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ADDED MASS STUDY OF PLANE STRUCTURES AT THEIR VARIOUS MOTIONS

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Abstract. Using FE analysis, this work fully devoted to added masses determination of the plate in infinite liquid in the case of different movements. Various forms and sizes of the plate are considered. The problem consists of two parts: first is studying of the plate motion in liquid as rigid; second is studying of the plate vibrations in its own modes, fixed with a hard screen and contacting with liquid by one side. The condition observance of liquid environment continuity leads to the Laplace equation, which is solved by using numerical approach. Finite element analysis of steady state problem is performed for added masses determination using thermal analogy. A compact analytical solution of the two considered problems is presented. The results of finite element analysis and corresponding analytical results of the problems are compared for the case of infinite liquid environment.

Keywords: added mass, finite element analysis, vibration mode, Laplace equation, thermal analogy, velocity potential

1. Introduction

The knowledge of the added masses of bodies contacting with liquid is necessary for solving various applied problems of hydromechanics, such as steady and transient motion of rigid bodies, vibration of bodies in liquid, local vibration of constructions in a fluid environment.

In all equations determining the parameters of ship motion, their general vibrations and hull shell vibration, the inertial properties of liquid are expressed by the added masses. For the last 50 years, a lot of journal articles [1] and sections in monographs [2-4] on theoretical and applied hydromechanics have been devoted to the added mass determination. The studies of the ships dynamics and their particular components lead to the need of solving these problems.

The method of thermal analogy, which is used in this work, is widely known in scientific community [1]. This method is very convenient for solving the Laplace equation, which is used for calculating the added masses of liquid. The motion of any structures as rigid, immersed in liquid, are widely represented in the articles and books, but there are not a lot of publications describing the movements with vibrations and deformations. Thus, for the complete added masses representation for the various structures movements in liquid both types of movements are studied in the work: vibrations and motion as rigid body.

The purposes of the work are the added masses determination of the plate immersed in liquid by numerical method and analyzing the influence on result structure dimensions and liquid environment.

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2. Added masses of bodies

Rigid motion of a body. We will calculate the added masses in the case of translational motions of a circular disk of radius r_d, immersed in infinite liquid. There is analytical solution of the problem [4]. We suppose that a rigid body of volume V has an external surface S. This body starts to move from initial position in infinite ideal homogeneous liquid without vortexes. Then the flow caused by the body motion will be potential during the motion. We need to introduce some hypotheses to solve this problem [5]:

- We consider only small oscillations of a plate in liquid, which are described by linearized equations. There are small displacements of the plate, and liquid is moving along with the plate in perfect contact;
- We assume that the liquid is incompressible;
- We suppose that shape modes of the plate in liquid environment and empty space are identical.

Thus, the liquid influence on the plate oscillations will be inertial and the mass of the plate is increasing, therefore its own frequencies are decreasing.

We will use coordinate system z, associated with the body. There are no vortexes, so that there is a potential function $\Phi(x, y, z, t)$, which is characterizing velocities of flow, which is arising in the case of structure vibrations in liquid. The components of liquid velocity are determined by the following relations:

$$v_x = \frac{\partial \Phi}{\partial x}; v_y = \frac{\partial \Phi}{\partial y}; v_z = \frac{\partial \Phi}{\partial z}. \quad (1)$$

In this case the condition of observance of liquid environment continuity leads to Laplace equation:

$$\Delta\Phi = 0. \quad (2)$$

The boundary condition for equation (2) is:

1. There is no leak on the external surface S:

$$\frac{\partial \Phi}{\partial n}| = v_n, \quad (3)$$

i.e. the projection of the liquid particle velocity on the external normal n to the surface is equal to the projection of the body point velocity on the same normal;

2. The absence of liquid particles motion at an infinite distance from the body:

$$Lim_{r\to\infty}\frac{\partial \Phi}{\partial x} = \underset{r\to\infty}{Lim}\frac{\partial \Phi}{\partial y} = \underset{r\to\infty}{Lim}\frac{\partial \Phi}{\partial z} = 0, \quad (4)$$

where $r^2 = x^2 + y^2 + z^2$ is – the distance from the origin to an arbitrary point of liquid;

