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TRANSMISSION OF ELASTIC WAVES THROUGH AN INTERFACE BETWEEN DISSIMILAR MEDIA WITH RANDOM AND PERIODIC DISTRIBUTIONS OF STRIP-LIKE MICRO-CRACKS

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Abstract. The present work investigates wave propagation through a damaged interface between two elastic media. Dynamic behaviour of the deterioration or damage of an interface is described using a random distribution of strip-like micro-cracks of different sizes, a periodic array of strip-like cracks or via a distributed spring model. The wave-field scattered by cracks is calculated using a boundary integral equation method. The spring model assumes introduction of the spring boundary conditions, where stresses are proportional to the displacement jump with the proportionality given by the spring stiffness. Components of the spring stiffness matrix are defined in terms of the concentration of the defects, their typical size and the elastic properties of the contacting materials. Numerical analysis of reflection and transmission for the considered models of the damaged interface is provided.

Keywords: imperfect contact; crack; interface; spring model; elastic wave; periodic array; random distritution.

1. Introduction

The propagation of elastic waves through an interface is usually assumed with perfect contact, i.e. the displacement and traction are continuous at the interface. If the interface is damaged in some way this is no longer true and the boundary conditions have to be modified. The simplest case is that of an open crack, but in many cases, more diffuse damage occurs. This can for instance be in the form of micro-cracks. This situation has in the past been modelled by deriving a (distributed) spring boundary condition, meaning that the traction is continuous while the jump in displacement is proportional to the traction [1, 2]. The proportionality factor (the spring constant) can be derived in various ways, the more fundamental approach is to derive it from a random distribution of small cracks [3 – 6].

Moreover, a periodic distribution of cracks has been assumed, and at low frequencies this can be expected to work well also [7 – 9]. In previous derivations of spring boundary conditions for a random distribution of cracks [3 – 6], it is assumed that the cracks are all of the same size. In reality, this is an oversimplification, and it is interesting to investigate the effects of a size distribution on the spring constant and this is thus the primary aim of the present contribution. As a comparison, also a periodic distribution of cracks is considered. The investigations are restricted to in-plane elastic waves (2D vector problem).

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2. Wave propagation through an interface with a distributed spring

Consider the propagation of time-harmonic waves with the angular frequency ω through an interface between two dissimilar elastic isotropic half-spaces. Cartesian coordinates $x = (x_1, x_2, x_3)$ are introduced in such a way that the interface is situated in the plane $x_3 = 0$. The material properties of two media V_1 $(x_3 < 0)$ and V_2 $(x_3 > 0)$ are determined by the mass densities ρ_j and Lamé constants λ_j and μ_j or by velocities $v_{j\mathrm{L}} = \sqrt{(\lambda_j + 2\mu_j)/\rho_j}$ and $v_{j\mathrm{T}} = \sqrt{\mu_j/\rho_j}$ of longitudinal and transverse waves, respectively. Here the subscript $j = 1, 2$ corresponds to the lower V_1 and upper V_2 half-spaces. The displacement vector $\boldsymbol{u}_j$ obeys the Lamé equation:

$$k_{j\mathrm{L}}^{-2}\,\nabla\nabla\cdot\boldsymbol{u}_j - k_{j\mathrm{T}}^{-2}\,\nabla\times(\nabla\times\boldsymbol{u}_j) + \boldsymbol{u}_j = 0$$

written in terms of wave numbers $k_{j\mathrm{L}} = \omega/v_{j\mathrm{L}}$ and $k_{j\mathrm{T}} = \omega/v_{j\mathrm{T}}$. Stresses and displacements are related by Hooke's law, written via the Kronecker delta δ_{im}:

$$\sigma_{im} = \lambda\delta_{im}\,\mathrm{div}\,\mathbf{u} + \mu\left(\frac{\partial u_i}{\partial x_m} + \frac{\partial u_m}{\partial x_i}\right).$$

