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APPLICATION OF THE ULTRASONIC WAVES FOR DETECTION OF EXFOLIATIONS BETWEEN THE SOLID INCLUSIONS AND THE ELASTIC MATRIX OF A METAMATERIAL

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Abstract. There is developed an algorithm to determine position of possible exfoliations of a triple-periodic system of elastic inclusions from an elastic matrix of a metamaterial after its production. For this aim there is used the irradiation of each side of a cubic specimen of the metamaterial by short impulses with a tone filling with the ultrasonic longitudinal waves. The algorithm implies the experimental and the theoretical study of the propagation of a high-frequency wave through the periodic system of inclusions. The theoretical solution is constructed by methods of the geometrical diffraction theory. Based on comparison of the experimental data and the theoretical calculations there is evaluated the presence of exfoliations and their position inside the metamaterial.

Keywords: metamaterials; elastic wave propagation; elastic medium with spherical inclusions; geometrical diffraction theory.

1. Introduction

At the present time there is an intensive study of properties of the metamaterials, which are produced by introducing of solid arrays into elastic media. Such metamaterials can be used, in particular, as wave filters. In [1] there are presented some results of natural experiments on filtration properties of a triple-periodic system of solid spheres, located inside a cube with an elastic matrix made of a hardened epoxy, under propagation of high-frequency elastic waves of various frequencies. In [2 – 7] there are developed analytical and numerical methods to solve two-dimensional problems about propagation of acoustic and elastic, mainly low-frequency, waves through the periodic systems of obstacles. The three-dimensional problems of this kind are studied in [7 – 10]. A three-dimensional short-wave diffraction of acoustic waves by a periodic system of solid obstacles is studied in [11]. Double reflections and transformations of short waves are studied in [12].

2. Problem Formulation

The sample is of a cubic shape, being made of metamaterial, which consists of an elastic matrix, with rigid spherical inclusions of the same radius, whose centers are located at the nodes of a triple-periodic system with the same step along each of the three directions naturally associated with edges and faces of the cube, see Fig.1. The cube is related to a global Cartesian coordinate system $OX_1X_2X_3$, whose coordinate axes originate from a certain vortex, so that the cube is fully situated inside the first octahedron. The minimal distance of the spheres from respective faces is b. In this coordinate system the spherical boundary surfaces of the system of inclusions are defined by the following equations:

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$$\left(x_1 - m_1(2a+b)\right)^2 + \left(x_2 - m_2(2a+b)\right)^2 + \left(x_3 - m_3(2a+b)\right)^2 = a^2;\ m_1, m_2, m_3 = \overline{1, M}\ . \quad (1)$$

Let us consider the case when in the process of cube's production there may arise some exfoliations of the matrix material from some parts *S*(*m*1, *m*2, *m*3) of the surfaces of respective spherical inclusions. The problem is to detect the presence of the exfoliations and to determine their positions inside the specimen.

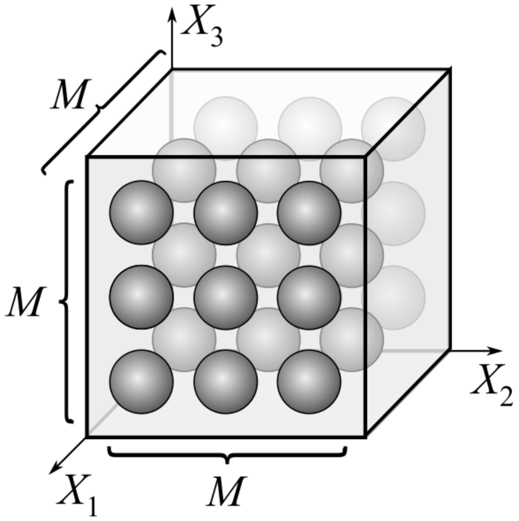

Fig.1. Triple-periodic system of 3 × 3 × 3 = 27 (M = 3) equal spherical obstacles, located inside cube with elastic matrix: the isometric view.

