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THE COMPARATIVE EVALUATION OF PRECISION OF CLASSICAL AND NUMERICAL SOLUTIONS OF CONTACT PROBLEMS

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Abstract. It is shown that the true reason of existing inconsistency at comparison of the Hertz–Belyaev classical solutions of contact problems with the numerical solutions, obtained by finite element method (FEM) for canonical-formed elastic bodies contact more often is not the unsubstantially declared poor precision of the numerical method, but the substantial distortion of the contact elements curvature, which causes accuracy of classical solutions. The criteria of reliable accuracy evaluation of classical and numerical solutions of contact problems at measuring the maximal shear stresses and the depth of its occurrence are given.
Keywords: depth of the maximal shear stress; elastic cylinders; contact mechanics; curvature; maximal contact stress; maximal shear stress.

1. Introduction

The classical solutions of Hertz–Belyaev [1, 2] are usually considered to be of high precision not only in the context of contacting bodies canonical shape, but also in general engineering practice, dealing with great variety of real contact units: at least, bearings to involute gears. As a result, the classic contact theory contradicts and is ineffective in explaining their behavior, but practice turns out has significant economic losses, for instance, in transport engineering.

The aim of this paper is to prove reliable assessment criteria for the classical contact problems and numerical solutions, at determining maximal shear stresses and their depth in the contact area of canonical shape bodies (elastic circular parallel infinite cylinders) with initial linear contact between them.

The problem of determining the maximal shear stresses and their depth in the contact area for the case of canonical shape bodies was solved by Belyaev [2] basing on the Hertz contact problem and the Boussinesq – Cerruti solutions with replacing a cylinder under study with a half-space. Belyaev managed not only to explain the reason of unusual (not according to Hertz) destruction of contact units, but also to reduce a complicated and cumbersome theory to simple and absolutely precise (in the context of his solution) correlations of the main factors of contact problem (Table 1). The solution [2] leads to widely used conclusions (Table 1):

1) the correlations of the main factors of the Hertz–Belyaev contact problems $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b$ are defined with relatively insignificant influence of a single parameter (b / a) of the shape of the contact area;

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2) the contacting cylinders have not only identical maximal normal stresses, but also identical maximal shear stresses (as well as identical depth values of its occurrence).

Hereinafter: $\tau_{\max}$ is the maximal shear stress; $\sigma_{z\max}$ is the maximal contact stress; $h_{\tau\max}$ is the depth of the maximal shear stress; b is the semi-minor axis of the elliptic contact area (or the half-width of the contact zone of parallel cylinders); a is the semi-major axis of the elliptic contact area.

That is why the invention of this classical solution has played a great role not only in the development of contact theory, but also in engineering. Unfortunately, the solutions by Belyaev are often attributed to some of his followers or (as it can be seen from papers [1, 3]) even are referred as the Hertz solutions.

Table 1. Relative values and correlations of the N. M. Belyaev classical solution.

Relative values and correlations	Initial-pointed contact						Initial-linear contact
b/a	1.00	0.95	0.80	0.60	0.20	0.10	0
$\tau_{\max}/\sigma_{z\max}$	0.31	0.31	0.31	0.32	0.32	0.31	0.300
$h_{\tau\max}/b$	0.50	0.50	0.55	0.62	0.75	0.77	0.786

2. Numerical verification and ACM-DS

In the papers [3, 4], different (in choosing the control method, as well as in the gained results) and rather representative comparative assessments of the Hertz–Belyaev solution are found. In the paper [3], this comparison is performed by using the finite elements method (FEM), with help of ANSYS software. The contradictions, obtained are assigned exceptionally to the faults of the numerical method.

The paper [4] gives comparison with analytical contactless method. The method deals with the problem of deep stresses in contacting bodies with canonical shapes. It is based on the solution by N.I. Muskhelishvili of the problem on compressing a single elastic disk by oppositely directed forces [5]. The contradictions of the comparison in [4] (unlike the conclusions of [3]) are assigned exceptionally to the faults of the classical method. Hereafter, an abbreviation ACM-DS is used for Analytical Contactless Method of Deep Stresses determination.