It is common to assume perfect contact boundary conditions at interfaces so that stresses and displacements are continuous. A more general type of boundary condition is the distributed spring or spring boundary conditions, in which the traction vector is a product of a diagonal stiffness matrix $\boldsymbol{\kappa}$ and the crack opening displacement (COD) vector [1]:

$$\boldsymbol{\tau}_1^{\mathrm{in}}(\mathbf{x}) = \boldsymbol{\tau}_2^{\mathrm{in}}(\mathbf{x}) = \boldsymbol{\kappa}\left(\mathbf{u}_1^{\mathrm{in}}(\mathbf{x}) - \mathbf{u}_2^{\mathrm{in}}(\mathbf{x})\right). \tag{1}$$

Spring stiffness can be frequency dependent, which usually implies complex values of stiffness and introduction of energy dissipation into the model [10]. The real part of $\boldsymbol{\kappa}$ gives a spring force, while imaginary part corresponds to a viscous damping force. The values of the elements of the spring stiffness matrix indicate the degree of damage of an interface from the perfect contact $(\kappa_{ii} \to \infty)$ to the absence of contact at all $(\kappa_{ii} = 0)$. For strip-like delamination, i.e. in the case of in-plane motion, the diagonal matrix $\boldsymbol{\kappa} = \mathrm{diag}\{\kappa, 0, \kappa\}$ has two equal non-zero elements [4, 8].

The wave-fields in the media for normally incident longitudinal $(s = \mathrm{L})$ and transverse $(s = \mathrm{T})$ waves can be represented in the following form:

$$\boldsymbol{u}_s^{\mathrm{in},\kappa} = \begin{cases} \boldsymbol{p}_s\left(\mathrm{e}^{\mathrm{i}k_{1s}x_3} + R_s^-(\kappa)\,\mathrm{e}^{-\mathrm{i}k_{1s}x_3}\right), & x_3 < 0 \\ \boldsymbol{p}_s T_s^-(\kappa)\,\mathrm{e}^{\mathrm{i}k_{2s}x_3}, & x_3 > 0. \end{cases} \tag{2}$$

The amplitude reflection and transmission coefficients

$$R_s^-(\kappa) = \frac{k_{1s}c_{1s} - k_{2s}c_{2s} - k_{1s}c_{1s}k_{2s}c_{2s}\cdot\frac{1}{\kappa_s}}{\mathrm{i}k_{1s}c_{1s} + k_{2s}c_{2s} - k_{1s}c_{1s}k_{2s}c_{2s}\cdot\frac{1}{\kappa_s}},$$

$$T_s^-(\kappa) = \frac{2\,k_{1s}c_{1s}}{\mathrm{i}k_{1s}c_{1s} + k_{2s}c_{2s} - k_{1s}c_{1s}k_{2s}c_{2s}\cdot\frac{1}{\kappa_s}} \tag{3}$$

are calculated for wave field (2) in accordance with boundary conditions (1); $c_{j\mathrm{L}} = \lambda_j + 2\mu_j$, $c_{j\mathrm{T}} = \mu_j$. Here $\boldsymbol{p}_s$ is a unit vector, giving the polarization of the incident plane wave, for the longitudinal wave $\boldsymbol{p}_\mathrm{L} = (0, 0, 1)$, while for the transverse wave $\boldsymbol{p}_\mathrm{T} = (1, 0, 0)$.

3. Wave scattering at an interface with distribution of strip-like cracks

Let us consider an infinite set $\Omega = \cup\Omega_i$ of strip-like interface cracks of length $2l_j$ with centres, situated at the points $(\chi_i, 0)_j$, see Fig. 1. At the surface of each crack the stress-free boundary conditions are assumed, while i-th the crack occupies the domain
$\Omega_i = \{|x_1 - \chi_i| < l_i,\ |x_2| < \infty, x_3 = 0\},\quad \Omega_i \cap \Omega_j = \emptyset.$

The total displacement field $\boldsymbol{u}$ in the considered media with interface cracks is a sum of the wave-field $\boldsymbol{u}^{\text{in}}$ in the absence of cracks and the scattered wave-field $\boldsymbol{u}^{\text{sc}}$. The wave-field $\boldsymbol{u}^{\text{sc}}$ scattered by the crack Ω_i has continuous stresses $\boldsymbol{\tau}^{\text{sc}}$ at the interface, while the displacement field $\boldsymbol{u}^{\text{sc}}$ has a jump. Accordingly, these wave-fields satisfy the following boundary conditions:

$$\begin{cases} \boldsymbol{u}_1^{\text{sc}} = \boldsymbol{u}_2^{\text{sc}}, \quad \boldsymbol{\tau}_1^{\text{sc}} = \boldsymbol{\tau}_2^{\text{sc}}, \quad x \notin \Omega_i, \\ \boldsymbol{\tau}_1^{\text{sc}} = \boldsymbol{\tau}_2^{\text{sc}} = -\boldsymbol{\tau}_1^{\text{in}}, x \in \Omega_i. \end{cases}$$