3. The existence of a free surface S_1on which the body moves (additional condition [3]):

$$\Phi|_{S_1} 0. \quad (5)$$

Potential Φ can be represented in the following form

$$\Phi = v_{Ox}\varphi_1 + v_{Oy}\varphi_2 + v_{Oz}\varphi_3 + \omega_x\varphi_4 + \omega_y\varphi_5 + \omega_z\varphi_6. \quad (6)$$

Formula (6) shows that $\varphi_i(i = 1, \dots, 6)$ are the potentials of liquid flows in the cases of the body motion along the $x, y, z -$ axes or rotation about these axes $i = 4, 5, 6$ with the unit linear (or angular) velocities. Thus, the problem of the body movements in infinite ideal liquid consists of solving six problems. These problems are formulated in the following way: we need to solve the Laplace equation $\Delta\varphi_i = 0$ with the boundary conditions:

$$\frac{\partial \varphi_1}{\partial n}|_S = \alpha ,$$

$$\frac{\partial \varphi_2}{\partial n}|_S = \beta ,$$

$$\frac{\partial \varphi_3}{\partial n}|_S = \gamma ,$$

$$\frac{\partial \varphi_4}{\partial n}|_S = (y\gamma - z\beta),$$

$\frac{\partial \varphi_5}{\partial n}|_S = (z\alpha - x\gamma)$,
$\frac{\partial \varphi_6}{\partial n}|_S = (x\beta - y\alpha)$,
$\lim_{r\to\infty} \frac{\partial \varphi_i}{\partial x} = \lim_{r\to\infty} \frac{\partial \varphi_i}{\partial y} = \lim_{r\to\infty} \frac{\partial \varphi_i}{\partial z} = 0$,
where $r^2 = x^2 + y^2 + z^2$.

In the work we are studying translational motion along $z -$axis and rotation about x and $y -$ axes. Thus, we need to solve three problems $\Delta\varphi_{3,4,5} = 0$ with the following boundary conditions:

$\frac{\partial \varphi_3}{\partial n}|_S = \gamma = \cos(n, z) = 1$, (Fig. 1)
$\frac{\partial \varphi_4}{\partial n}|_S = (y\gamma\text{-}z\beta) = y\gamma = y$, (Fig. 2)
$\frac{\partial \varphi_5}{\partial n}|_S = (z\alpha\text{-}x\gamma) = \text{-}x\gamma = \text{-}x$, (Fig. 3)

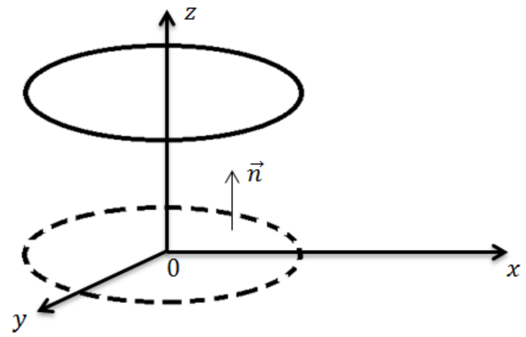

Fig. 1. Translational motion along z – axis

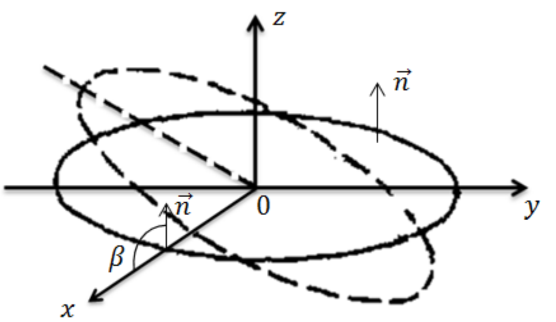

Fig. 2. Rotation about x- axis

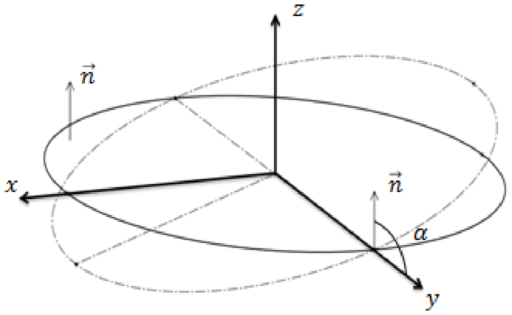

Fig. 3. Rotation about $y -$ axis

$Lim_{r\to\infty} \frac{\partial \varphi_{3,4,5}}{\partial x} = \underset{r\to\infty}{Lim} \frac{\partial \varphi_{3,4,5}}{\partial y} = \underset{r\to\infty}{Lim} \frac{\partial \varphi_{3,4,5}}{\partial z} = 0$,
where $r^2 = x^2 + y^2 + z^2 -$ the distance from the origin to an arbitrary point of liquid.