First, let us look on the results of the paper [4], since the ACM-DS method is a very successful tool for the reliable assessment of the classical and numerical solutions of the problem in question for parallel bodies with canonical forms. Thus, the ACM-DS is much stricter than classical and numerical solutions and its application as the control method significantly facilitates the detection of specific reasons of these traditional methods' faults. ACM-DS combines such peculiarities as the exclusion of one of the key method of classical contact problems (Boussinesq-Cerruti solutions) and on the contrary the strict accomplishment of the whole complex of other classical simplifications, including Hertz's law of contact pressure. It provides uniform consideration for three compared solutions, namely ACM-DS, classical and numerical ones. Moreover, ACM-DS excludes key problems (interpreted in [3] as sources of faults) of the numerical FEMs, namely the extraction of the half or a quarter of a cylinder for the quickening of its analysis, the conditions of its decomposition on finite elements in the contact area etc. It significantly decreases the manifestation risks of hard-to-explain faults and makes it notably easier to assess the FEM precision.

The creation of ACM-DS [4] was preceded by the development (also on the basis of a problem [5]) of the determination method of the strain state of contacting parallel cylinders and the detection of previously unknown effects of their rapprochement [6]. In both cases, it is kept the correct comparison with classical solutions (according to the solution [5]) and such their requirements as isotropy, homogeneity and ideal elasticity (in sense of Hooke's law) of the materials of compressed bodies, and also stated absence of dynamic effects, friction (sliding and rolling) and the intermediate lubricating layer. In order to add the Hertz's law of a contact load, the superposition principle of forces is implemented. The adding the Hertz's law of a contact pressure let narrow the comparison conditions of classical and numerical methods to the influence assessment of the cylinder change technique by the half-space on the precision of the deflected mode.

The analysis [4] is accomplished for the contact of two cylinders – the studied cylinder with varying radius R_c and the base cylinder with radius $R_b = const$.

Values r_1, ω_1, r_2, ω_2 (Fig. 1) are calculated for every rated point $(x;z)$ using the formulae:

$L_{p1} = R_c \cdot \sin\alpha$; $L_{p2} = R_c \cdot (1-\cos\alpha)$; $L_{pp} = 2 \cdot R_c \cdot \cos\alpha$; $L_{p3} = L_{pp} + L_{p2}$;

$\omega_1 = arctg(\dfrac{x - L_{p1}}{z - L_{p2}})$; $\omega_2 = arctg(\dfrac{x - L_{p1}}{L_{p3} - z})$; $r_1 = \dfrac{x - L_{p1}}{\sin\omega_1}$; $r_2 = \sqrt{(x - L_{p1})^2 + (L_{p3} - z)^2}$.

