangles and get an infinite linear algebraic system to determine unknown coefficients in all general solutions.

2. Superposition method

Consider an elastic rectangle $0 \le x \le h$, $0 \le y \le b$ in a rectangular Cartesian coordinate system Oxy. A sum of two solutions each of which can satisfy any boundary conditions on two opposite sides of elastic rectangle may be considered as a general solution for the rectangle. Every solution has a number of unknowns which can be determined meeting the

http://dx.doi.org/10.18720/MPM.3612018_15

boundary conditions on two opposite sides. Using the general solution these unknowns are determined when meeting the boundary conditions on four rectangular sides.

The MIF solution [8] with the initial functions in the form of trigonometric series on the line $x=0$ can satisfy any boundary conditions on two opposite sides $x=0,h$ of the orthotropic rectangle

$$\mathbf{U}^1=\sum_{m=0}^{\infty}\tilde{\mathbf{L}}_m^1\mathbf{U}_m^{1,0}. \quad (1)$$

Here $\mathbf{U}^1=\{u,v,\sigma_x,\sigma_y,\tau_{xy}\}$ is a vector of the stress-strain components, $\mathbf{U}_m^{1,0}=\{u_m^1\sin(\alpha_m y),v_m^1\cos(\alpha_m y),s_m^1\sin(\alpha_m y),t_m^1\cos(\alpha_m y)\}$ $(\alpha_m=m\pi/a)$ are vectors of initial functions determined on the line $x=0$, $\tilde{\mathbf{L}}_m^1=[\tilde{L}_{ij,m}^1(x,A_{kl})]$ ($i=1,\dots,5$, $j=1,\dots,4$) is a matrix of the values of the MIF operators and A_{kl} are moduli of elasticity of an orthotropic plate.

Second solution is obtained by the MIF when the initial functions are determined on the line $y=0$

$$\mathbf{U}^2=\sum_{n=0}^{\infty}\tilde{\mathbf{L}}_n^2\mathbf{U}_n^{2,0}, \quad (2)$$

where $\mathbf{U}^2=\{u,v,\sigma_x,\sigma_y,\tau_{xy}\}$, $\mathbf{U}_n^{2,0}=\{u_n^2\cos(\beta_n x),v_n^2\sin(\beta_n x),s_n^2\sin(\beta_n x),t_n^2\cos(\beta_n x)\}$ $(\beta_n=n\pi/h)$ are vectors similar to vectors in the solution (1), $\tilde{\mathbf{L}}_n^2=[\tilde{L}_{ij,n}^2(y,A_{kl})]$ ($i=1,\dots,5$, $j=1,\dots,4$) is a matrix of the values of the MIF operators.

The general solution is built as a sum of these two MIF solutions (1), (2) with the unknown coefficients u_m^1, v_m^1, s_m^1, t_m^1, u_n^2, v_n^2, s_n^2, t_n^2 ($m,n=0,\dots,\infty$) [8,9]

$$\mathbf{U}=\mathbf{U}^1+\mathbf{U}^2. \quad (3)$$

With the general solution (3) we can analyze a linearly-elastic rectangle with arbitrary boundary conditions on its edges one of the following types:

- kinematic (displacements u and v)
- force (normal and shearing stress)
- mixed (shearing stress and normal displacement)
- mixed (normal stress and tangential displacement)

To do this it is necessary to calculate components of the strain and stress state (SSS) defined on all four edges $x=0,h$, $y=0,a$ (two components on each edge) and then to use different technologies for finding unknown coefficients in the general solution: equating the coefficients in the trigonometric series for the boundary conditions obtained from the general solution and the given ones, the collocation method, various forms of the least squares method, and so on. In this paper expanding into trigonometric series is applied.