In the case, when the plate is moving on the surface of liquid as was noted earlier the additional condition of free surface existence is introduced
$\varphi_{3,4,5}|_{S_1} = 0$.

The kinetic energy of liquid enclosed between the surface of a moving body S and a stationary spherical surface $\boldsymbol{\Sigma}$ of radius a surrounding the body and the volume of liquid V adjacent to it, is defined as

$$T = -\frac{1}{2}\rho \iint_S \Phi \frac{d\Phi}{dn} dS, \quad (7)$$

where ρ is the liquid density.

Using the expression for the velocity potential (6), and denoting
$v_{Ox} = v_1; v_{Oy} = v_2; v_{Oz} = v_3; \omega_x = v_4; \omega_y = v_5; \omega_z = v_6$,
we have

$$T = \frac{1}{2}\sum_{\substack{i=1\\k=1}}^{6} \lambda_{ik} v_i v_k. \quad (8)$$

The added masses of liquid adjoining to the surface of the moving body can be determined by the following formulas

$$\lambda_i = \rho \iiint_V \left[\left(\frac{d\varphi_i}{dx}\right)^2 + \left(\frac{d\varphi_i}{dy}\right)^2 + \left(\frac{d\varphi_i}{dz}\right)^2 \right] dxdydz, \quad (9)$$

$$\lambda_{ik} = -\rho \iint_S \varphi_k \cdot \frac{d\varphi_i}{dn} dS\ . \quad (10)$$

For the numerical computation in the ANSYS software in the cases of translational motion along $z -$ axis and rotation about x and $y -$ axes we can use the thermal analogy of steady state problem [1], i.e.

$\Delta\varphi_i = 0 <=> \Delta T = 0.$

The condition of the function assignment $\frac{d\varphi_i}{dn}|_S$ on the surface S *is* equal to the condition of the heat flux existence $-k\frac{dT}{dn}|_S$ on the same surface of the body, i.e.

$\frac{d\varphi_i}{dn}|_S = -k\frac{dT}{dn}|_S = q_n|_S,$

where k – a thermal conductivity coefficient. When $k = 1$,

$$q_n|_S = \frac{d\varphi_i}{dn}|_S,$$

where functions $\frac{d\varphi_i}{dn}|_S$ depend on the body motion types.

Additional condition (4), having the form

$Lim_{r\to\infty}\frac{\partial\varphi_i}{\partial x} = \lim_{r\to\infty}\frac{\partial\varphi_i}{\partial y} = \lim_{r\to\infty}\frac{\partial\varphi_i}{\partial z} = 0,$

is equivalent to the condition that the temperature gradient is zero at the infinity from the moving body, i.e.

$\nabla T|_{r\to\infty} = 0 => q|_{r\to\infty} = 0.$

Additional condition (5) of the free surface existence is equivalent to the condition that the temperature is zero on the liquid free surface, i.e.

$T|_{S_1} = 0.$

Tables 1 and 2 show the results of added masses. These results have calculated by two different ways such as integration over the outer surface (*S*), which is external surface of considered domain, and volume integration (*V*). The convergence of added masses on the size of liquid environment is presented in Figs. 4, 5.

Table 1. Added masses in the case of translational motion along $z -$ axis

Formulation	**Added mass, kg**		
Analytical estimation			
A.I. Korotkin, 1986 [4]	21 333.33		
Numerical results ($\boldsymbol{r_d = 2\ m}$)			
$\boldsymbol{q_n\|_S = 1}$ $\boldsymbol{q_n\|_{r=R} = 0}$ $\boldsymbol{T = 0}$ on the cut surface	R, m	S	V
	10	20 626.16	20 311.90
	20	20 566.76	20 250.92
	30	20 560.40	20 239.33