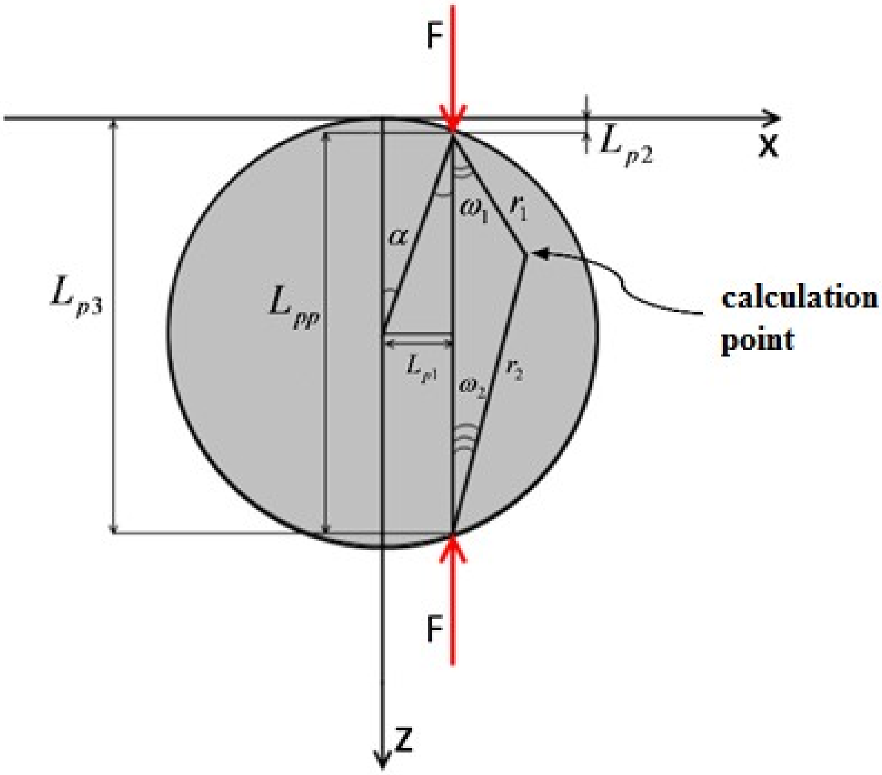

Fig. 1. Cylinder, compressed with opposite forces F, directed along axis Oz.

According to the problem solution [4] for the action of the Hertz contact pressures on the cylinder, let us write down the expressions for normal stresses, directed simultaneously along the *x*-axis and *z*-axis (Fig. 1) and for shear stresses, respectively:

$$\sigma_x = -\int_{-\alpha_b}^{\alpha_b} \frac{\sigma_H(\alpha)}{\pi}\left[2\cdot\frac{\sin^2\omega_1(\alpha)\cdot\cos\omega_1(\alpha)}{r_1(\alpha)} + 2\cdot\frac{\sin^2\omega_2(\alpha)\cdot\cos\omega_2(\alpha)}{r_2(\alpha)} - L_{p\alpha}(\alpha)\right]R_c d\alpha\,; \quad (1)$$

$$\sigma_z = -\int_{-\alpha_b}^{\alpha_b} \frac{\sigma_H(\alpha)}{\pi} \cdot \left[2 \cdot \frac{\cos^3 \omega_1(\alpha)}{r_1(\alpha)} + 2 \cdot \frac{\cos^3 \omega_2(\alpha)}{r_2(\alpha)} - L_{p\alpha}(\alpha) \right] R_c d\alpha \,; \tag{2}$$

$$\tau = -2 \cdot \int_{-\alpha_b}^{\alpha_b} \frac{\sigma_H(\alpha)}{\pi} \left[\frac{\sin \omega_1(\alpha) \cdot \cos^2 \omega_1(\alpha)}{r_1(\alpha)} + \frac{\sin \omega_2(\alpha) \cos^2 \omega_2(\alpha)}{r_2(\alpha)} \right] R_c d\alpha \,. \tag{3}$$

Here α_b is the central angle of the circle sector with the radius R_c, encircling the circular arc with the length b_H; $\sigma_H(\alpha)$ is the pressure, corresponding Hertz's law $\sigma_H(\alpha R_c)$.

In Table 2 (also in Figs. 2 and 3) the comparative analysis of the results, obtained in the work [4], is partially presented. The calculations are given for steel cylinders with the same Young's moduli: $E_1 = E_2 = 2 \cdot 10^5$ MPa and Poisson's ratios: $\mu_1 = \mu_2 = 0,28$.

In the result, the deviations (in regard to classical values) of relative values of the maximal shear stresses and the depth of its occurrence have been revealed. Moreover, the criterial proportion $\frac{b_H}{R_c}$, which determines the level of these deviations is offered.