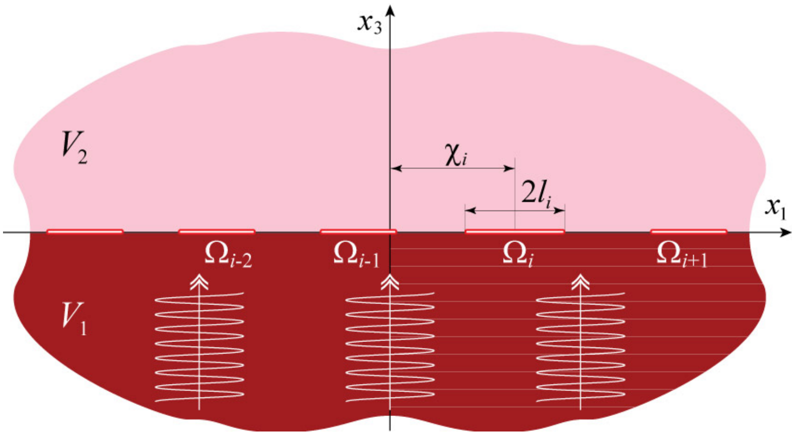

Fig. 1. Geometry of the problem: distribution of cracks.

Stochastic distribution of cracks. Consider next the propagation of the elastic plane wave through the interface $x_3 = 0$ with a random distribution of open cracks Ω_i. The random distribution is assumed to be translationally invariant. The degree of damage of an interface can be described by the crack density $C = S_{\text{dam}}/S_{\text{total}}$ of the distribution of cracks, where S_{dam} is the overall area of the cracks, S_{total} is considered damaged area. The exact scattered wave-field by a random distribution is impossible to determine. If the interaction between the defects is neglected, the ensemble averaging technique can be applied in order to describe a random distribution of cracks [3].

Far away from the interface, the total scattered field can be represented in the form of outgoing plane waves with amplitudes $P_s^{\pm}$ propagating in the $\pm x_3$ direction:

$$\langle \boldsymbol{u}_s^{\text{sc}} \rangle = \boldsymbol{p}_s \begin{cases} P_s^- \,\mathrm{e}^{-\mathrm{i}k_{1s}x_3}, & x_3 < 0, \\ P_s^+ \,\mathrm{e}^{\mathrm{i}k_{2s}x_3}, & x_3 > 0. \end{cases} \tag{4}$$

Each set of cracks is equiprobable. In this case, the crack density is

$$C = \frac{\sum_{i=1}^{N} l_i}{x_0} = \frac{N \cdot \bar{l}}{x_0},$$

where the interval $(-x_0; x_0)$ is under study and $\bar{l}$ is the average crack length.

The Betty–Rayleigh reciprocal relation is applied (see [3] for more details), and it leads to the following amplitudes of the scattered wave-field (4):

$$P_s^{\pm} = -\frac{1}{2}\left(1 \pm R_s^{-}\right) C \mathbf{p}_s \, \Delta \bar{\mathbf{u}}_s.$$

The latter is expressed in terms of the reflection coefficient R_s^{-} for the interface without cracks and the average value of the COD for a single crack:

$$\Delta \bar{\boldsymbol{u}}_s = \sum_{i=1}^{N} \int_{-l_i}^{l_i} \Delta \boldsymbol{u}_s(x_1) \mathrm{d} x_1 \cdot \left(\sum_{i=1}^{N} 2 l_i\right)^{-1}, \tag{5}$$

calculated using the boundary integral equation method [11] and the integral approach [12].