3. Analytical decomposition method

Suppose that the structure can be divided into contiguous rectangles. In this case it is possible to construct for each of the rectangles a general solution of the form (3) with its own set of unknown coefficients. For each *k*-th rectangle a general solution in the form (3) can be written as

$$\mathbf{U}^k=\mathbf{U}_1^k+\mathbf{U}_2^k, \quad (4)$$

where $\mathbf{U}^k=\{u^k,v^k,\sigma_x^k,\tau_{xy}^k,\sigma_y^k\}$ is a vector of local SSS components, $\mathbf{U}_1^k=\{u_1^k,v_1^k,\sigma_{x,1}^k,\tau_{xy,1}^k,\sigma_{y,1}^k\}$ and $\mathbf{U}_2^k=\{u_2^k,v_2^k,\sigma_{x,2}^k,\tau_{xy,2}^k,\sigma_{y,2}^k\}$ are vectors of solutions of the type

(1) and (2) respectively. This solution has indeterminate coefficients $u_m^{1,k}$, $v_m^{1,k}$, $s_m^{1,k}$, $t_m^{1,k}$ and $u_n^{2,k}$, $v_n^{2,k}$, $s_n^{2,k}$, $t_n^{2,k}$ $(n,m=0,\ldots,\infty)$ entering correspondingly into solutions $\mathbf{U}_1^k$ and $\mathbf{U}_2^k$.

For finding the concrete values of all indeterminate coefficients in all solutions for all rectangles it should use the boundary conditions of the whole body and joining conditions on the corresponding edges of contiguous rectangles preliminarily expanded the SSS components on their sides into trigonometric series. An infinity linearly algebraic system will be got by equating coefficients of the same harmonic in expanding of corresponding SSS components. The reduction method can be used to get an approximation to the solver of this infinite system.

The decomposition of the plate with the crack on the eight contiguous rectangles is presented in Fig. 1.

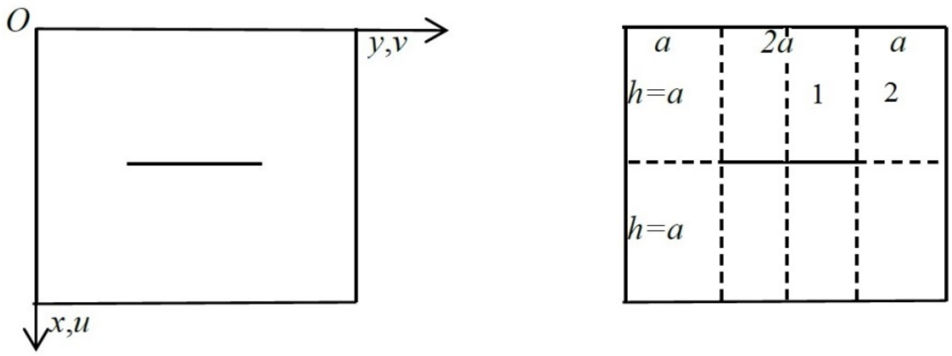

Fig. 1. Decomposition of the plate with the crack.

The boundary conditions on the sides of the contiguous rectangles (dashed lines) can be any but for this problem the displacements and normal and shear stresses should be equal. And on the sides belonging to the boundary of the original structure (solid lines) the boundary conditions should be equal to ones given on the original structure.