The correspondence to the classical solution is confirmed (Table 2) only for parameters, approximately limited by an area $b_H / R_c \le 2 \cdot 10^{-4}$. For instance, for bodies with big radius of curvature and/or minor contact stresses $\sigma_{z\max}$. In Table 2, $\Delta(\tau_{\max} / \sigma_{z\max})$ is the deviation of $\tau_{\max} / \sigma_{z\max}$ from the classical value 0.300 and $\Delta(h_{\tau\max} / b_H)$ is the deviation of $h_{\tau\max} / b_H$ from the classical value 0.786.

The analysis [4] showed that the values of $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b_H$ for any radius of a basic cylinder do not remain constant at the curvature radius change of the studied cylinder.

The calculations results (Figs. 2 and 3) show that the values of $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b_H$ do not remain constant on R_c and $\frac{b_H}{R_c}$ change, and the deviations from Table 1 data of values $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b_H$ have clearly defined laws:

1) the ratios $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b$ keep its values at invariable ratio of b_H / R_c;

2) the ratios $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b_H$ (and the level of its deviation from the classical solutions) increase so far as the value of b_H / R_c increases.

Figures 2 and 3 show the deviation dependencies of the ratios $\tau_{\max} / \sigma_{z\max}$ and $h_{\tau\max} / b_H$ on the studied cylinder radius R_c (Fig. 2), and on the ratio b_H / R_c (Fig. 3). For instance, with the maximum permitted (according to the contact area smallness) value of $b_H / R_c = 0{,}14$ the classical solution gives the results understated for values of $\tau_{\max} / \sigma_{z\max}$ (by 10.6 %) and $h_{\tau\max} / b_H$ (by 5.3 %).

In this case the dependence of $\tau_{\max} / \sigma_{z\max}$ on R_c in the context of the flat problem ($b_H / a = 0$) is notably stronger then traditionally considered universal dependence $\tau_{\max}$ only on the contact area form throughout the whole range.

Table 2. Assessment results of the Hertz–Belyaev classical solutions precision at the deep stresses determination in the body of the studied cylinder R_c (contact with the cylinder $R_b = 10^9$ mm) via ACM-DS for small contact areas.

R_c, mm	$\frac{b_H}{R_c}$	$\sigma_{z\max}$,MPa	$\frac{\tau_{\max}}{\sigma_{z\max}}$	$\frac{h_{\tau\max}}{b_H}$	$\Delta\left(\frac{\tau_{\max}}{\sigma_{z\max}}\right)$,%	$\Delta\left(\frac{h_{\tau\max}}{b_H}\right)$,%
10^7	≤ 0.0002	≤ 11	0.300	0.786	≤ 0.1	0
	0.02	1086	0.304	0.791	1.5	0.7
	0.06	3258	0.313	0.802	4.4	2.0
	0.08	4345	0.318	0.808	5.9	2.8
	0.10	5431	0.322	0.814	7.4	3.6
	0.12	6517	0.327	0.821	9.0	4.4
	0.14	7603	0.332	0.828	10.6	5.3
10^5	≤ 0.0002	≤ 11	0.300	0.786	≤ 0.1	0
	0.02	1085	0.304	0.791	1.5	0.7
	0.06	3256	0.313	0.802	4.4	2.0
	0.08	4341	0.318	0.808	5.9	2.8
	0.10	5426	0.322	0.814	7.4	3.6
	0.12	6511	0.327	0.821	9.0	4,4
	0.14	7596	0.332	0.828	10.6	5.3
10^3	≤ 0.0002	≤ 11	0.300	0.786	≤ 0.1	0
	0.02	1085	0.304	0.791	1.5	0.7
	0.06	3255	0.313	0.802	4.4	2.0
	0.08	4340	0.318	0.808	5.9	2.8
	0.10	5425	0.322	0.814	7.4	3.6
	0.12	6510	0.327	0.821	9.0	4.4
	0.14	7595	0.332	0.828	1.6	5.3
10	≤ 0.0002	≤ 11	0.300	0.786	≤ 0.1	0
	0.02	1085	0.304	0.791	1.5	0.7
	0.06	3255	0.313	0.802	4.4	2.0
	0.08	4340	0.318	0.808	5.9	2.8
	0.10	5425	0.322	0.814	7.4	3.6
	0.12	6510	0.327	0.821	9.0	4.4
	0.14	7595	0.332	0.828	10.6	5.3