Subsequently, the ensemble average of the total transmission coefficient for the distribution of cracks becomes

$$\tilde{T}_s = T_s^{-} + P_s^{+} = T_s^{-}\left(1 - \tfrac{1}{2} \boldsymbol{p}_s \, \Delta \bar{\boldsymbol{u}}_s\right), \tag{6}$$

where $T_s^{-} = 1 + R_s^{-}$. Thus, the total transmission coefficient in the case of randomly distributed interface micro-cracks is expressed in terms of the material constants, the length of the cracks and the parameter *C*, describing the crack density.

Single strip-like crack. It is necessary to consider the wave scattering problem for a single crack to find the average value of the COD for the transmission coefficient in the form (6). To solve this two-dimensional problem for a single strip-like crack of width 2*l*, located at the interface, the wave-field can be expressed via the frequency spectrum using Fourier transform over one horizontal coordinate. The numerical solution for the scattered wave-field is performed using the representation of $\boldsymbol{u}$ through the convolution of the Green's matrix and the vector-function, composed of normal and tangential stresses at the interface.

The solution of the corresponding boundary-value problem can be formulated via the introduction of the unknown COD. Substitution of the integral representation for $\boldsymbol{\tau}^{\mathrm{sc}}$ into the inhomogeneous boundary condition at the crack surface (7) gives an integral equation for the unknown crack-opening displacement:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{L}(\alpha) \cdot \Delta \boldsymbol{U}(\alpha) \cdot \mathrm{e}^{-\mathrm{i}\alpha x_1} \, \mathrm{d}\alpha = -\boldsymbol{\tau}^{\mathrm{in},0}(x_1, 0). \tag{7}$$

A detailed description of the derivation of Green's matrices and the Fourier transform $\mathbf{L}(\alpha)$ of the kernel of the integral equation (7) was given in [12, 13].

For the use of the Bubnov–Galerkin scheme, the crack-opening displacement is expanded into a series of Chebyshev polynomials of the second kind, forming a complete set on the interval [–*l*, *l*]. Inserting this expansion into the integral equation (7) and projection on the same polynomials leads to an infinite system of linear algebraic equations.

Based on the assumption that the sizes of the cracks are much less than a wavelength of the incident wave, an asymptotic solution for a single crack can be obtained in analytic form [3]. On assuming that $k_{1\mathrm{T}} \, l \ll 1$, the average COD for strip-like crack has the following form, respectively, for incident longitudinal and transverse waves:

$$\Delta \bar{\boldsymbol{u}}_{\mathrm{L}} = \frac{\pi l H_{\mathrm{L}}}{4} \begin{pmatrix} \mathrm{i} \mathrm{g}_1 \\ \mathrm{g}_1 \end{pmatrix}, \quad \Delta \bar{\boldsymbol{u}}_{\mathrm{T}} = \frac{\pi l H_{\mathrm{T}}}{4} \begin{pmatrix} \mathrm{g}_2 \\ -\mathrm{i} \mathrm{g}_2 \end{pmatrix}, \tag{8}$$

$$\mathrm{g}_1 = \frac{\lambda_1 + 2\mu_1}{\mu_1(\lambda_1 + \mu_1)} + \frac{\lambda_2 + 2\mu_2}{\mu_2(\lambda_2 + \mu_2)}, \quad \mathrm{g}_2 = \frac{1}{\lambda_1 + \mu_1} + \frac{1}{\lambda_2 + \mu_2}, \quad H_s = \mathrm{i} \frac{c_{1s} k_{1s} c_{2s} k_{2s}}{c_{1s} k_{1s} + c_{2s} k_{2s}}.$$

Periodic distribution of cracks. The damaged interface can also be modelled as a periodic array of cracks. Let us consider plane wave scattering by a periodic array of strip-like cracks of length 2*l* with spacing *w* between the centres of adjacent cracks. The COD $\Delta \boldsymbol{u}(\boldsymbol{x})$

at the interface is, of course, the sum of the CODs $\Delta\mathbf{u}_n$, $n = 0, \pm 1, \pm 2$, for all the cracks Ω_n. Thus, the integral Equation (7), written for a single crack, can be modified as follows [14]:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{L}(\alpha)\cdot\sum_{n=-\infty}^{\infty}\Delta\boldsymbol{U}_n(\alpha)\cdot \mathrm{e}^{-\mathrm{i}\alpha x_1}\,\mathrm{d}\alpha = -\boldsymbol{\tau}^{\mathrm{in},0}(x_1, 0).$$

It follows from the periodicity of the problem that the COD for the *n*-th crack for normally incident waves can be expressed in terms of the COD for the reference crack: $\Delta\mathbf{u}_n(x_1) = \Delta\mathbf{u}_0(x_1 - wn)$.