3. Method of solution

One of efficient methods to detect defects in the elastic media is the ultrasonic non-destructive testing. To detect exfoliations in a specimen, one can use sensors, which work on longitudinal waves of high frequencies. For each of three pairs of the cubic faces, there are performed the following experiments. Sequentially, from each side of the cube there are introduced equal impulses with a tonal filling of 5 – 10 periods of a plane high-frequency harmonic in time longitudinal elastic wave. Such a structure of the introduced impulse permits calculation of characteristics of the passed impulse in the time-harmonic regime. Since the impulse is filled with the longitudinal wave, whose speed is higher than the propagation speed of the transverse wave, it is natural that the first registered impulse on the opposite cube's face is a certain modified impulse filled of longitudinal wave. Let the results of the natural measurements of the displacements on the opposite cube's faces all are known in the full set of six experiments. On the base of these six impulses one should determine, firstly whether there is any exfoliation, and secondly what are their positions in the sample. Such a problem is related to the so called inverse problems of mathematical physics. The solution of the inverse problem is constructed on the base of the direct problem on the propagation of a plane elastic longitudinal wave through a triple-periodic system of solid spherical inclusions in the cube free of any exfoliation.

Solution of the direct problem. The solution of the direct problem is constructed by the methods of the short-wave diffraction theory of elastic waves. The discretization of the incident plane wave permits its representation by a set of point spherical sources of the longitudinal wave. Each spherical source under the discretization is represented as a set of the radial propagating rays of the longitudinal waves. The problem is reduced to the study of the short-wave diffraction of elastic waves in a local formulation, by taking into account multiple reflections of the longitudinal wave by a system of rigid spherical inclusions. The solution of the problem consists of the two following steps. On the first step there is solved a geometrical problem – to find trajectories for each ray, arrived at the opposite face. On the second step, after having found the trajectories and the points of mirror reflections, in frames of the Geometric Diffraction Theory (GDT), the algorithm determines the displacements at the passed wave over the receiving face.

First step. The resulting field of the propagating elastic longitudinal waves, over the receiving face, is a sum of the through-transmitted rays, which can be of one of the following types: (i) the rays passed across the system of spherical obstacles without any diffraction; (ii) the rays reflected from the system of obstacles or from the faces of the cube only once; (iii) the rays of multiple reflection from the system of solid spherical inclusions and from the sides of the cube.

The rays, passed through the system of inclusions without diffraction, are the rays, which propagate from the point sources of the spherical waves in the horizontal layers of thickness b between the neighbor layers of the spherical obstacles.

In the case of the gently sloping incidence to the side obstacles, which are tangential to the upper and the lower boundary planes of the elastic layer, there is a possibility to reflect only once; after that the reflected longitudinal wave falls to the receiving face.

When constructing the trajectories of multiple reflections, the diffraction field is very complex. This is connected with the fact that on the boundary surfaces of the solid inclusions and of the cube's faces, additionally to the reflections of the longitudinal waves there is also a transformation of the longitudinal wave to the transverse one, and *vice versa*. However, for the trajectories of the through-transmitted ray at the points of mirror reflections we take into account only reflection of the longitudinal wave again to the longitudinal wave. Under this condition, we take into account only those longitudinal waves, whose direction of the propagation subtends an acute angle with the direction of the initial plane wave.

Second step. On this step the solution to the problem for the displacements can be written out explicitly, at the registering point, for each of the three types of waves discussed above.

1. For the rays passing across the cube without diffraction, in the high-frequency regime of oscillations, as $k_p \to \infty$ and $k_s \to \infty$, along ray direction $\mathbf{q}$, let us write out the leading asymptotic term of the representation for the wave at the receiving point:

$$\mathbf{u}_{\mathbf{q}}^{(p)}(y) = Q_{\mathbf{q}}\mathbf{q}\frac{k_p^2}{4\pi\rho\omega^2}\frac{e^{ik_pR_0}}{R_0}\left[1 + \mathrm{O}\left(\frac{1}{k_p R_0}\right)\right], \quad R_0 = | x_0 - y |, \tag{2}$$

where $Q_{\mathbf{q}}$ is the projection of the point source $\mathbf{Q}$ at point x_0 on the direction of the incident wave $\mathbf{q}$, ρ is the mass density, λ, μ are the Lame coefficients, ω is the frequency of the oscillations, $k_p = \omega / c_p$, $k_s = \omega / c_s$, c_p, c_S are wave numbers and waves speeds, for the longitudinal and the transverse wave.

