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A BOUNDARY ELEMENT APPROACH FOR 3D TRANSIENT DYNAMIC PROBLEMS OF MODERATELY THICK MULTILAYERED ANISOTROPIC ELASTIC COMPOSITE PLATES

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Abstract. A Laplace-domain boundary element approach for transient dynamic analysis of three-dimensional (3D) moderately thick multilayered (piecewise homogeneous) anisotropic linear elastic composite plates is presented. The boundary element formulation is based on the system of weakly singular displacement boundary integral equations. The spatial discretization is based on collocation method and mixed representation of geometry and boundary functions. To obtain time-domain solutions, the Convolution Quadrature Method with the Runge-Kutta method as an underlying time stepping method is used as a numerical technique for inverse Laplace transform. To improve the computational efficiency of the boundary element formulation a parallelization scheme is implemented. Boundary element results for the test example are provided to validate the proposed approach.
Keywords: multilayered plates; anisotropic linear elasticity; boundary element method; dynamic analysis.

1. Introduction

Anisotropic elastic materials are extensively used in the design of high performance structural elements such as multilayered composite plates. Therefore, for their correct utilization it is essential to assess accurately their response due to various transient dynamic loadings.

For the transient analysis of the multilayered anisotropic elastic composite plates with the complicated geometrical structures and complex boundary conditions, it is necessary to employ a numerical simulation method like Finite Element Method (FEM) or Boundary Element Method (BEM) since finding analytical closed form solutions is often impossible. Although throughout its development history, BEM in various formulations has been successfully applied to elastodynamic problems [1, 2], literature on BEM implementations for three-dimensional anisotropic elastic plates is somewhat scarce. Main reason for that is the lack of the closed form expressions of the dynamic anisotropic fundamental solutions. In the framework of the Dual Reciprocity Boundary Element Method (DRBEM) [3] only the static anisotropic fundamental solutions are employed even in the situations when the boundary integral equations contain domain integral, such as if non-zero body force is present. Though several DRBEM formulations were proposed (e.g. Gaul et al. [4]), the necessity of additional collocation nodes inside the domain and uncertainty of choice of the approximation functions still remain.

In this paper, the multi-domain boundary element approach is used for the transient dynamic analysis of the three-dimensional problems of moderately thick multilayered

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anisotropic elastic composite plates. The Laplace domain boundary integral equations (BIEs) are regularized by the static part of the traction fundamental solution. Integral representations of static singular and dynamic regular parts of anisotropic fundamental solutions are employed. To obtain time-domain solutions the Convolution Quadrature Method (CQM) with the Runge-Kutta method [5] as an underlying time stepping method is used as a numerical technique for inverse Laplace transform. To improve the computational efficiency of the proposed BEM formulation a parallelization scheme is implemented. A boundary element results for test example are provided to validate the proposed approach.

2. Problem statement and BEM formulation

Let $\Omega \subset R^3$ and $\Gamma = \partial\Omega$ denote the volume and the boundary of an arbitrary sub-domain of a three-dimensional multi-domain linear and anisotropic elastic solid. Neglecting the body forces and assuming vanishing initial conditions, the Laplace transformed equations of motion are given as

$$C_{ijkl}\bar{u}_{k,lj}(\mathbf{x},s) = \rho s^2 \bar{u}_i(\mathbf{x},s),\ \ \mathbf{x}\in\Omega,\ \ i,j,k,l=\overline{1,3}, \tag{1}$$

where s is the complex number Laplace variable, C_{ijkl} are the elastic constants, $\bar{u}_i$ is the displacement vector, ρ is the mass density.

The corresponding boundary conditions are given as follows

$$\bar{u}_i(\mathbf{x},s) = \bar{u}_i^*(\mathbf{x},s),\ \ \mathbf{x}\in\Gamma_u, \tag{2}$$

$$\bar{t}_i(\mathbf{x},s) = \bar{t}_i^*(\mathbf{x},s),\ \ \mathbf{x}\in\Gamma_t, \tag{3}$$

where $\bar{t}_i$ is the traction vector, $\bar{u}_i^*$ and $\bar{t}_i^*$ are the prescribed displacements and tractions, respectively.