The basic steps of solving the integral Equation (7) are described in Refs. [4, 13].

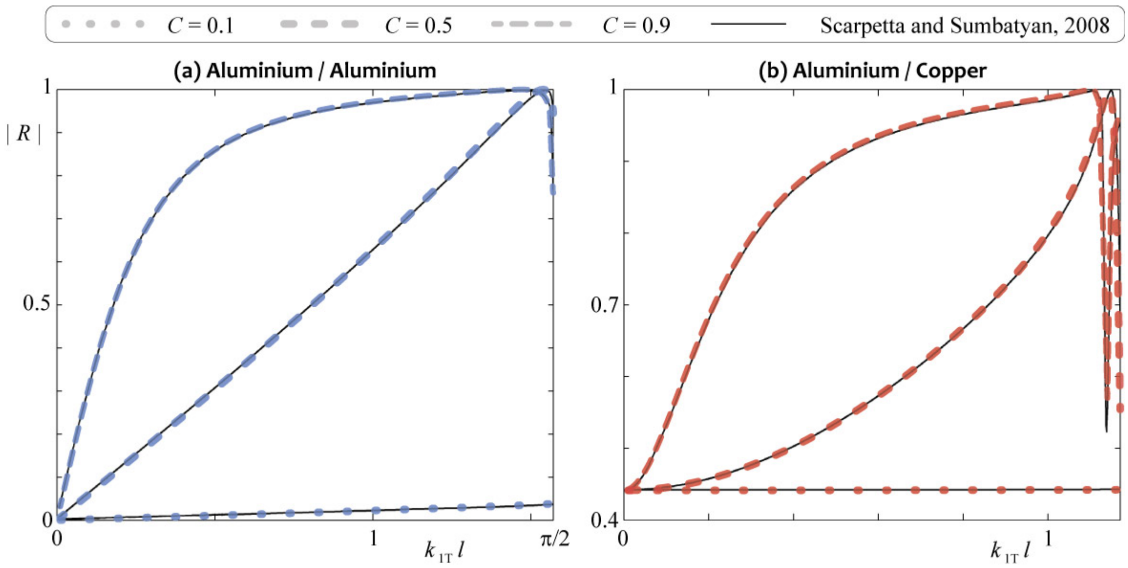

Fig. 2. Reflection coefficient |R| for the periodic array of interface cracks.

Once the crack-opening displacement is determined, it is straightforward to calculate the transmission and reflection coefficients $|T|$ and $|R|$. The reflection coefficient, calculated for a periodic array of interface cracks $(w = 2, l = 0.9)$, is depicted in Fig. 2. Two cases have been studied: distribution of cracks in homogenous space made of aluminium (Fig. 2a) and cracks between two dissimilar half-spaces of aluminium and copper (Fig. 2b). The material properties of the considered materials are given in Table 1. Fig. 2 also exhibits a good agreement of the present approach with the method described in Ref. [9].

Table 1. Properties of elastic materials.

Material	Longitudinal wave velocity, v_L [km/s]	Transverse wave velocity, v_T [km/s]	Density, ρ [kg/m^3]
Aluminium	6.42	3.04	2700
Copper	5.01	2.27	8930

Equivalency of two models: distributed spring and stochastic distribution of cracks. It has been assumed that the distributed spring model and the model of the random distribution of cracks should provide the same transmission coefficients. Thus, the transmission coefficients (3) equals to the transmission coefficient (6). From this equality, the value of spring stiffness can be determined as:

$$\kappa_s = 2H_s\left(\frac{1}{C\,\boldsymbol{p}_s\Delta\bar{\boldsymbol{u}}} - \frac{1}{2}\right). \tag{9}$$