Even though in the works [3, 4] diametrically opposite conclusions are made on the comparison of classical method and numerical one, practically these works have absolutely the same results. Thus, let us compare these results, considering that R_1 and R_b are for the base cylinder radius, and R_2 and R_c are for the studied cylinder radius. Then we can see that under conditions that $E_1 = E_2 = 2\cdot 10^5$ MPa; $\mu_1 = \mu_2 = 0,3$; $R_1 = 94,06$ mm; $R_2 = 91,89$ mm (conditions [3]) and $E_1 = E_2 = 2\cdot 10^5$ MPa; $\mu_1 = \mu_2 = 0,28$; $R_b = 10^9$ mm; $R_c = 10...10^7$ mm (conditions [4]), the analyses [3, 4] almost equally reflect traditional idea of approximate boundaries by the data of contact area smallness for classical solutions. Furthermore, even in these boundaries the analyses [3, 4] equally showed great contradictions in the classical calculation of maximal shear stresses: 10% using numerical method [3] and 10.6% using analytical ACM-DS method [4].

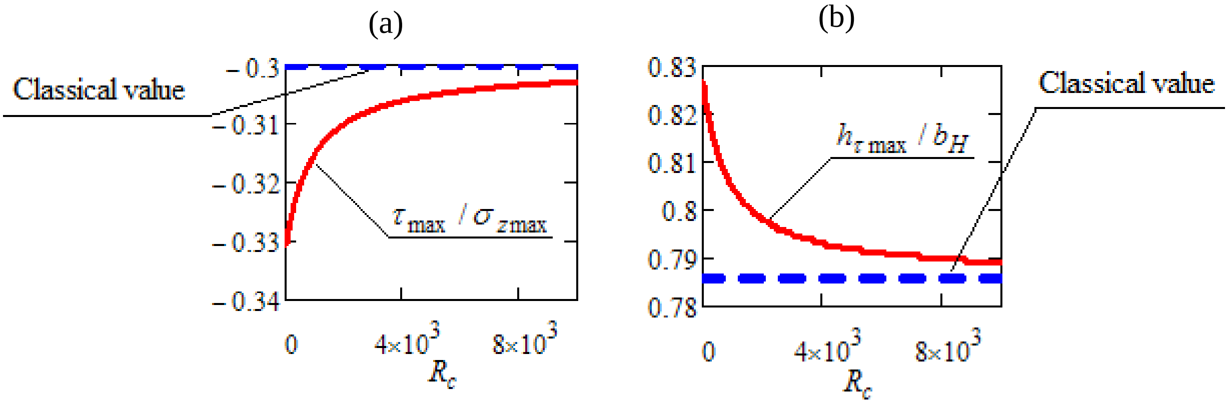

Fig. 2. Assessment results of the determination by using classical solutions for maximal shear stresses (a) and depth of its occurrence (b) according to the radius of the examined cylinder by using ACM-DS method.