To implement a Boundary Element Method, the boundary-value problem (1) − (3) is formulated as the Laplace domain displacement boundary integral equations, regularized by the static part of the traction fundamental solution as follows

$$\int_\Gamma \left[\bar{u}_k(\mathbf{y},s)\bar{h}_{jk}(\mathbf{r},s) - \bar{u}_k(\mathbf{x},s)h_{jk}^S(\mathbf{r})\right]\mathrm{d}\Gamma(\mathbf{y}) - \int_\Gamma \bar{t}_k(\mathbf{y},s)\bar{g}_{jk}(\mathbf{r},s)\mathrm{d}\Gamma(\mathbf{y}) = 0,\ \ \mathbf{x}\in\Gamma, \tag{4}$$

where $\mathbf{r}=\mathbf{y}-\mathbf{x}$, $\mathbf{x}$ and $\mathbf{y}$ are the source and the field points, respectively; $\bar{g}_{jk}$ and $\bar{h}_{jk}$ denote three-dimensional Laplace domain displacement and traction fundamental solutions, respectively.

In the Laplace domain three-dimensional displacement fundamental solutions for anisotropic elastodynamics can be represented as [6, 7]

$$\bar{g}_{ij}(\mathbf{r},s) = g_{ij}^S(\mathbf{r}) + \bar{g}_{ij}^D(\mathbf{r},s), \tag{5}$$

where

$$g_{ij}^S(\mathbf{r}) = \frac{1}{8\pi^2|\mathbf{r}|}\int_{|\mathbf{d}|=1}\Gamma_{ij}^{-1}(\mathbf{d})dL(\mathbf{d}), \tag{6}$$

$$\bar{g}_{ij}^D(\mathbf{r},s) = -\frac{1}{8\pi^2}\int_{\substack{|\mathbf{n}|=1\\ \mathbf{n}\cdot\mathbf{r}>0}}\sum_{m=1}^{3}\frac{k_m E_{im}E_{jm}}{\rho c_m^2}\exp\left(-k_m\,|\,\mathbf{n}\cdot\mathbf{r}\,|\right)dS(\mathbf{n}), \tag{7}$$

with (see Fig. 1)

$$c_m = \sqrt{\frac{\lambda_m}{\rho}},\ \ k_m = \frac{s}{c_m},\ \ \Gamma_{ij}(\mathbf{d}) = C_{kijl}d_k d_l,\ \ \Gamma_{ij}(\mathbf{n}) = C_{kijl}n_k n_l, \tag{8}$$

$$dL\left(\mathbf{d}(\varphi)\right)\in D^S = \{0\le\varphi\le 2\pi\}, \tag{9}$$

$$dS\left(\mathbf{n}(b,\varphi)\right)\in D^D = \{0\le b\le 1;\ 0\le\varphi\le 2\pi\}, \tag{10}$$

$$\mathbf{n}(b,\varphi)=\sqrt{1-b^2}\,\mathbf{d}+b\mathbf{e},\ \ \mathbf{e}=\mathbf{r}/|\mathbf{r}|,\ \ \mathbf{e}=[e_1,e_2,e_3], \tag{11}$$

$$\mathbf{d}(\varphi)=\left[e_2\cos\varphi+e_1e_3\sin\varphi,-e_1\cos\varphi+e_2e_3\sin\varphi,-\left(1-e_3^2\right)\sin\varphi\right]\Big/\sqrt{1-e_3^2}\ , \tag{12}$$

where eigenvalues and the corresponding eigenvectors of $\Gamma_{jk}(\mathbf{n})$ are denoted by λ_m, E_{jm}.

The traction fundamental solutions are obtained from the following expression:

$$\overline{h}_{jp}(\mathbf{r},s)=C_{ijkl}\overline{g}_{kp,l}(\mathbf{r},s)n_i(\mathbf{y}),\ \ j,k,p=\overline{1,4},\ \ i,l=\overline{1,3}, \tag{13}$$

with $n_i(\mathbf{y})$ is the unit outward normal vector to the boundary at the field point $\mathbf{y}$.

