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# STRESS-STRAIN STATE IN THE CORNER POINTS OF A CLAMPED PLATE UNDER UNIFORMLY DISTRIBUTED NORMAL LOAD

**Alexander V. Matrosov[1*], Vladislav A. Suratov[1]**

[1]Saint Petersburg State University, 7-9, Universitetskaya nab.,St. Petersburg, 199034, Russia

*e-mail: avmatrosov@mail.ru

**Abstract.** The bending of a rectangular clamped thin plate under the uniformly distributed transverse load is considered. The solution of the Sophie Germaine equation is constructed by the method of initial functions (MIF). On two opposite sides the boundary conditions are satisfied exactly. Then, on the two remaining ones, the boundary conditions are satisfied approximately by the collocation method. The results of calculations of the stress-strain state at the corner points of the plate are given.

**Keywords:** method of initial functions; bending of a plate clamped; corner points.

## 1. Introduction

The history of building analytical solutions for the problem of bending of a clamped thin plate is dated over 140 years and a very good review of various approaches to this problem is presented in [1]. However this theme continues to attract attention of modern scientists. Various analytical approaches to get exact or approximate solution for the clamped thin elastic plane in the context of solution of the biharmonic equation appear nowadays [2-7]. These investigations have a special attention in getting such solution which can model a behavior of the clamped bending plate in the neighborhood of the corner point. The results presented in this paper are based on the solution received by the method of initial function. It should be noted that under the approach presented the solution satisfies exactly the differential equation of bending of the plate and also boundary conditions on the two opposite sides of the plate, while boundary conditions on other pair of the plate's sides are satisfied approximately. So it can judge the accuracy of the solution built on the accuracy of satisfying the boundary conditions on this pair of sides.

## 2. MIF solution

Consider an isotropic rectangular clamed thin plate in the rectangular coordinate system 0xy with original in the center of the plate $-a/2 \le x \le a/2$, $-b/2 \le y \le b/2$ with a thickness $\delta << \max(a,b)$. A deflection $w$ of the middle surface of a thin plate is defined by the biharmonic differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \tag{1}$$

where $q$ is an intensity of a normal load subjected to the upper plate's surface, $D = E\delta^3 / 12\left(1-\nu^2\right)$ is a cylindrical rigidity, $E$ is a modulus of elasticity and $\nu$ is a Poisson's ratio. The boundary conditions are as follows

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$$
\begin{aligned}
&x=\pm a/2: \quad w(\pm a/2,y)=\theta_x(\pm a/2,y)=0,\\
&y=\pm b/2: \quad w(x,\pm b/2)=\theta_y(x,\pm b/2)=0.
\end{aligned} \tag{2}
$$

In (2) $\theta_x(\pm a/2,y)$, $\theta_y(x,\pm b/2)$ are the angles of rotation respectively of the sections $x=\pm a/2$ and $y=\pm b/2$.

The solution of the differential equation (1) under the assumption that it is regarded as an ordinary differential equation with respect to the variable $x$ with the symbolic parameter of differentiation with respect to the variable $y$ can be received by the MIF in the operator form

$$
\begin{aligned}
w(x,y)=L_{ww}(\beta,x)w^0(y)+L_{w\theta}(\beta,x)\theta_x^0(y)+L_{wM}(\beta,x)M_x^0(y)+\\
+L_{wV}(\beta,x)V_x^0(y)+w_{part},
\end{aligned} \tag{3}
$$

where $w^0(y)$, $\theta_x^0(y)$, $M_x^0(y)$, $V_x^0(y)$ are respectively the deflection, the angle of rotation, the bending moment, the generalized shearing force on the initial line $x=0$ (so called the initial functions), $L_{ww}(\beta,x)$, $L_{w\theta}(\beta,x)$, $L_{wM}(\beta,x)$, $L_{wV}(\beta,x)$ are the MIF operator-functions, $\beta$ is a symbol of differentiation in the respect of the variable $y$ and $w_{part}$ is a partial solution of the nonhomogeneous equation (1).

The MIF operators in (3) have the following form [8]

$$
\begin{aligned}
&L_{ww}(x)=\cos(\beta x)-1/2\sin(\beta x)x\beta\nu+1/2\sin(\beta x)x\beta,\\
&L_{w\theta}(x)=(\nu+1)\sin(\beta x)/(2\beta)+(-\nu/2+1/2)\cos(\beta x)x,\\
&L_{wM}(x)=-\sin(\beta x)x/(2D\beta),\\
&L_{wV}(x)=(\cos(\beta x)\beta x-\sin(\beta x))/(2D\beta^3).
\end{aligned}
$$

The partial solution is taken as $w_{part}=qx^4/(24D)$.