2. In [13], on the base of the physical Kirchhoff diffraction theory, there is developed an integral Somiljana's representation [14] for the displacements along the direction $x_0 - y^* - x$, in the longitudinal wave, at the wave one-time reflected from arbitrary smooth surface. Let us write out it for the radial displacement $u_r^{(p)}(x)$ (after taking all slowly varying functions out of the integral in the high-frequency approximation), in the local spherical coordinate system $r,\ \theta,\ \psi$ at point y^* of the boundary surface:

$$u_r^{(p)}(x) = \frac{Q_{\mathbf{q}}}{4\pi\mu}\, i\, \frac{k_p^2}{k_s^2} \cdot \frac{k_p}{2\pi}\, \frac{\cos\gamma^*}{L_0 L} Z(y^*)\ V_{pp}(y^*) \iint_S e^{ik_p\varphi_{pp}} dS\,, \tag{3}$$

$$u_\theta^{(p)}(x) = 0, \quad u_\psi^{(p)}(x) = 0\,, \ \varphi_{pp} = | x_0 - y | + | y - x |, \ L_0 = | x_0 - y^* |, \ L = | y^* - x |.$$

The leading asymptotic term of the diffraction integral (3) is obtained by a two-dimensional stationary phase asymptotic estimate [15]. The so obtained explicit expression corresponds to the geometrical diffraction theory:

$$u_r^{(p)}(x)=\frac{B\times Z(y^*)\times V_{pp}(y^*)\exp\left[i\,k_p\left(L_0+L\right)+\frac{\pi}{4}\left(\delta_2^{(pp)}+2\right)\right]}{\sqrt{\left(L_0+L\right)^2+2L_0L\left(L_0+L\right)\left(2H\cos^2\gamma^*+\tilde{k}\sin^2\gamma^*\right)\cos^{-1}\gamma^*+4L_0^2L^2K}},$$

$$B=\frac{Q_{\mathbf{q}}}{4\pi\mu}\frac{k_p^2}{k_s^2}. \tag{4}$$

Here $K=k_1k_2$ is the Gaussian curvature, $H=(k_1+k_2)/2$ is the average curvature of the surface at the point of mirror reflection y^*, and $\tilde{k}$ is the curvature of the normal cut of the reflecting surface by the plane of the ray x_0-y^*-x. The curvature $\tilde{k}$ is defined by Euler's formula: $\tilde{k}=k_1\cos^2\tilde{\varphi}+k_2\sin^2\tilde{\varphi}$ ($\cos\tilde{\varphi}=\frac{\cos\alpha^*}{\sin\gamma^*}$, $\sin\tilde{\varphi}=\frac{\cos\beta^*}{\sin\gamma^*}$), which represents the curvature of arbitrary normal cut in terms of its principal curvatures k_1, k_2 and angle $\tilde{\varphi}$, which is subtended by the tangential line to this normal cut with the first principal direction. In formula (3), V_{pp} is the coefficient of $p-p$ reflection. Parameter $Z(y^*)$ takes the value +1 or −1, depending in which part of the spherical surface, the point of the mirror reflection is located. Following [16], $Z(y^*)=+1$, if y^* belongs to the surface of the corresponding sphere, in the zone of braking engagement of the respective sphere with the matrix, and $Z(y^*)=-1$, if y^* belongs to the free surface $S(m_1,m_2,m_3)$ of the respective exfoliation.

3. The multiple diffraction of the high-frequency longitudinal wave is studied in frames of an integral representation. The base for this integral representation for the radial displacement of N times reflected elastic wave, forms a modification of the integral representation [17], in the reflected wave, in frames of the Kirchhoff physical diffraction theory, realized earlier in the two-dimensional case of multiple reflection [18].