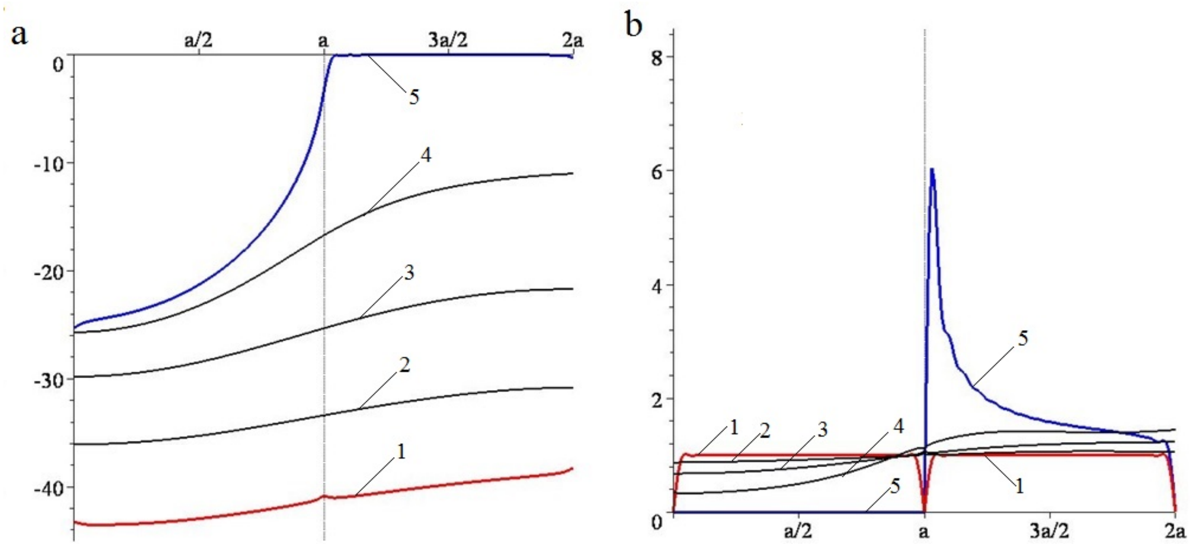

Fig. 2. Displacements uA_{22}/qh (a) and normal stresses σ_x/q (b) in the horizontal cross-sections.

4. Computational experiments

The analysis of the plate from a graphic epoxide (its geometry see Fig. 1) stretched by a uniformly distributed normal loads of intensity q applied to its sides $x = 0, 2h$. Technical elastic constants of the material are as follows: $E_x = 181GPa$, $E_y = 10.34GPa$, $G = 7.17GPa$, $\nu_x = 0.016$, $\nu_y = 0.28$. Elastic constants are expressed through technical ones by the following formulas: $A_{12} = \left(E_x\nu_y/\left(1-\nu_x\nu_y\right) + E_y\nu_x/\left(1-\nu_x\nu_y\right)\right)/2$, $A_{11} = E_x/\left(1-\nu_x\nu_y\right)$, $A_{22} = E_y/\left(1-\nu_x\nu_y\right)$, $A_{66} = G$. Since the problem of stretching a plate with a crack is symmetric with respect to the lines of the plate passing through the center and parallel to the axes Ox and Oy, the calculation is performed for the right upper quarter of the plate (rectangles 1 and 2 in Fig. 1).

Figure 2 shows the graphs of dimensionless displacement uA_{22}/qh and normal stresses σ_x/q (*b*) in the horizontal cross-sections $x = 0, h/4, h/2, 3h/4, h$. The graph number 1 corresponds the section $x = 0$ and the graph number 5 corresponds the section $x = h$, and the other graphs correspond to the intermediate sections. It can be seen that at the crack tip the normal stress σ_x/q tends to a large value.