Since the problem is symmetric about the axes *Ox* and *Oy* it is sufficient to perform the analysis of the plate in one of the quarters of the coordinate system, for example, in the first one. In this case the boundary conditions will be as follows

$$
\begin{aligned}
&x=0,a/2: \quad \theta_x(0,y)=\theta_x(a/2,y)=0, \quad V_x(0,y)=w(a/2,y)=0,\\
&y=0,b/2: \quad \theta_y(x,0)=\theta_y(x,b/2)=0, \quad V_y(x,0)=w(x,b/2)=0.
\end{aligned} \tag{4}
$$

Here $\theta_y$ and $V_y$ are respectively the angle of rotation and the generalized shearing force on the sections $y=const$ of the plate.

Assuming in (1) that $\theta_x^0$ and $V_x^0$ are zero (according with the boundary conditions (4)), calculating the displacement and the rotation angle on the side $x=a/2$ and setting them equal to zero, a system of differential equations for finding the unknown initial functions $w^0$, $M_x^0$ will be obtained

$$
\begin{aligned}
&L_{ww}\big|_{x=\frac{a}{2}}w^0+L_{wM}\big|_{x=\frac{a}{2}}M_x^0=-\frac{q}{384D}a^4,\\
&L_{\theta_x w}\big|_{x=\frac{a}{2}}w^0+L_{\theta_x M}\big|_{x=\frac{a}{2}}M_x^0=-\frac{qa^3}{24D}.
\end{aligned} \tag{5}
$$

The solution of (5) is a sum the solutions of the homogeneous system $w_{hom}^0(y)$, $M_{x,hom}^0(y)$ and the partial solutions $w_{part}^0(y)$, $M_{x,part}^0(y)$ of the nonhomogeneous system. The homogeneous solutions can be received using one function $\phi(y)$

$$w^0_{hom}(y) = \left[\frac{\sin(\beta/2)}{\beta}\right]\phi(y),$$
$$M^0_{x,hom}(y) = \left[4\cos(\beta/2) + (1-\nu)\beta\sin(\beta/2)\right]\phi(y). \tag{6}$$

The function $\phi(y)$ satisfies an ordinary differential equation

$$\left(1+\frac{\sin(\beta)}{\beta}\right)\phi(y) = 0. \tag{7}$$

The common integral of the equation (7) has a form

$$\phi(y) = \sum_{i=1}^{\infty} C_i \exp(k_i y). \tag{8}$$

Here $C_i$ are arbitrary constants and $k_i$ are the roots of the characteristic transcendental equation $1+\sin(k)/k = 0$. Note that these roots are complex quantities.

The partial solutions can be taken as $w^0_{part}(y) = qa^3/384D$, $M^0_{x,part}(y) = qa^3/24D$. So the solution of the system (5) taking into account (6) and (7) is obtained in the form

$$w^0(y) = \left[\frac{\cos(\beta/2)}{\beta}\left(\sum_{i=1}^{\infty} C_i\,\mathrm{e}^{k_i y}\right)\right] + \frac{q}{384D}a^3,$$
$$M^0_x(y) = \left[4\cos(\beta/2) + (1-\nu)\beta\sin(\beta/2)\right]\left(\sum_{i=1}^{\infty} C_i\,\mathrm{e}^{k_i y}\right) + \frac{qa^3}{24D}. \tag{9}$$

Substituting (9) into (3) with performing the operations of differentiation with respect to the variable $y$ the representation of the function $w(x,y)$ containing arbitrary constants $C_i$ is received. Note that solution (9) satisfies exactly the biharmonic differential equation (1) and the boundary conditions (4) on the sides $x = 0, a/2$. For satisfying the boundary conditions (4) on the sides $y = 0, b/2$ of the plate the collocation method is used.