If the disposition of the boundary surfaces is such that the trajectory of the ray $x_0-y_1^*-y_2^*-\ldots-y_N^*-x_{N+1}$ results in $p-p-p-\ldots-p$ reflection, then the amplitude of the radial displacement in N times reflected ray at point x_{N+1}, relatively the local spherical coordinate system $r,\ \theta,\ \psi$ at point y_N^* of the boundary surface, is represented by the Somiljana multiple integral [14] , with a certain strongly-oscillating function of many variables:

$$u_r^{(p)}\left(x_{N+1}\right)=i^N\ \frac{Q_{\mathbf{q}}}{4\pi\mu}\frac{k_p^2}{k_s^2}\left(\frac{k_p}{2\pi}\right)^N\left(\frac{k_p^2}{2k_s^2}\right)^N L_0^{-1}\prod_{n=1}^{N}Z_n(y_n^*)\ L_n^{-1}\cos\gamma_n^*\ V_{pp}\left(y_n^*\right)\times$$

$$\times\iint\limits_{S_N}\iint\limits_{S_{N-1}}\ldots\iint\limits_{S_2}\iint\limits_{S_1}e^{ik_p\varphi}dS_1\ dS_2\ldots dS_{N-1}\ dS_N\ , \tag{5}$$

$$u_\theta^{(p)}\left(x_{N+1}\right)=0,\quad u_\psi^{(p)}\left(x_{N+1}\right)=0,$$

$$\varphi_p=\left|\ x_0-y_1\ \right|+\left|\ y_1-y_2\ \right|+\ldots+\left|\ y_{N-1}-y_N\ \right|+\left|\ y_N-x_{N+1}\ \right|,$$

$$L_0=\left|\ x_0-y_1^*\ \right|,\ L_{n-1}=\left|\ y_{n-1}^*-y_n^*\ \right|,\quad n=2,\ldots,N;\ \ L_N=\left|\ y_N^*-x_{N+1}\ \right|.$$

Let us write out the leading asymptotic term for the radial displacement, which is obtained by the asymptotic estimate, as $k_p\to\infty$, of the diffraction $2N$-fold integral (5) by the multidimensional ($2N$ -fold) stationary phase method [14]:

$$u_r^{(p)}\left(x_{N+1}\right)=\frac{Q_{\mathbf{q}}}{4\pi\mu}\frac{k_p^2}{k_s^2}\prod_{n=1}^{N} Z_n(y_n^*)\cos\gamma_n^* V_{pp}\left(y_n^*\right)\frac{\exp\left\{i\left[k_p\sum_{n=0}^{N}L_n+\frac{\pi}{4}\left(\delta_{2N}^{(p)}+2N\right)\right]\right\}}{\prod_{n=0}^{N}L_n\sqrt{\left|\det\left(D_{2N}^{(p)}\right)\right|}}. \qquad (6)$$

In expression (6), $V_{pp}\left(y_n^*\right)$ are the reflection coefficients of p – wave at points of mirror reflection y_n^*, $n=1, 2, ..., N$. Parameter $Z_n(y_n^*)$ takes the values +1 or – 1, depending on the factor, in which part of the spherical surface the point of mirror reflection is located. Following [16], $Z_n(y_n^*)=+1$, if y_n^* belongs to the surface of respective sphere in the zone of its braking engagement with the matrix and $Z_n(y_n^*)=-1$, if y_n^* belongs to the part of the free surface $S(m_1,m_2,m_3)$ of respective exfoliation. Parameter $\delta_{2N}^{(p)}=\operatorname{sign} D_{2N}^{(p)}$ is the sign of the Hessian matrix $D_{2N}=(d_{ij})$, $i, j=1, 2, 3, ..., 2N$, which is a banded matrix (with the width of the band equal to seven), symmetric $d_{ij}=d_{ji}$, with the following nontrivial elements d_{ij}, $i\le j$:

$$\begin{aligned}
d_{2n-1,2n-1} &= \left(L_{n-1}^{-1}+L_n^{-1}\right)\left(1-\left(\mathbf{q}_0^{(n)},\mathbf{e}_\theta^{(n)}\right)^2\right)-2a^{-1}\left(\mathbf{q}_0^{(n)},\mathbf{e}_r^{(n)}\right),\ n=\overline{1,N}\\
d_{2n,2n} &= \left(L_{n-1}^{-1}+L_n^{-1}\right)\left(1-\left(\mathbf{q}_0^{(n)},\mathbf{e}_\varphi^{(n)}\right)^2\right)-2a^{-1}\left(\mathbf{q}_0^{(n)},\mathbf{e}_r^{(n)}\right),\ n=\overline{1,N}\\
d_{2n-1,2n} &= \left(L_{n-1}^{-1}+L_n^{-1}\right)\left(\mathbf{q}_0^{(n)},\mathbf{e}_\theta^{(n)}\right)\left(\mathbf{q}_0^{(n)},\mathbf{e}_\varphi^{(n)}\right),\ n=\overline{1,N}\\
d_{2n-1,2n+1} &= L_n^{-1}\left(\left(\mathbf{q}_0^{(n)},\mathbf{e}_\theta^{(n)}\right)\left(\mathbf{q}_0^{(n+1)},\mathbf{e}_\theta^{(n+1)}\right)-\left(\mathbf{e}_\theta^{(n)},\mathbf{e}_\theta^{(n+1)}\right)\right),\ n=\overline{1,N-1}\\
d_{2n-1,2n+2} &= L_n^{-1}\left(\left(\mathbf{q}_0^{(n)},\mathbf{e}_\theta^{(n)}\right)\left(\mathbf{q}_0^{(n+1)},\mathbf{e}_\varphi^{(n+1)}\right)-\left(\mathbf{e}_\theta^{(n)},\mathbf{e}_\varphi^{(n+1)}\right)\right),\ n=\overline{1,N-1}\\
d_{2n,2n+1} &= L_n^{-1}\left(\left(\mathbf{q}_0^{(n)},\mathbf{e}_\varphi^{(n)}\right)\left(\mathbf{q}_0^{(n+1)},\mathbf{e}_\theta^{(n+1)}\right)-\left(\mathbf{e}_\varphi^{(n)},\mathbf{e}_\theta^{(n+1)}\right)\right),\ n=\overline{1,N-1}\\
d_{2n,2n+2} &= L_n^{-1}\left(\left(\mathbf{q}_0^{(n)},\mathbf{e}_\varphi^{(n)}\right)\left(\mathbf{q}_0^{(n+1)},\mathbf{e}_\varphi^{(n+1)}\right)-\left(\mathbf{e}_\varphi^{(n)},\mathbf{e}_\varphi^{(n+1)}\right)\right),\ n=\overline{1,N-1}.
\end{aligned} \qquad (7)$$

The elements of the Hessian matrix (7) are written out for the considered case of the triple-periodic system of spherical obstacles, related to a global Cartesian coordinate system, formed by the edges of the cube, originated from a certain vortex. In the numerical treatment, to form the trajectory of multiple reflection at the mirror reflection points, it is also necessary to take into account local spherical coordinate systems. In formulae (7), $\left(\mathbf{e}_r^{(n)},\mathbf{e}_\theta^{(n)},\mathbf{e}_\varphi^{(n)}\right)$ is the orthonormal basis of a local spherical coordinate system at the mirror reflection point $y_n^*\left(x_{1n}^*,x_{2n}^*,x_{3n}^*\right)$, $\mathbf{q}_0^{(n)}=\left\{q_{01}^{(n)},q_{02}^{(n)},q_{03}^{(n)}\right\}$ is the unit vector $\mathbf{q}^{(n)}$, defining the direction of incidence of the wave in the global Cartesian system at point y_n^* of the sphere. Let us calculate the local spherical coordinates $\left(a,\theta_n,\varphi_n\right)$ of the mirror reflection point $y_n^*\left(x_{1n}^*,x_{2n}^*,x_{3n}^*\right)$, located over the boundary surface of respective sphere, with certain parameters k_1,k_2,k_3:

$$\theta_n=\theta_n^{(k_1,k_2,k_3)}=\arccos((x_{3n}^*-2k_3a)/a), \qquad 0\le\theta_n\le\pi$$

$$\varphi_n=\varphi_n^{(k_1,k_2,k_3)}=\begin{cases}\arccos((x_{1n}^*-2k_1a)/a), & x_{2n}^*-2k_2a\ge 0\\ \pi+\arccos((x_{1n}^*-2k_1a)/a), & x_{2n}^*-2k_2a<0\end{cases} \qquad 0\le\varphi_n\le\pi$$

Let us write out the expressions for the local basis vectors $\mathbf{e}_r^{(n)},\mathbf{e}_\theta^{(n)},\mathbf{e}_\varphi^{(n)}$ in terms of the global basis vectors $\mathbf{i},\mathbf{j},\mathbf{k}$ at point y_n^*:

$$\mathbf{e}_r^{(n)} = \mathbf{i}\sin\theta_n\cos\varphi_n + \mathbf{j}\sin\theta_n\sin\varphi_n + \mathbf{k}\cos\theta_n,$$
$$\mathbf{e}_\theta^{(n)} = \mathbf{i}\cos\theta_n\cos\varphi_n + \mathbf{j}\cos\theta_n\sin\varphi_n - \mathbf{k}\sin\theta_n,$$
$$\mathbf{e}_\varphi^{(n)} = -\mathbf{i}\sin\varphi_n + \mathbf{j}\cos\varphi_n.$$