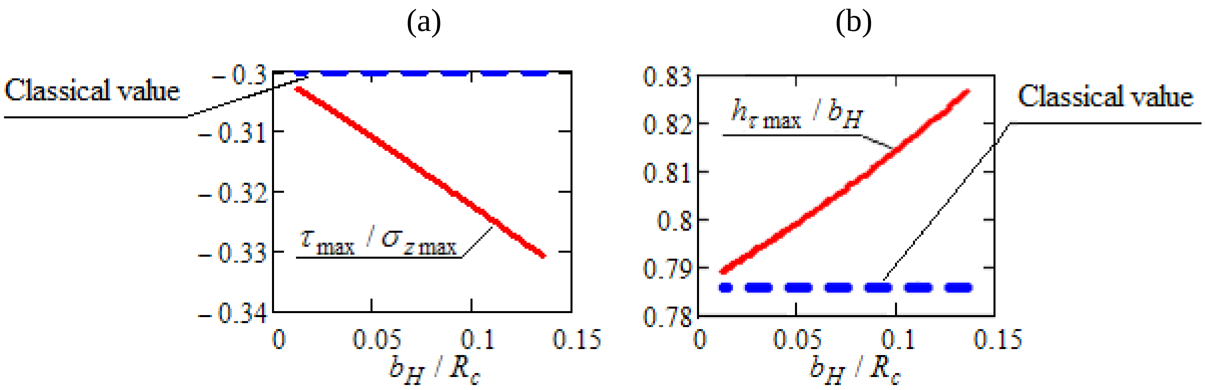

Fig. 3. Assessment results of the determination by using classical solutions of maximal shear stresses (a) and depth of its occurrence (b) according to the parameter b_H / R_c by using ACM-DS method.

At the same time the only calculated values of maximal shear stresses shown in [3] (ANSYS – 268.61 MPa, and the classical solution – 266.97 MPa) represent the fault, which is about 0.61%. It states the high precision of methods at comparison in the conditions of small values of b_H / R_c, that completely corresponds to the ACM-DS results (Table 2). The explanation of other (which are not presented in [3]) results with the fault within the 10% range via the numerical method shortcomings has no reasons. For instance, the problem solution [3] by way of two semi-cylinders (instead of full cylinders) is a common practice of the ANSYS method [10] and cannot be the reason of big ANSYS mistakes. Furthermoreб the authors themselves [3] showed its small (about 0.61%) fault. The calculation results via ACM-DS method (Table 2) showed the reasons of such faults manifestation. They relate to the poor precision of the classical solutions, concerning the curvature calculation of the contact elements of canonical bodies [4, 6, 7, 9] and to a much greater extent of the real contact assemblies [7, 8].

Unfortunately, the paper's authors [3] obviously having (by way of not presented in the article results of numerical method) a serious evidential ground of the poor precision of classical solutions made conclusions, which totally correspond to traditional ideas, but contradict the results, obtained by them.

The results of [3] are particularly interesting because they are provided for a case, which is assumed to fit well the Hertzian theory. Earlier, analogous studies were carried out (often in details) for the wheel-rail contact problems. For example, the paper [11] is devoted to investigating applicability of the Hertz contact theory, and the results of the classical solution are compared to the ones, obtained with three-dimensional FEM modeling for linear elastic and elastic-plastic materials. The paper [12] continues the studies of [11], but considers also possible changes in profile radii of the rail, which can be caused by different reasons. Both papers [11] and [12] conclude, that the numerical solution agrees well with the theoretical one in the framework of specific maximum external load and the requirement for the surface curvature of the rail remains unchanged. The similar results are also observed in the paper [13], where particular emphasis is placed on material parameters.

In [14], authors investigated the influence of geometric parameters on contradictions between numerical simulations and analytic solutions, based on the Hertzian theory. It is remarkable, that, according to the results of this paper, the FEM modeling provided maximal 4.62% deviation from the theoretical solution for the small rail radius of curvature, but the reasons why such a deviation took place were not discussed explicitly.

Conclusions

1) The genuine reason of the contradiction facts in the calculation results of deep stresses contact, observed usually between classical and numerical methods is the poor precision of classical solutions concerning the role of contact elements curvature;
2) analytical solutions provide (while refuse to use the Boussinesq-Cerruti solutions) precision and reliability, but are limited in the variety consideration of geometric shapes and other parameters of contacting bodies;
3) numerical methods better consider the variety of geometric shapes and other parameters of contacting bodies, but they are more cumbersome and less reliable by using;
4) the excess value of maximal shear stress (regarding the one obtained with help of classical solution) reaches 10.6% (in the context of small contact areas) and can be much more within the framework of common using;
5) the criterial parameter is defined, which describes the distortion level of the classical solution.

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