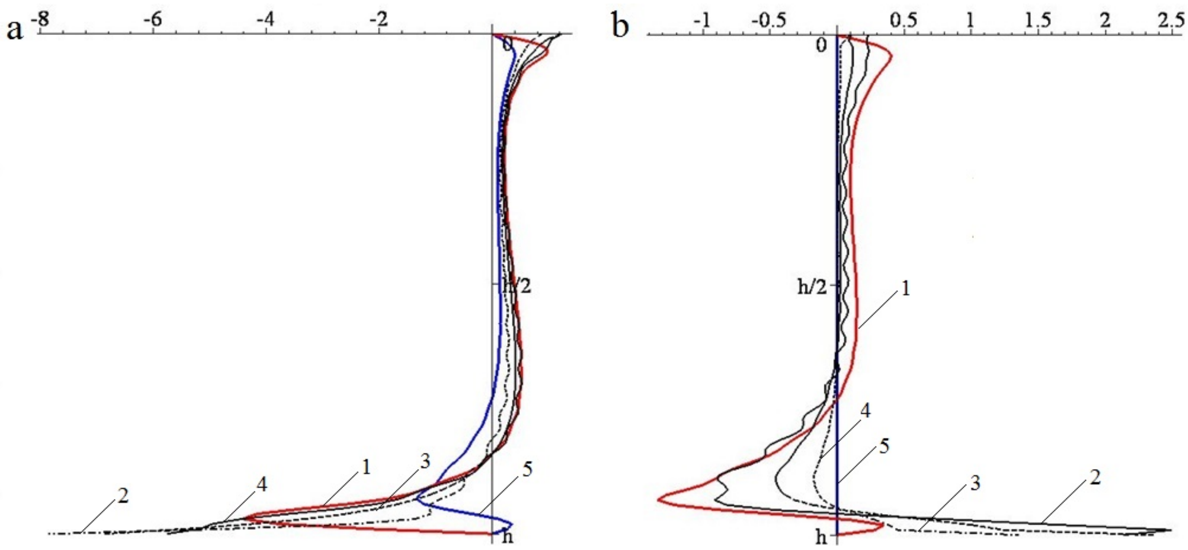

Fig. 3. Normal stresses σ_y/q in the vertical cross-sections above the crack (a) and to the right of the crack (b).

Figure 3 shows normal dimensionless stresses σ_y/q in the vertical cross-sections $y = a(i-1)/4$, $i = 1,\ldots,5$ of the simple bodies 1 (a) and 2 (b). It can be seen that the upper shore of the crack is compressed that can lead to local buckling of the plate in the vicinity of the crack.

5. Conclusion

The general solution constructed for an elastic rectangle was successfully used for analyzing the stress and strain state of an orthotropic rectangle clamped on two opposite edges. It's not difficult using the software designed in Maple to analyze stresses and displacements in a rectangular with other boundary conditions, e.g. free-supported two opposite edges, shear loading and so on.

On the base of this general solution an algorithm of analytical decomposition to analyze a geometrically complex elastic structures composed from a number of rectangles was developed. Its application for computing a stretch of an orthotropic rectangle with a crack showed its effectiveness to compute elastic structures with singularities. In conclusion, it should be noted that the MIF use nowadays in the superposition method [10] to analyze 3D single- and multi-layer thick plates.

References

[1] Y. Tokovyy, C.-C. Ma // *International Journal of Engineering Science* **92** (2015) 1.
[2] G. Vitucci, G. Mishuris // *European Journal of Mechanics, A/Solids* **65** (2017) 195.
[3] P. Samea, M. Eskandari, S.F. Ahmadi // *Applied Mathematical Modelling* **52** (2017) 458.
[4] V.V. Meleshko, A.P.S. Selvadurai // *Journal of Engineering Mathematics* **46(3-4)** (2003) 19.
[5] M.D. Kovalenko, T.D. Shulyakovskaya // *Mechanics of Solids* **46(5)** (2011) 721.
[6] A.P. Kerzhaev, M.D. Kovalenko, I.V. Menshova, In: *ECCOMAS Congress 2016 - Proceedings of the 7th European Congress on Computational Methods in Applied Sciences and Engineering.* Vol. 2, p. 3107.
[7] Z. Zhuravlova // *Applied Mathematics and Mechanics (English Edition)* **38(7)** (2017) 935.
[8] A.V. Matrosov, In: *2015 International Conference on "Stability and Control Processes" in Memory of V.I. Zubov, SCP 2015 – Proceedings 7342156*, p. 414.
[9] V.V. Meleshko, Y.V. Tokovyy // *Journal of Engineering Mathematics* **78(1)** (2013) 143.
[10] A.V. Matrosov, G.N. Shirunov, In: *2014 International Conference on Computer Technologies in Physical and Engineering Applications, ICCTPEA 2014 – Proceedings 6893300 (IEEE Catalog number CFP14BDA-USB, 2014)*, p. 96.