If randomly distributed strip-like cracks are of different sizes and the asymptotic representation (8) is applied to calculate the average value of the COD, then the relation (9) gives

$$\kappa_{11} = \kappa_{33} = \frac{4(1+d)}{\pi g_1 C l^*} - \mathrm{i} H_s,$$

$$d = \sqrt{1 + \tfrac{\mathrm{i}}{2}\pi g_1 C l^*}, \quad l^* = \frac{s^2 + (\bar{l})^2}{\bar{l}}, \quad s^2 = \frac{\sum_{i=1}^N l_i^2}{N} - \left(\frac{\sum_{i=1}^N l_i}{N}\right)^2.$$

Here s^2 is the crack length variance. In the case of cracks of equal sizes ($l_i = l$) the variance $s^2 = 0$ and mean $l^* = l$, so the stiffness constant:

$$\kappa_{11} = \kappa_{33} = \frac{8}{\pi g_1 C l} - \mathrm{i} H_s.$$

4. Comparison of the approaches

Consider plane wave propagation through a damaged interface, modelled by distributed spring, periodic and random distribution of cracks. The reflection coefficients R for the three approaches in the case of incoming longitudinal plane wave are depicted in Figs. 3 and 4 for crack densities $C = 0.1$ and $C = 0.5$. The left plots correspond to a plane with damage, while right plots exhibit reflection coefficient for dissimilar media (aluminium/copper).

The reflection by the considered kinds of damages is very similar at lower frequencies. The frequency range, where the reflection coefficients are compatible, changes with crack density and elastic properties of the contacting media V_1 and V_2. Scattering by a periodic array of cracks is accompanied by occurrence of cut-off frequencies, depending on elastic moduli, and the reflection coefficient has sharp peaks close to them. Naturally, any other kind of damage does not provide such peaks, related to the cut-off frequencies. Nevertheless, plane waves, scattered by a random distribution, have amplitudes quite close to those, reflected by a periodic array at the frequencies below the first cut-off frequency. Difference between random distribution and spring model arises due to the CODs for a random distribution are calculated for each frequency, while frequency-independent asymptotic solution is used for the spring stiffness in the case of a distributed spring.

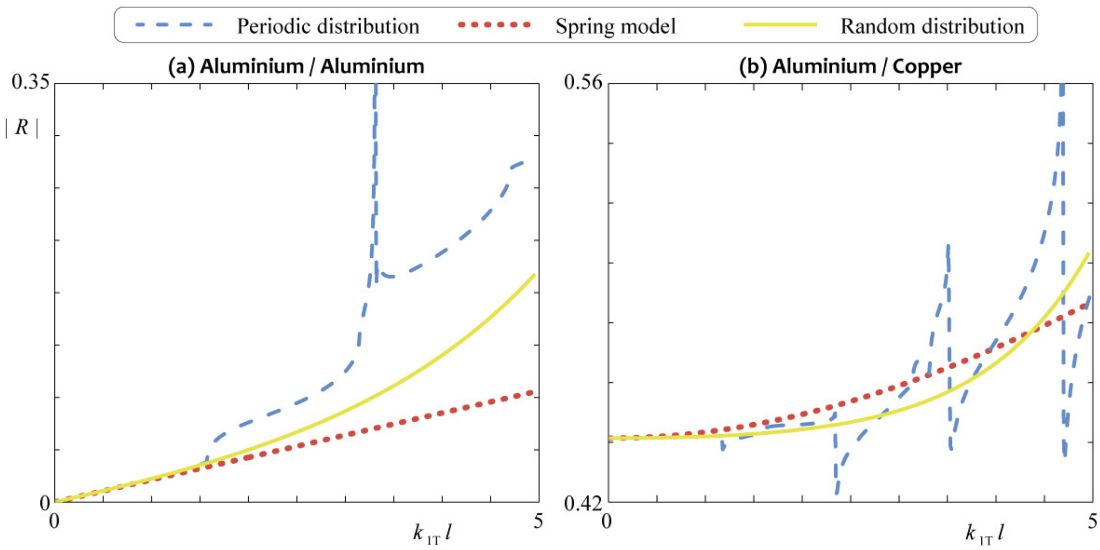

Fig. 3. Reflection coefficient R for longitudinal wave, scattered by damaged interface ($C = 0.1$).