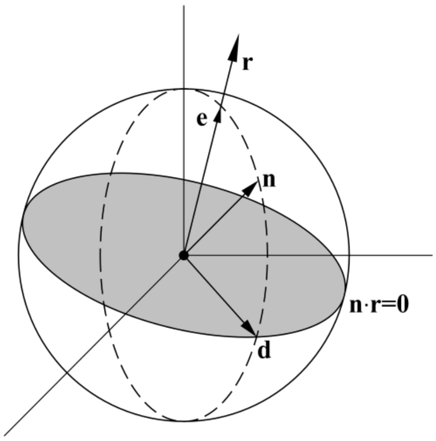

Fig. 1. Schematic depiction of vectors **r**, **e**, **n** and **d**.

Present BEM formulation uses quadrangular quadratic elements for the discretization of the surface Γ. The boundary fields, namely the displacements $\overline{u}_k$ and the tractions $\overline{t}_k$, are approximated on boundary elements according to the idea of mixed elements with linear and constant shape functions, respectively. In case of the multi-domain structural configuration, the continuity conditions are enforced on the each interface between adjacent sub-domains. After spatial discretization of all sub-domains, application of the nodal collocation method and invoking the corresponding boundary conditions a complex-valued system of linear algebraic equations is obtained for a particular value of a Laplace transform parameter:

$$\overline{\mathbf{A}}(s)\overline{\mathbf{f}}(s)=\overline{\mathbf{b}}(s), \tag{14}$$

where $\overline{\mathbf{A}}(s)$ is the rearranged system matrix, $\overline{\mathbf{b}}(s)$ and $\overline{\mathbf{f}}(s)$ are the known and unknown vectors, respectively.

In order to obtain a time domain solution we need to assemble and solve system (14) for a set of Laplace transform parameters. Two-dimensional numerical integration, which is required to calculate the regular parts of the dynamic fundamental solutions for a 3D general anisotropic elastic material, is a very time-consuming process. To accelerate overall computation process we employ an OpenMP-based parallelization scheme. Laplace domain boundary element formulation allows solving the problem on different frequencies independently. Thus, we split the total set of Laplace transform parameters evenly among all OpenMP threads. While this parallelization strategy is very simple and easy to implement, it has a downside: memory, required to store the influence matrix, is needed to be allocated for each thread, which seriously limits total number of unknowns in case of in-core calculations.

Finally, the CQM with a Radau IIA Runge-Kutta method (RKCQM) [5] is used as a numerical technique for inverse Laplace transform, which yields results in time domain.

If m-stage Runge-Kutta method is given by its Butcher tableau $\frac{\mathbf{c}|\mathbf{A}}{\mathbf{b}^T}$ where $\mathbf{A} \in R^{m\times m}$, $\mathbf{b}, \mathbf{c} \in R^m$ then for the time period $[0, N\Delta t]$ discretized with the N equidistant time steps Δt, RKCQM approximation of the time-domain response $f(t)$ of the known Laplace transformed function $\overline{f}(s)$ is given as follows

$$f(0) = 0, \quad f((n+1)\Delta t) = \mathbf{b}^T \mathbf{A}^{-1} \sum_{k=0}^{n} \omega_{n-k}(s\overline{f}(s)), \tag{15}$$

$$\omega_n(s\overline{f}(s)) = \frac{R^{-n}}{L} \sum_{p=0}^{L-1} s_p \overline{f}\left(s_p\right) e^{-in\phi_p}, \quad n = \overline{0, N-1}, \tag{16}$$

$$s_p = \frac{\gamma(z_p)}{\Delta t}, \quad z_p = Re^{i\phi_p}, \quad \phi_p = 2\pi \tfrac{p}{L}, \tag{17}$$

$$\gamma(z_p) = \mathbf{A}^{-1} - z_p \mathbf{A}^{-1} \mathbf{I} \mathbf{b}^T \mathbf{A}^{-1}, \quad \mathbf{I} = (1, ..., 1)^T, \tag{18}$$

where $0 < R < 1$ is a CQM parameter.