The local coordinates of the direction of wave incidence at point y_n^* are defined by the relations:

$$q_r^{(n)} = \left(\mathbf{q}_0^{(n)}, \mathbf{e}_r^{(n)}\right), \quad q_\theta^{(n)} = \left(\mathbf{q}_0^{(n)}, \mathbf{e}_\theta^{(n)}\right), \quad q_\varphi^{(n)} = \left(\mathbf{q}_0^{(n)}, \mathbf{e}_\varphi^{(n)}\right).$$

For the reflected ray

$$q_r^{(n+1)} = -q_r^{(n)}, \quad q_\theta^{(n+1)} = q_\theta^{(n)}, \quad q_\varphi^{(n+1)} = q_\varphi^{(n)}.$$

The coordinates of the reflected ray $\mathbf{q}^{(n+1)} = \left\{q_1^{(n+1)}, q_2^{(n+1)}, q_3^{(n+1)}\right\}$ in the global coordinate system:

$$q_1^{(n+1)} = -q_r^{(n)}\sin\theta_n\cos\varphi_n + q_\theta^{(n)}\cos\theta_n\cos\varphi_n - q_\varphi^{(n)}\sin\varphi_n,$$
$$q_2^{(n+1)} = -q_r^{(n)}\sin\theta_n\sin\varphi_n + q_\theta^{(n)}\cos\theta_n\sin\varphi_n + q_\varphi^{(n)}\cos\varphi_n,$$
$$q_3^{(n+1)} = -q_r^{(n)}\cos\theta_n - q_\theta^{(n)}\sin\theta_n.$$

If the exfoliations are totally absent in the specimen, then the deviation of the theoretical prediction from the practical measurements is insignificant. In the case of a significant difference of these results, to find spatial location of the exfoliation, we analyze and compare the ranges of the deviation for the through-transmitted wave, over each of the six faces of the registered impulse.

Solution of the inverse problem. The direct problem results in the values of the displacement amplitudes in the registered longitudinal wave, over each point of the opposite face. Then we construct over this face some two-dimensional domains, which indicate the difference between the theoretical calculations and the experimental data. The boundary contours of these domains form the directional lines to the cylindrical surfaces with a generatrix orthogonal to the receiving face. Generally, with each such a face there may be connected even a set of such surfaces. Then we evaluate common domains of these cylinders' intersection. In this way one can find the location of the largest exfoliations of the spherical inclusions from the elastic matrix, which can significantly influence the physical properties of the metamaterial. To improve the precision of the location and the form of each such an exfoliation domain, one should use the solution of the direct problem on the through-transmission of the ultrasonic impulse by taking into account the location and the exfoliation' geometry already stated at this stage. This is attained by applying formulae (2), (4), (6) with the values of parameter $Z_n(y_n^*)$ at each point of mirror reflection, taking into account the detected exfoliation zones on the preceding step. Further, this iteration process can be continued up to an acceptable result. With so doing, at each iteration step the results of the calculation are compared with experimental data. Further improvement of the location and the shape of the exfoliations can be performed by applying over each face of the cube some sloped sensors, with the angle of incidence of the longitudinal wave equal to 45 degrees.

4. Conclusion

In frames of the geometrical short-wave diffraction theory, there is developed an algorithm to determine presence and location of possible exfoliations of the periodic system of solid inclusions from the elastic matrix of the metamaterial. The algorithm is based on the irradiation from each face of the cubic specimen the short impulses with a tone filling with the ultrasonic longitudinal waves. The algorithm implies experimental and theoretical study of the through-transmission of the high-frequency wave across the periodic system of inclusions. The theoretical solution is constructed by methods of the geometrical diffraction theory. On the comparison of experimental data and the theoretical calculations, there is detected the presence of the exfoliations and their location, with further refinement of the exfoliations' shape, in frames of the proposed iteration process.

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