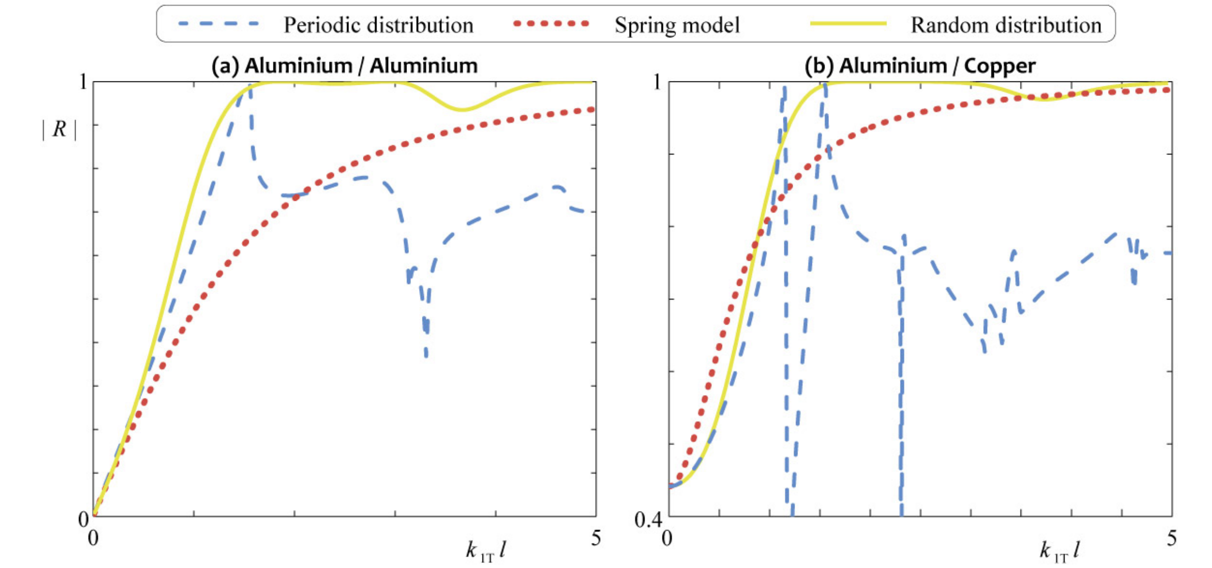

Fig. 4. Reflection coefficient R for longitudinal wave, scattered by damaged interface ($C = 0.5$).

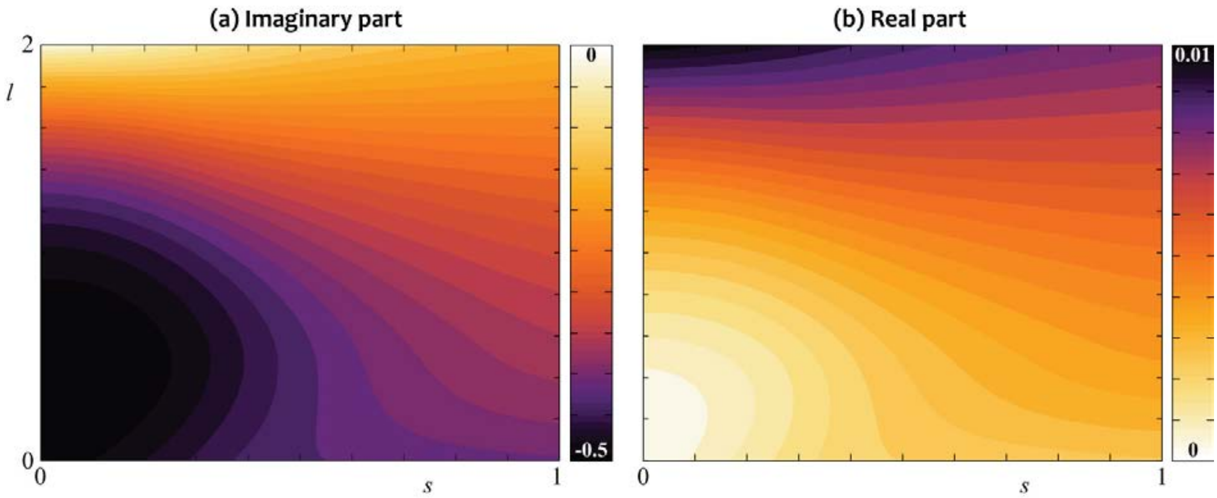

Fig. 5. Imaginary (a) and real (b) parts of the average value of the COD at $k_{1T}s = 0.2$.