Table 2. Added masses in the case of rotation about x or $y -$ axes

Formulation	**Added mass, kg**		
Analytical estimation			
A.I. Korotkin, 1986 [4]	11 377.78		
Numerical results ($\boldsymbol{r_d = 2\ m}$)			
$\boldsymbol{q_n\|_S = -x}$ $\boldsymbol{q_n\|_{r=R} = 0}$ $\boldsymbol{T = 0}$ on the cut surface	R, m	S	V
	10	10 662.49	10 391.97
	20	10 662.06	10 390.99
	30	10 661.97	10 390.14

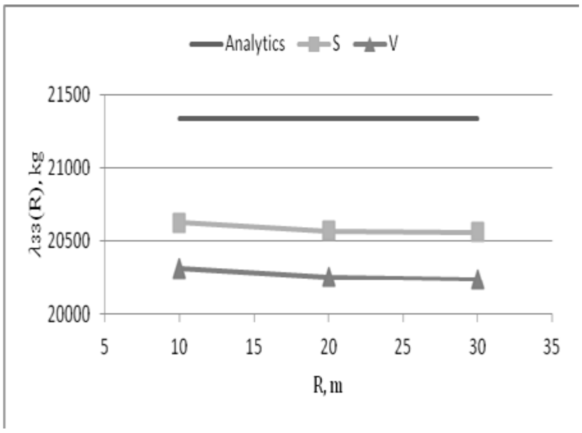

Fig. 4. Convergence of the added masses on the size of liquid environment, the calculations being done by integration over an outer surface and by volume integration

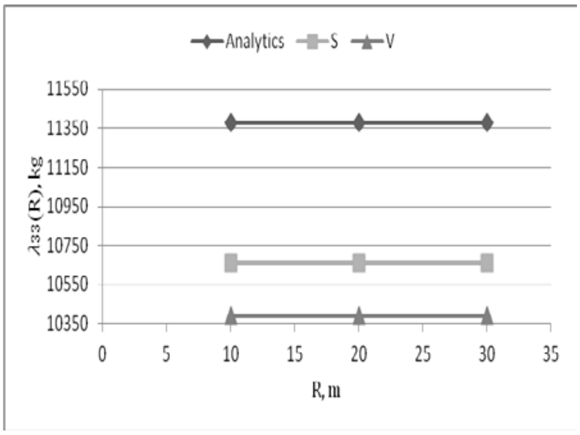

Fig. 5. Convergence of the added masses on the size of liquid environment, the calculations being done by integration over an outer surface and by volume integration

Vibrations of a fixed rectangular plate. In the second part of the work we are solving more difficult problem of the rectangular plate vibrations in an infinite liquid. The plate dimensions are $a\ x\ b$. Here, as the motion of the plate, we will consider its vibrations by 1st mode shape. In the first part of the work we introduced the hypothesis that the liquid is incompressible. The influence of the liquid compressibility on vibrations is determined by dimensionless parameter, which is called the Strouhal number:

$$S_t = \omega l / c_0, \qquad (11)$$

where c_0 is the sound speed in water; l is the characteristic size, i.e. is the distance between the nodes of a vibration mode; ω is the vibrations frequency.

We need to use previous formulas (9) and (10) for determining the added masses in the case of the plate vibrations in an infinite liquid. We are solving the problem in the case, when a part of the volume boundary, adjoining to the plate, is represented by rigid screens S_1, for which

$$\frac{d\Phi}{dn} = 0.$$

In connection with the new problem formulation, the boundary conditions will change.

We suppose that the displacements of the plate wetted surface along the normal n are characterized by the vector of generalized coordinates $\boldsymbol{a}(t)$ and by the vector of coordinate functions $\boldsymbol{f}(x, y, z)$, so that the displacement $w_n(x, y, z, t)$ can be found as the scalar product

$$w_n(x, y, z, t) = \boldsymbol{a}(t)^T \cdot \boldsymbol{f}(x, y, z).$$

The motion velocity $v_n(x, y, z, t)$ can be found as the scalar product

$$v_n(x, y, z, t) = \dot{\boldsymbol{a}}(t)^T \cdot \boldsymbol{f}(x, y, z),$$

where $x, y, z -$ the coordinates of points on the surface S.