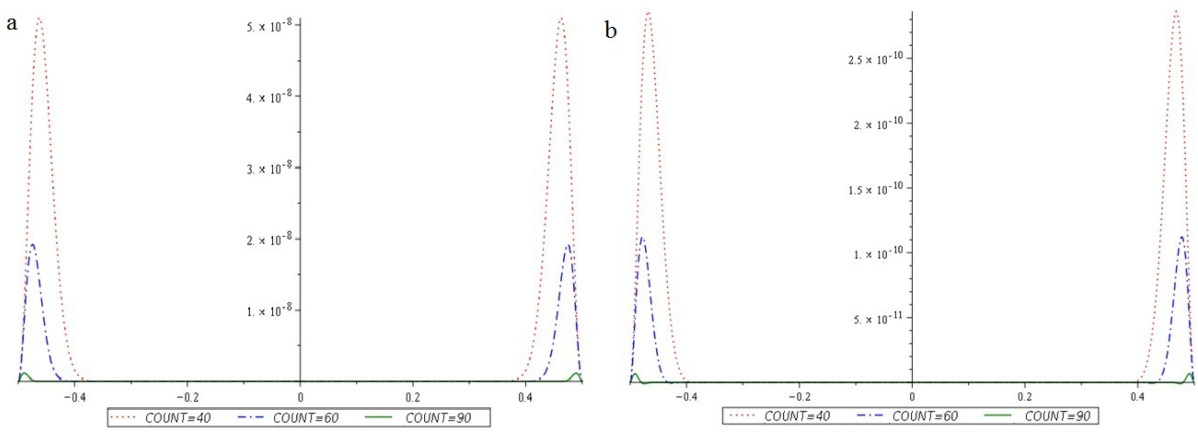


**Fig. 1.** The dimensionless displacement $wD$ (a) and the rotation angle $\theta_y$ (b) along the side $y = b/2$ (b).

For this the displacement $w(x,b/2)$, the rotation angles $\theta_y(x,b/2)$, $\theta_y(x,0)$ and the shear force $V_y(x,0)$ on the indicated sides are evaluated using (3) taking into account the expressions obtained for all initial functions. Then the real parts of the expressions obtained (they are complex) are calculated. After this using collocation method for the components of the stress-strain state on the sides $y = 0, b/2$ a linear algebraic system for determining coefficients $C_i$ is built. The number of the collocation points should be such that the resulting system of linear algebraic equations with respect to a finite number of arbitrary constants $C_i$ stored in the solution (9) turned out to be closed. Have solved this system an approximate analytical representations for all components of the stress-strain state of a rigidly clamped plate will be obtained.

### 3. Numerical results

Using this approach the analysis of a square isotropic plate with dimensions $a = b = 1m$ has been performed ($E = 2\cdot10^5 MPa$, $\nu = 0.3$, $q = 1N/m^2$).

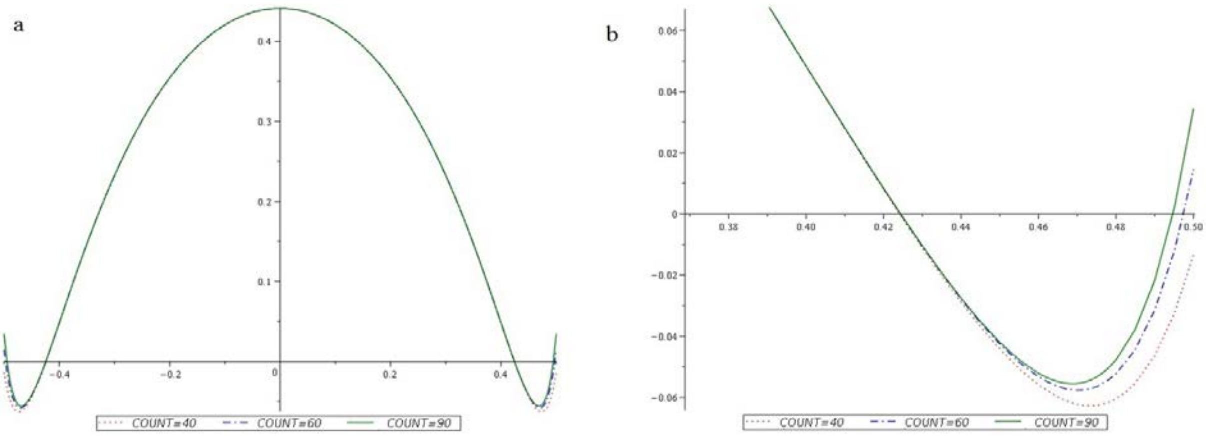


**Fig. 2.** The generalized shear force $V_y$ on the side $x = a/2$ (a); in a neighborhood of a corner point $(a/2, b/2)$ (b).

As noted earlier our solution satisfies exactly as the biharmonic equation of the problem as the boundary conditions on the sides $x = 0, a/2$. Thus in order to judge its accuracy it is necessary to check meeting the boundary conditions on the sides $y = 0, b/2$. Fig. 1 shows the graphics of the dimensionless displacement $wD$ and the rotation angle $\theta_y$ (b) along the side $y = b/2$ with different number of roots $k_i$ of the transcendental equation (40, 60, 90). Note that the maximum value of the displacement is in the center $x = y = 0$ of the plate and its dimensionless value is equal to 0.001265 and the maximum value of the angle of rotation is of the order $10^{-7}$. So the boundary conditions on this side can be considered as equal to zero.

Fig. 2 presents the graphics of the generalized shear force $V_y$ along the side $y = b/2$. It should be noted that its maximum value at the center of the indicated side coincides with calculations by other methods, for example, [2, 5, 6]. However, the behavior in the vicinity of the angular point differs somewhat from that of other authors. In [2] the value of this force in the corner point is assumed to be zero, whereas in [5] this assumption is not made and the computed value coincides with the solution presented in this paper when 40 roots of the presentation (8) are taken into account.

The general solution in [5] is constructed in the form of a series of eigenfunctions of the operator of the problem, and calculations were performed with retention of 37 eigenfunctions. However, the results are not given when retaining a larger number of eigenfunctions. We expect that the graphs will behave similarly as graphs taking into account 60 and 90 roots in Fig. 2 and won't be equal to zero.

## 4. Conclusion

The results are in agreement with other studies but for the generalized shear force there are differences in the neighborhood of the corner point.

Here one of the possible methods for calculating an elastic system using MIF is presented. In combination with the superposition method, the MIF is used to solve complex elastic systems [9, 10] with satisfaction of arbitrary boundary conditions.

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