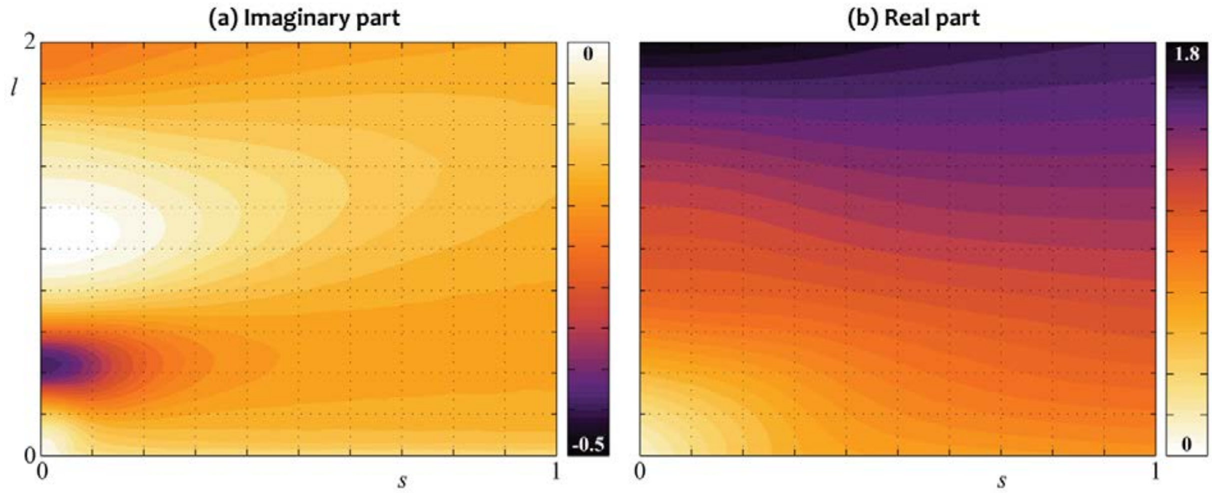

Fig. 6. Imaginary (a) and real (b) parts of the average value of the COD at $k_{1T}s = 4$.

To evaluate the case of randomly distributed cracks of various lengths, it is enough to estimate the average values of the COD for single cracks of different lengths. The coefficient H_s in the relation (9) is pure imaginary; therefore, the real part of the COD describes attenuation, while the imaginary part of the COD gives the stiffness value. A data set, consisting of $N = 10000$ cracks of lengths $l_i \in [0.01; 1.994]$, is considered. The lognormal distribution with mean value $\bar{l}$ and variance s^2 is used. The average value of the COD is calculated via (5) with the expansion of the COD into a series of the Chebyshev polynomials of the second kind. Influence of the variation of the crack sizes of strip-like interfacial cracks between two dissimilar media (Aluminium/Copper) on the averaged values of the COD is illustrated in Figs. 5 and 6, where surfaces, demonstrating averaged values of the COD, are shown. The dependence of the imaginary and real parts of the averaged COD on the variance s^2 is demonstrated by colour: greater absolute values of the real and imaginary part of the COD are depicted the darker colour.

Small mean values $\bar{l}$ correspond to the cracks of large sizes with the nearly zero probability of generation in a set. Increase of the value of $\bar{l}$ causes the probability to increase. One can see that the imaginary part of the average COD decreases with frequency, for some values of the distribution parameters to zero. The values of the real part of the average COD *vice versa* is increasing for higher frequency, however, that have a similar general behaviour regardless of the frequency of the incident wave.

5. Conclusions

The present investigation compares three different ways of modelling damage at an interface between two different media. The most detailed model is by a random distribution of micro-cracks, where also the influence of different crack lengths is taken into account. The second model is when the cracks are of the same length and periodically distributed, and the third and simplest is when the first model is used to estimate a (distributed) spring boundary condition between the two media. It is noted that the spring constants can be given explicitly by simple form. For low frequencies, the three models give a reflection from the interface which is almost the same.

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