Using boundary condition (3), the velocity potential Φ can be found similarly as the scalar product

$$\Phi = \boldsymbol{\tau}(t)^T \cdot \boldsymbol{\varphi}(x, y, z). \qquad (12)$$

We insert (12) into (3)

$$\frac{\partial \phi}{\partial n}|_S = v_n =>$$

$$\boldsymbol{\tau}(t)^T \cdot \frac{\partial \phi}{\partial n}|_S = \dot{\boldsymbol{a}}(t)^T \cdot \boldsymbol{f}(x, y, z) =>$$

$$\boldsymbol{\tau}(t) = \dot{\boldsymbol{a}}(t),$$

$$\frac{\partial \phi}{\partial n}|_S = \boldsymbol{f}(x, y, z).$$

Thus, the velocity potential can be presented in the following form

$$\Phi = \dot{\boldsymbol{a}}(t)^T \cdot \boldsymbol{\varphi}(x, y, z). \quad (13)$$

The each function $\varphi_i, i = 1..N$, as was noted earlier, is the velocity potential in the case of the wetted surface motion with a unit generalized velocity. So, the vector $\boldsymbol{\varphi}_i$ is the vector of the unit potentials. There are the boundary conditions for series of N problems, N is the number of modes, which are describing the complex body motion.

Using equation (2) and boundary conditions (3) and (4), we can obtain the equation and the boundary conditions for the functions $\boldsymbol{\varphi}(x, y, z)$ sought. We insert (13) into (2):

$\Delta\Phi = 0$,

where

$$\Phi = \dot{\boldsymbol{a}}(t)^T \cdot \boldsymbol{\varphi}(x, y, z) =>, \dot{\boldsymbol{a}}(t)^T \cdot \Delta\boldsymbol{\varphi}(x, y, z) =>, \Delta\boldsymbol{\varphi}(x, y, z) = 0, \quad (14)$$

on condition that $\dot{\boldsymbol{a}}(t) \neq 0$;

$$\frac{\partial\Phi}{\partial n}|_S = v_n = \dot{\boldsymbol{a}}(t)^T \cdot \frac{\partial\varphi}{\partial n}|_S = \dot{\boldsymbol{a}}(t)^T \cdot \boldsymbol{f}(x, y, z) =>$$

$$\frac{\partial\varphi}{\partial n}|_S = \boldsymbol{f}(x, y, z); \quad (15)$$

$$(4) \Leftrightarrow Lim_{r\to\infty} \frac{\partial\varphi}{\partial x} = \underset{r\to\infty}{Lim} \frac{\partial\varphi}{\partial y} = \underset{r\to\infty}{Lim} \frac{\partial\varphi}{\partial z} = 0, \quad (16)$$

where $r^2 = x^2 + y^2 + z^2$ – the distance from the origin to an arbitrary point of liquid.

The condition of rigid screen existence is presented in following form

$$\dot{\boldsymbol{a}}(t)^T \cdot \frac{\partial\varphi}{\partial n}|_{S_1} = 0 => \frac{\partial\varphi}{\partial n}|_{S_1} = 0. \quad (17)$$

We can obtain the kinetic energy of liquid from formula (7)

$$T = -\tfrac{1}{2}\rho \iint_S \dot{\boldsymbol{a}}^T \cdot \boldsymbol{\varphi} \frac{d\varphi}{dn} \cdot \dot{\boldsymbol{a}} dS\ ,$$

or

$$T = \tfrac{1}{2}\dot{\boldsymbol{a}}^T \cdot [\lambda] \cdot \dot{\boldsymbol{a}},$$

where $[\lambda]$ − the matrix of added masses

$$[\lambda] = -\rho \iint_S \boldsymbol{\varphi} \frac{d\varphi^T}{dn} \cdot dS\ . \quad (18)$$

The problem consists in determining the added masses for two-mode vibrations of the plate:

- $f_z(x, y) = \sin\left(\pi \frac{\left(x - \frac{a}{2}\right)}{a}\right) \cdot \sin\left(\pi \frac{\left(y - \frac{b}{2}\right)}{b}\right)$ − simple-supported plate;
- $f_z(x, y) = \frac{1}{4}\left[1 - \cos\left(\pi \frac{\left(x - \frac{a}{2}\right)}{\frac{a}{2}}\right)\right] \cdot \left[1 - \cos\left(\pi \frac{\left(y - \frac{b}{2}\right)}{\frac{b}{2}}\right)\right]$ − clamped-edge plate.

For confirmation of incompressible liquid hypothesis we have solved an additional problem of searching the first natural frequency for these plates (Table 3).