3. Numerical example

Consider a rectangular two-layer cross-ply square anisotropic elastic plate (see Fig. 2) with lamination [0/90]. The plate has the thickness $h = 0.25$ m with the side length $a = 1.0$ m. The plate is clamped at bottom side of the second layer and subjected to the Heaviside-type impact loading $t_3 = t_3^* H(t)$, $t_3^* = -10^5$ Pa at the top side of the first layer. The rest of plate's surface is considered free of tractions. We consider an anisotropic elastic material with mass density $\rho = 3227$ kg/m^3 and with the following elastic constants:

$$\mathbf{C} = \begin{bmatrix} 257 & 90 & 65 & 0 & 0 & 0 \\ 90 & 200 & 79 & 0 & 0 & 0 \\ 65 & 79 & 203 & 0 & 0 & 0 \\ 0 & 0 & 0 & 62 & 0 & 0 \\ 0 & 0 & 0 & 0 & 74 & 0 \\ 0 & 0 & 0 & 0 & 0 & 65 \end{bmatrix} \text{GPa}. \tag{19}$$

It is clear from the plots that are shown in Figures 3 and 4 that the boundary element results, produced by the present boundary element formulation, are accurate and are in the good agreement with the corresponding FEM solutions.

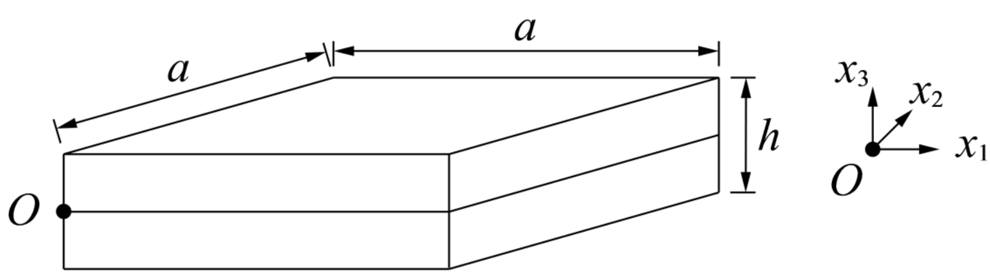

Fig 2. Geometry of the rectangular two-layer thick plate.

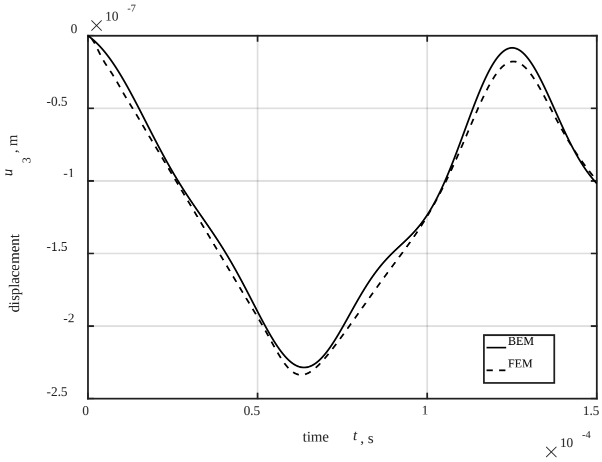

Fig 3. Vertical displacements at the point (0.5, 0.5, 0.25) m.

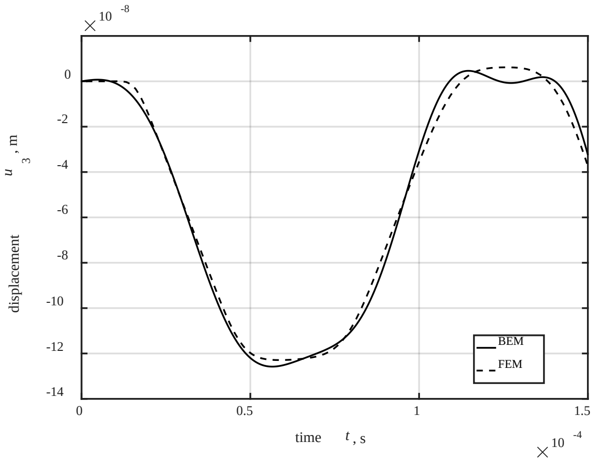

Fig 4. Vertical displacements at the point (0.5, 0.5, 0.125) m.

4. Conclusions

In this work, a Laplace domain direct boundary element formulation is applied to the transient dynamic problems of the three-dimensional anisotropic elastic multilayered moderately thick composite plates. Integral expressions of the anisotropic fundamental solutions are employed. The RKCQM, reformulated as a numerical inverse Laplace transform, is used to obtain solutions in time-domain. Present BEM formulation is verified by comparing obtained boundary element results of the test example to those by FEM.

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