Table 3. First natural vibration frequency of the square plate ($a = b = 4\ m$)

Analytical estimation, Rad/s	Numerical results, $\boldsymbol{Rad/s}$
freely-supported plate	
1.937	1.937
clamped-edge plate	
3.652	3.531

Knowing these frequencies, we can use formula (11) for determining the Strouhal number. Here parameter $l = 4\ m$ corresponds to the first mode shape, the sound speed in water is $c_0 = 1500\ m/s$. Then we obtained Strouhal numbers for two types of the plate modes which are much less than one. Our hypothesis is confirmed.

For the numerical computation in the ANSYS software for the plate modes, simple-supported and clamped-edge plate, we can use the thermal analogy, i.e.

$\Delta\boldsymbol{\varphi} = 0 <=> \Delta T = 0.$

Condition (15) of the function assignment $\frac{\partial \varphi}{\partial n}|_S$ on the surface S is equivalent to the condition of the heat flux existence $-k\frac{dT}{dn}|_S$ on the same surface of the body, i.e.

$-k\frac{dT}{dn}|_S = q_n|_s = \boldsymbol{f}(x, y, z),$

where $k -$ the thermal conductivity coefficient. When $k = 1$:

$q_n|_s = \boldsymbol{f}(x, y, z),$

where functions $\boldsymbol{f}(x, y, z)$ depend on the body vibration types.

Condition (16)

$Lim_{r\to\infty}\frac{\partial \varphi_i}{\partial x} = \lim_{r\to\infty}\frac{\partial \varphi_i}{\partial y} = \lim_{r\to\infty}\frac{\partial \varphi_i}{\partial z} = 0$

is equivalent to the condition that the temperature gradient is zero at infinity, i.e.

$\nabla T|_{r\to\infty} = 0 => q|_{r\to\infty} = 0.$

The condition of rigid-screen existence (17) is equivalent to the condition that the temperature gradient is zero on the rigid-screen surfaces, i.e.

$\nabla T|_{S_1} = 0 => q|_{S_1} = 0.$

For the numerical solution, we need to change the boundary conditions, in order to avoid the Neumann boundary condition, i.e. to assume that the velocity potential is zero at infinity

$\varphi|_{r\to\infty} = 0 => T|_{r\to\infty} = 0.$

Tables 4 and 5 show the comparison of the analytical and numerical results; the latter are obtained for different sizes of liquid environment R. The convergence of the added masses for the freely-supported plate on the size of liquid environment is presented in Figs. 6, 7.

Table 4. Added masses for a freely-supported plate

Formulation	**Added mass, kg**		
Analytical estimation			
V.V. Davydov, N.V. Mattes, 1974 [9]	6 560		
Yu.A. Shimanskiy, 1963 [2]	6 720		
V.A. Postnov, 1983 [3]	6 880		
Numerical results ($\boldsymbol{a = 4\ m, b = 4\ m}$)			
$\boldsymbol{q_n|_S = \sin\left(\pi\frac{\left(x-\frac{a}{2}\right)}{a}\right)\cdot\sin\left(\pi\frac{\left(y-\frac{b}{2}\right)}{b}\right)}$ $\boldsymbol{T|_{r=R} = 0}$ $\boldsymbol{q_n = 0}$ on the cut surface	R, m	S	V
	10	5 867.96	6 477.48
	40	6 020.83	6 082.67
	100	6 109.81	6 037.70
	200	6 141.15	6 011.49
	250	6 147.41	6 008.96

Table 5. Added masses for a clamped-edge plate

Formulation	Added mass, kg		
Analytical estimation			
V.V. Davydov, N.V. Mattes, 1974 [9]	2 970		
Yu.A. Shimanskiy, 1963 [2]	2 970		
V.A. Postnov, 1983 [3]	2 970		
Numerical results ($\boldsymbol{a} = \mathbf{4}\ \boldsymbol{m}, \boldsymbol{b} = \mathbf{4}\ \boldsymbol{m}$)			
$\boldsymbol{q_n}\|_{\boldsymbol{S}} = \frac{1}{4}\left[1 - \cos\left(\pi\frac{(x-2)}{2}\right)\right] \cdot \left[1 - \cos\left(\pi\frac{(y-2)}{2}\right)\right]$	R, m	S	V
$\boldsymbol{T}\|_{\boldsymbol{r}=\boldsymbol{R}} = \mathbf{0}$	*10*	2 769.97	3 008.26
$q_n = 0$ on the cut surface	*40*	2 855.67	2 842.77
	100	2 892.87	2 819.87
	200	2 905.50	2 811.47
	250	2 908.03	2 807.11

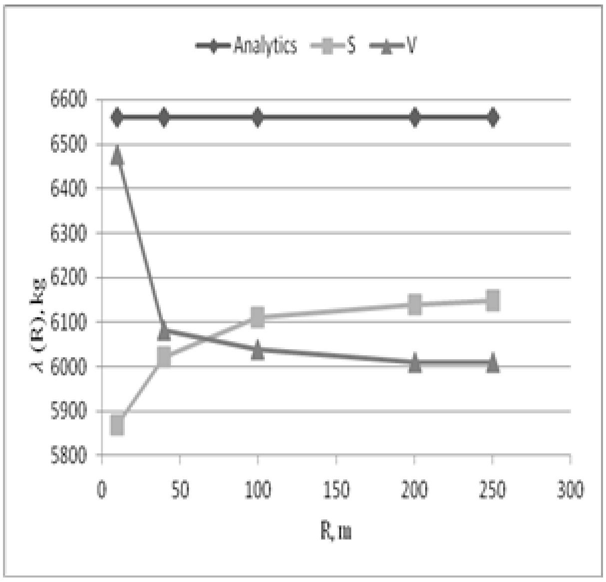

Fig. 6. Convergence of the added masses on the size of liquid environment, the calculations being done by integration over an outer surface and by volume integration

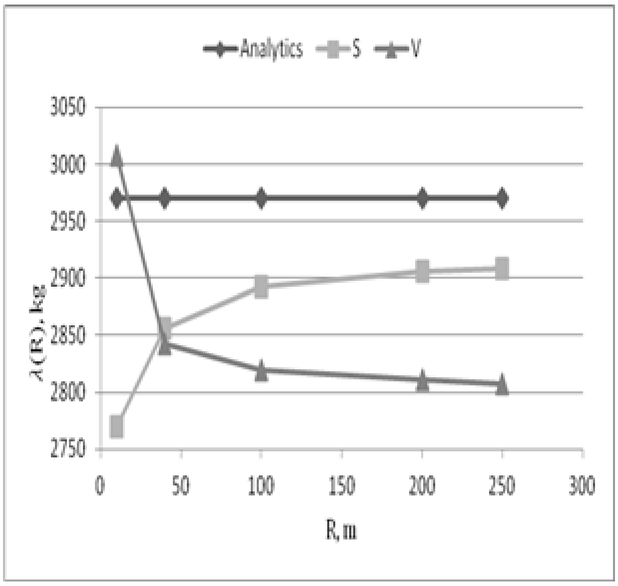

Fig. 7. Convergence of the added masses on the size of liquid environment, the calculations being done by integration over an outer surface and by volume integration

3. Conclusions

Rigid motion of a body.

1. The differences between the results of numerical solution and analytical result are less than 5%.
2. Increasing the size of liquid environment leads to computational complicating. For the circular plate motion, there is a convergence of the added masses in the case of increasing the liquid environment size R. There is the ultimate size of liquid environment $R = 10a$, where a is the radius of a plate.
3. For determining added masses on the plate translational motion, we used two methods of integration: integration over an outer surface of the plate and integration through the liquid volume. Both methods are acceptable for solving the problem; one can use any of them. However, it should be emphasize that the finite element model for volume integration requires a regular finite element mesh through the overall volume.

Vibrations of a fixed rectangular plate.

1. There is a convergence of the added masses for vibrations of a rectangular plate in the case of increasing the liquid environment size R.
2. One can select a reasonable limit for the liquid-environment size, above which there is no need to increase it. The difference of added masses for the liquid-environment sizes $R = 200\ m$ and $R = 250\ m$ is in the range 0.05 – 0.1%. Therefore the most suitable size is $R = 50a$, where a is the biggest side of a plate. Further increasing the size leads to computational complications without improving the results.
3. For vibrations of the rectangular plate in an infinite liquid, the differences between the numerical solution results and analytical one are no more than 6 %.
4. For solving this problem, as in the previous section, we used two methods of integration for determining added masses. As before, both methods are acceptable, but the finite element model requires again a regular finite element mesh through the overall volume.

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